A Theory of Factor Shares and Endogenous Growth

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Abstract

In US data factor shares progress in recursive cycles: a phase when the labor share declines is always followed by a labor share increasing phase. Labor share decline is generally associated with a fast-growing labor productivity, and slow-growing wage rate and employment. We provide a theoretical framework in which endogenous interaction between relative factor prices and labor-saving innovations generates both long run growth and medium run factor share cycles. In our model, capital and labor are complementary inputs; technological progress is labor saving and embodied in capital goods. Capital accumulation, i.e. expansion of an existing technology, tends to increase the labor income share, which, once reaching a certain threshold, provides incentives for firms to innovate to obtain a new and more advanced labor saving technology. Reallocation of labor from traditional to new technology consequentially reduces the labor income share. The model economy settles into recursive factor share cycles, while generating movements between labor productivity, wage and working hours, and factor shares consistent with empirical observation.

JEL classification: E25, O30, O40
Keywords: Factor share cycles ; Competitive innovation; Labor saving technical change; Growth cycles

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1 Introduction

The recent literature that examined the decline of the labor income share over past decades (Karabarbounis and Neiman, 2014; Piketty and Zucman, 2014; Elsby et al., 2013 etc.) has revived interest in macroeconomic theories of distribution. In this paper, we revisit the factor share series in the US and show that the labor income share always progresses in recursive medium-run cycles, i.e. a phase when the labor share declines is followed by a labor share increasing phase, which then precedes the next declining phase. While a constant labor share is usually viewed as a stylized fact in theorizing on economic growth, its fluctuation has been noticed since at least Solow (1958). The goal of this paper is to develop a long-run endogenous growth model that is consistent with the medium-run behavior of factor shares.

The post-World War II US economy, or generally the modern economic growth history, is characterized by a long run accumulation of capital relative to labor. Two key assumptions in our theory are that capital and labor are complementary inputs[1], and that technical progress is labor saving and embodied in capital goods[2]. There are growth episodes in which capital accumulation is mainly in the form of expansion of existing technologies, and others when replacement of old technologies with new ones is the dominant force. The first kind, capital widening, increases the labor income share under capital-labor complementarity, while the second, capital deepening, reduces the labor income share due to the labor saving nature of technological progress. In our model, under the interaction between relative factor prices and labor-saving innovations, the economy endogenously alternates between the two growth episodes, generating simultaneously long run growth and medium run factor share cycles.

We start with documenting empirical patterns related to the labor income share over time in the US. It has long been recognized that the labor income share behaves countercyclically–it rises during recessions (e.g. Boldrin and Horvath, 1995; Gomme Greenwood, 1995). We first show that the labor share, in addition to countercyclical rises in recession times, displays substantial variations during non-recession periods as well. We then smooth out business cycle fluctuations of the labor share by calculating moving averages of 2 to 4 years and the HP trend, and show that the labor share, across all sectors and in both gross and net terms, is characteristic of recursive decline-then-rise medium-run cycles. The labor share is never a constant, but always fluctuates within a certain interval. Our empirical investigation then turns to documenting the second set of facts that the labor share co-moves systematically with other labor market variables. In particular, we show that there is a significant and negative correlation between changes in labor share and labor productivity, that is, labor share declines typically in periods when

[1]That capital and labor are complementary inputs in the aggregate is supported by most empirical literature estimating the elasticity of substitution between the two production factors. See e.g. Antras (2004); Oberfield and Ravan (2020).

[2]Research on the biased-ness of technological progress and if the pace at which it advances should be made responsive to movements in factor prices dates back to at least Hicks (1932) and Kennedy (1964). Recent contributions include Acemoglue (2002) and Jones (2005), among others.

the labor productivity progresses at a fast rate. On the other hand, change in labor share is positively correlated with changes in wage rate and employment/working hours. The decrease of labor share tend to associate with a slow-growing labor price and quantity.[3]

Our model is one of vintage capitals. We start with a version of model in which labor supply is exogenous. Production requires two complementary inputs, capital and labor. More recent vintage of capital embodies more advanced and labor saving technology in the sense that it requires less labor to produce one unit of the final good. Each vintage of capital can be used to reproduce itself. It can also be employed by profit-maximizing firms in its innovation activities to produce the next vintage of capital–equivalently more advanced technology. In order to obtain one unit of next vintage capital good as output, firms need to use more than one unit of current vintage capital as input. That innovation is costly guarantees that it occurs only when it is economically convenient to do so.

We show that there exists a unique equilibrium path, in which the model economy settles into recurring cycles, each consisting of what we labeled a growth phase and a build-up phase. During the growth phase, two capital goods of consecutive qualities are simultaneously employed in producing the final good, and labor continuously reallocate from less to more advanced method of production. This labor reallocation process increases average labor productivity as more labor is employed by more advanced technologies over time. On the other hand, the real wage is stagnant as the productivity of the marginal labor employed in the old technology is not altered before the process finishes. Together, these lead to a declining labor share. The growth phase in our model corresponds to episodes of rapid technical progress in the real world. For example, the rapid progress in automation and information technologies over the past decades might leave productivity increasing faster than wage, and reduce the share of income that accrues to labor (Hubmer and Respetro, 2021).

The growth phase ends as all labor force is employed by the new technology. At that point, we show that the economy will not enter immediately to the next growth phase. That is, a even more advanced technology will not be introduced immediately to produce the final good. The reason is that capital is too expensive at that point which renders introducing a more capital intensive method of production not profitable. We show that the economy will then enter what we called build-up phase, in which current vintage capital is accumulated. Due to capital-labor complementarity, the accumulation of capital reduces capital price and increases wage rate and its share of income. Moreover the labor productivity is stagnant as old method of production repeats and new ones have not yet utilized. The build-up phase endogenous leads to the next growth phase. A rising wage rate and labor share during the build-up phase implies an increasing cost of using current method of production. Once the labor income share, which also resembles the unit labor cost, reaches a certain level, it becomes profitable for firms to innovate and obtain the next generation of capital, which embodies a technology that requires relatively less labor in

[3]It should be noted that the labor income share is defined as labor productivity divided by real wage rate. This per se, however, does not necessarily imply the correlations here, see details in the facts section.

production. This starts the next growth phase and a new cycle.

While the model with exogenous labor supply generates endogenous factor share cycles, it is silent on the positive correlation between factor shares and employment/working hours observed in data. We then move to a version of model in which labor supply is endogenous. Adding endogenous labor supply does not qualitatively change the model property of recurring factor share cycles. It, however, produces a positive correlation between change in labor share and change in employment/working hours. In particular, during a growth phase when the labor share declines, employment decreases because of a rising consumption and a stagnant wage. On the other hand, a rising wage during a build-up phase encourages workers to supply more working hours. This is consistent with the empirical evidence that employment drops upon arrival of a permanent technological improvement and start rising again only quite a few quarters later (e.g. Gali, 1999; Basu et al. 2006).

Endogenous growth in our model comes from decentralized technological change. As in other models with endogenous technological change (e.g. Romer, 1990; Grossman and Helpman, 1991; and Aghion and Howitt, 1992) innovations and the adoption of new methods of production are carried out by profit maximizing firms moving up a capital ladder. However, innovation is not driven by monopoly profits, but comes about as a response to changes in relative prices in competitive markets (Boldrin and Levine, 2001; 2008). In the model, it is desirable to take advantage of an existing technology until diminishing returns make it profitable to move on to a new technology. Different from endogenous growth models in which innovation is driven by monopoly power, as there is no market power led distortions, the competitive equilibrium is Pareto efficient in our model. In addition, innovation is biased towards labor and against capital in our model, which also distinguishes our framework from the models mentioned above.

A stationary yet fluctuating factor share is an equilibrium rather than an deviation from it. Our model economy features endogenously recurring medium-run cycles: neither aggregate productivity nor preference shocks are assumed. We are not the first to deal with it. To the best of our knowledge, though, we are the first to make the claim that a sound theory of endogenous growth cycles can be built upon the observation that firms expand productive capacity when they expect the adopted technology to yield a profit in future periods, while they reduce capacity and try to change their technology when they realize the latter is no longer profitable at the expected equilibrium prices.[4] In our model a relatively stable long run trend obtains, around which growth cycles occur; the endogenous interaction between relative factor prices and labor-saving innovations is the source of both long run growth and medium run cycles.

A few papers have studied the behavior of factor shares over the business cycle fre-

[4]Explicit mention should be made, though, of Goodwin (1967) and Reichlin (1986). The economic intuition underlying the endogenous oscillations in these two models is quite akin to ours. However, technological innovation being absent, there is no growth, either exogenous or endogenous, in either model.

quency (Boldrin and Horvath, 1995; Gomme and Greenwood, 1995; Young, 2004; Boldrin and Fernandez Villaverde 2005; Choi and Rios-Rull 2009, 2019; Rios-Rull and Santaeulalia-LIopis, 2010).[5] Boldrin and Horvath (1995) present a real business cycle model of contractual arrangements between risk averse employees and risk neutral employers. The optimal contract, which insures workers away from a fall of wage in recessions, generates a countercyclical labor share. Similarly, Gomme and Greenwood (1995) build a model where workers purchase insurance from the entrepreneurs through optimal contracts. Since our model focuses on fluctuations over frequencies longer than typical business cycles and assumes complete markets, none of the considerations used in those papers is directly pertinent to the mechanism explored here. Examining the factor share series, we find that the labor share not only behaves countercyclically, it also demonstrates a U-shape especially during some long expansions. A combination of risk-sharing contractual arrangements and the technological structure explored in our paper potentially generates a factor share pattern during expansions and recessions that aligns more with empirical observation.

The labor share decreases during a growth phase of the model economy, as labor reallocates from traditional to more advanced technologies. This technology view is shared by some recent studies on the decline of labor share in past decades (e.g. Hubmer and Respetro, 2021). One such example is the replacement of online retailers with traditional Mom-and-Pop stores; the expansion of the former, which admits a relative low labor share, drives down the aggregate labor share in the retailing industry (Boldrin and Zhu, 2021). One insight of our model is that forces that drives down the labor share endogenously and eventually lead to the rise of it. Extending the factor share series to the year of 2020, we find that the labor share has actually increased for a nontrivial magnitude from 2014 to 2019 and before a recession in 2020. For the whole economy, the gross and net labor income share increase for 1.7 and 2.6 percentage points respectively. For non-financial corporations' sectors, the labor share has risen for about 3.3 percentage points in that six year interval. While longer periods of data is needed for a firm conclusion, the recent rise of labor share seems not to preclude the possibility that the previous decline might be part of a larger decline-then-rise cycle. This looks even more feasible if we focus on the net labor share, which does not decrease as much as the gross one since the late 1990s.

The idea that technological changes both respond to and affect factor prices is close in spirit to the intuitive arguments given by Blanchard et al. (1997) and Caballero and Hammour (1998). Inspired by the European experience in the 1970s and 1980s, these papers have explored the dynamics over the medium-run induced by exogenous changes in real wages. After an initial increase in wages, due for example to an exogenous strengthening of the bargaining power of workers, the capital share goes down. However, over time firms react by adopting technologies that reduce the labor input per unit of output, leading to a recovery or even an increase in capital shares. The key difference in our

[5] Also related is the branch of literature focused on explanations based on models with imperfect competition and/or increasing returns to scale, e.g. Hornstein (1993), Ambler and Cardia (1998), Bils (1987), Rotemberg and Woodford (1999), and Hansen and Prescott (2005).

investigation is that we do not begin with an initial, exogenously given shock to wages (due to a change in technology, bargaining power or markup) and explore the aggregate dynamics after such a shock. We view the changes in capital income share as a systematic and recurrent feature of the economy: the main driving force behind the introduction of new technologies and, therefore, of sustained growth.

Growiec et al. (2018) decomposes the fluctuation of labor share into short ($\leq$ 8 years), medium ($8 - 50$ years) and long run ($\geq$ 50 years) frequencies, and find that medium-to-long run fluctuations account for about 80% of the total labor share fluctuation. Leon-Ledesma and Satchi (2019) studies the response of factor shares to productivity shocks over the short and medium run in an environment in which firms are allowed to choose technologies (Jones, 2005; Caselli and Coleman, 2006). When there is negative technology shock, employment declines, which generates a (counter-cyclical) rise in labor share under capital labor complementarity. Firms can switch technology but choose not to do that immediately due to existence of adjustment costs to technology choice. Over time, as firms choose to adopt more labor saving technologies, the labor share declines over the medium term.[6]. As in Leon-Ledesma and Satchi (2019), our model also produces an elasticity of substitution between capital and labor over the medium/long run is different from that over the short run, we differ in that the economy in our model endogenous alternate between phases in which capital formation substitutes labor in net and others when there is gross complementarity between the two production inputs.

Last, we note the similarities between some points of our model and the literature on directed technological change (e.g. Acemoglu (2002)). Three macroscopic differences are that (i) we claim growth cycles are 'caused' by labor-saving technological change, (ii) we focus on the fundamental bias (labor vs capital) in a perfectly competitive environment, and, (iii) we make the bias endogenous and not exogenous. In a recent paper, Acemoglu and Restrepo (2018) endogenizes the direction of innovations, either automating existing jobs or creating new labor intensive tasks, and allows it to be responsive to factor prices. A temporary increase in automation technology reduces the labor share in the short run, but also induce R&D efforts in creating new labor intensive tasks, which stabilize factor shares in the long run.[7] In the balanced growth of that model economy, the labor share is a constant. Different from that research, the labor income share in our model economy is stationary but not a constant, instead the economy endogenously alternates between a labor share declining phase and one with rising labor income share.

The rest of paper is organized as following: Section 2 presents the stylized facts. Sec-

[6] The result that the labor share responds differently to technology shock over time is consistent with Rios-Rull and Santaeulalia-Llopis (2010) which documents that a productivity shock produces a reduction of labor share at impact, making the short run response countercyclical, but it also produces a subsequent increase of labor share that overshoots its long run average. See Choi and Rios-Rull (2019) which incorporates putty-clay technology and search and matching frictions into a real business cycle model to reconcile the fall-then-rise response of the labor share to technology shocks.

[7] Growiec et al. (2018) also provides a model with directed technological change which carries similar intuition.

tion 3 outlines the model and characterizes the competitive equilibrium. A discussion on irregular cycles and output and population growth is provided in Section 4. Section 5 provides concluding remarks.

2 Facts

It has long been noticed that factor income shares fluctuate over the business cycle frequency (e.g. Boldrin and Horvath, 1995; Gomme Greenwood, 1995; Rios-Rull and Santaeulalia-LIopis, 2010). In this paper we focus on frequencies longer than those of usual business cycle models to provide a theory of long-run growth capable of explaining the recurrence factor share cycles. We first show that the labor share, in addition to rise countercyclically during recession, demonstrates substantial swings during non-recession periods. Then we smooth out business cycle fluctuations by taking moving averages (of 2 to 4 years) and obtaining the HP trend, and show that the labor share progresses in what we label 'medium run cycles'.

To calculate labor income share for the whole economy, we follow a common approach of dividing proprietor's income between labor and capital according to the share of capital income observed in the rest of economy (Cooley and Prescott, 1995). Therefore, output is defined as National Income less *proprietors income*, and the gross labor share is defined as,[8]

$$LS = \frac{\text{compensation of employees}}{\text{national income} + \text{depreciation} - \text{proprietors income}},$$

and the net labor share is defined as

$$LS^{net} = \frac{\text{compensation of employees}}{\text{national income} - \text{proprietors income}}.$$

By definition, the gross (*resp.* net) capital income share equals to 100% minus the gross (*resp.* net) labor income share.

Figure 2.1 plots the gross labor income share for the whole economy; also plotted is a HP trend with $\lambda = 1600$ and the NBER dating of recessions. A few remarks are in order. First, during a recession, the labor income share typically rises and reaches a local maximum. This confirms the well documented counter-cyclical property of the labor share in the literature. Second, the labor income share also shows substantial movements

[8]Data for our measures are taken directly from NIPA, Table 1.7.5 'Relation of gross domestic product, gross national product, net national product, national income, and personal income', and Table 1.12 'National income by type of income'. Since we only need percentages, we take nominal quantities that avoid distortions induced by price indices. Our sample of quarterly data goes from 1947-q2 to 2020-q4. In the benchmark, we don't subtract *taxes on production and imports* from value added in calculating the labor income share, as taxes data is not available for non-farm business sectors, or non-financial corporations sectors. We show in Figure 6.1 in appendix that adjusting for taxes has a negligible impact on the cyclical properties of labor income share in the aggregate economy.

in non-recession periods. For example, during the long expansions in 1960s, 1980s, 1990s and 2010s, the labor income share rises much earlier than when recessions kicks in. In middle-to-late 1960s, In two quarterly around the fourth quarter of 1969 when a recession which occurs, the labor share jumps by 0.95 percentage points. However, the labor share starts to rise much earlier and since the first quarter of 1964, which is almost 6 years earlier than the recession point. Similarly, in 2010s, the labor income share has already risen since 2015 and for about 5 years when a recession takes place in 2020. Third, that the labor income share demonstrates substantial variation at frequencies longer than typical business cycles, can be clearly seen from HP trend of the labor share in Figure 2.1. The labor share trend coincides with some long expansions, e.g. those in the 1960s and 2010s. However, it smooths out fluctuations in some small recessions. For example, during late 1950s and early 1960s, there are three moderate recessions, and labor share peaks in each of the recessions. The labor share trend, however, is monotonically decreasing in late 1950s and early 60s.

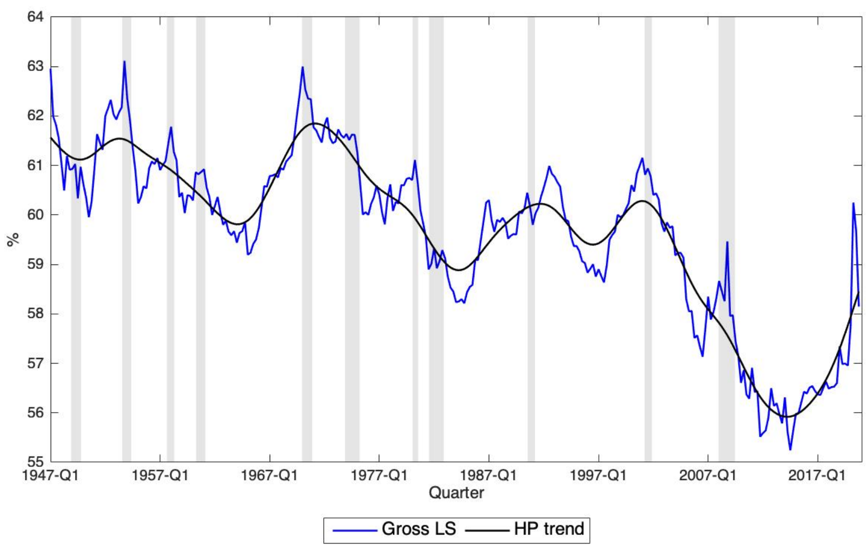


Figure 2.1: Gross labor share in the whole economy

While labor income consists of a single item 'compensation of employees' in national income and product accounts, capital income contains *rental income of persons*, *corporate profits*, *net interest and miscellaneous payments*, and *consumption of fixed capital*. We decompose the aggregate capital share into its four components and present the results in Figure 6.2 in appendix. As can be seen, *corporate profits* account for most of the cyclical pattern of the gross capital share.[9] Net interest is relatively a-cyclical. The rental income is also relatively smooth over time. The share of depreciation in gross income typically peak in

[9]The fact that corporate profits are the main driver of the factor share cycles supports our focus on the production side in the model part. It should be noted that the corporate profits in NIPA are accounting profits, a substantial portion of which corresponds to the opportunity cost of firms's owned capital from an economic point of view.

recessions, which is contrary to that of the gross capital share. Adjusting for depreciation, however, does not alter the cyclical properties of gross factor shares. As confirmed in Figure 6.3 in appendix, the cyclical pattern is not affected if we focus on the net labor (*equivalently* capital) income share.[10]

The labor income share co-moves systematically with other labor market variables over the medium run. Table 2.1[11] presents the correlation between growth in labor income share and growth rate in labor productivity (LP), wage rate (i.e. real compensation per working hour), employment, and working hours.[12] In addition to the original measure at the quarterly frequency, for each variable, we further calculate its 9-quarter (2-year), 13-quarter (3-year) and 17-quarter (4-year) moving average, and the H-P trend to smooth out typical business cycle fluctuations. We then calculate the growth (rate) of the moving averages and HP trend terms.[13]

Table 2.1: Corr. btw. growth in LS and other var.

Var.	LP	Wage	Emp	Hours
Quarterly	-0.50^{***}	0.44^{***}	-0.35^{***}	-0.39^{***}
9-quarter MA	-0.46^{***}	0.34^{***}	0.24^{***}	0.13^{**}
13-quarter MA	-0.39^{***}	0.38^{***}	0.28^{***}	0.17^{***}
17-quarter MA	-0.28^{***}	0.45^{**}	0.34^{***}	0.24^{***}
HP trend	-0.35^{***}	0.35^{**}	0.33^{***}	0.27^{***}

Note: $^{***}: p < 1\%$; $^{**}: p < 5\%$.

At all frequencies, there is a significantly negative correlation between growth in labor share and labor productivity, and a significantly positive correlation between growth in labor share and wage rate. It should be noted that by definition, labor share equals labor productivity divided by real wage. Though this equation imply growth in wage is slower than labor productivity in periods when labor share declines, it does not necessarily imply a negative (*resp.* positive) correlation between growth in LS and LP (*resp.* wage). For example, it is possible that labor productivity grows at a higher rate in labor share increasing periods than in labor share decreasing periods, as long as wage rate also grows

[10]Koh et al. (2020) documents the importance of changes in classification and accounting criteria for understanding the long run behavior of factors shares. The cyclical patterns, however, are not affected.

[11]For both the non-farm business sectors and non-financial corporations sector, the Bureau of Labor Statistics provides data on labor share, labor productivity, employment and working hours for both the non-farm business sectors and non-financial corporations sector. The correlations showed here is for the non-financial corporations sectors, however, as presented in Table 6.1 in appendix, the similar correlations also hold for the non-farm business sector.

[12]Note that the labor share is already in percentage, therefore we calculate its growth, instead of growth rates. The correlations using growth rate for labor share are qualitatively the same. For consistency and without causing confusion, we also use 'growth in X' to denote the relative change in X (non-LS variables).

[13]We focus on the correlation between growth in LS and growth in other variables as it is the empirical counterpart to our model, a point we will illustrate more in the model section.

faster in the former. From the facts documented above, however, the labor income share declines typically in periods when the labor productivity grows at a faster rate and wage rate is relatively stagnant. Figure 2.2 plots the growth in 3-year moving average of LS and LP, and Figure 2.3 plots that for LS and wage rate. By looking at the 3-year moving average, it also becomes clear that the correlation between growth in LS and LP (Wage) is not mainly driven by business cycle fluctuations. Actually, most co-movement between LS and wage (*resp.* opposite movement between LS and LP) occurs during expansions.

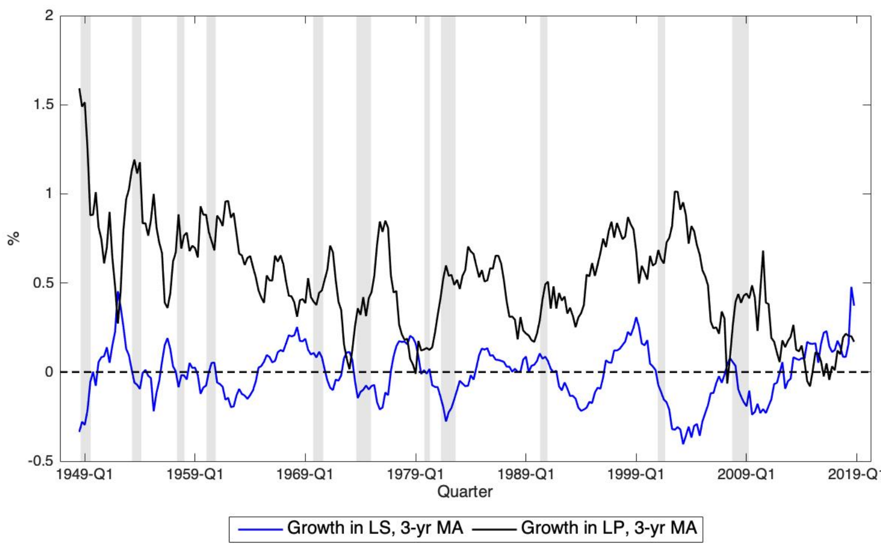


Figure 2.2: Growth in LS and LP, 3-yr MA

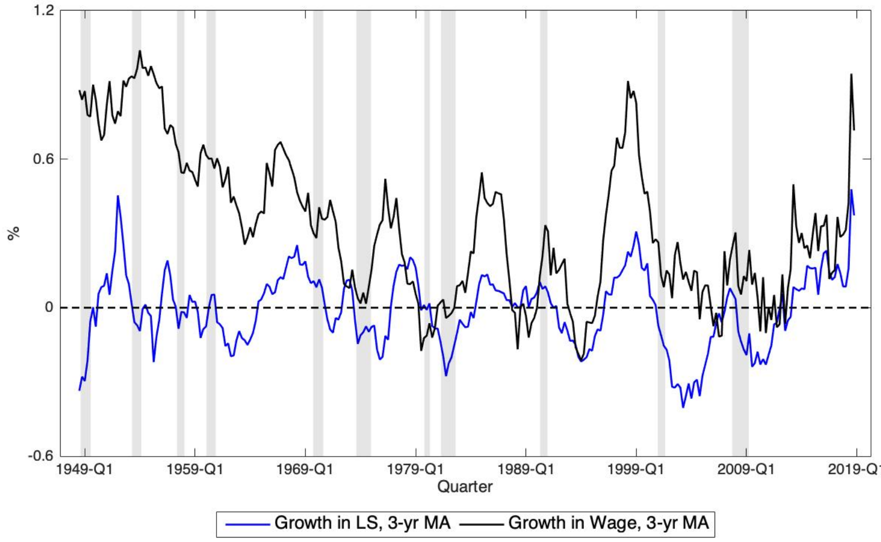


Figure 2.3: Growth in LS and wage, 3-yr MA

On the other hand, the correlation between labor share and employment/working hours changes from business cycle frequency to lower frequencies. The correlation is significantly negative for quarterly data, due to the fact that labor income share is at a local maximum during recessions when employment/hours is at a local minimum. However, when we move to lower frequencies to smooth out business cycle fluctuations, there is a robust positive correlation between growth in labor share and employment/hours. That is, during periods when the labor income share declines, employment and working hours in average grows at a relatively slow pace. Figure 2.4 plots the 3-year moving average of growth in labor share and growth rate in employment.[14]

The above change in signs of correlation is mainly due to the following two reasons. First, the moving average (or HP trend) of LS rises significantly less during recessions. For example, the maximum change of quarterly labor share, relative to the previous quarter, during the 2007-09 financial crisis is about 1.8%; At 3-year moving average, the maximum change is however significantly smaller, at 0.1%. Second, the labor income share co-moves with employment/working hours during non-recession periods. One example is the 2000s when the labor share declines for most of the time, while employment also declines or grows at a low rate. The clear contrast in the correlation between labor share and employment/working hours suggest that the medium run dynamics we focus on in this paper differs from fluctuations over the business cycle frequency. A CES production function with general complementarity between capital and labor has been used in the literature to generate counter-cyclical labor shares–i.e. a negative correlation between employment and labor share, the positive correlation between the two variables rules out

[14]Figures 6.6-6.9 in appendix present the similar correlation but for HP trend of each variable; and Figures 6.10-6.13 present the correlation for raw quarterly data of each variable.

the importance of this mechanism over the medium run.

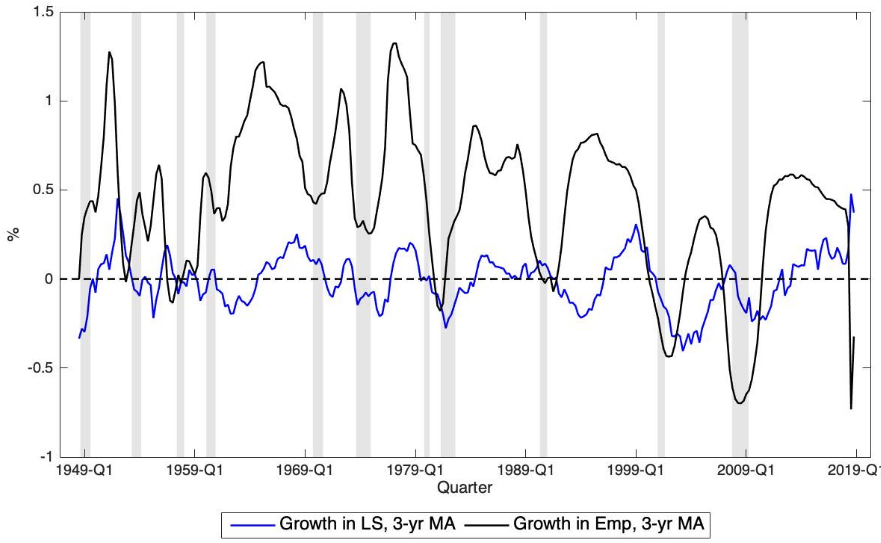


Figure 2.4: Growth in LS and Employment, 3-yr MA

A consequence of focusing on moving averages and HP trend terms is that decline in real value added during recession periods is essentially eliminated. As shown in Figure 6.14 in appendix, the 3-year moving average and the HP trend term of real value added in non-financial corporations sectors increases in almost all recessions, but the recent financial crisis and possibly the coronavirus recession as more data is available. Therefore models that rely on a sticky wage, due to e.g. contract, and declining income during recession times (Boldrin and Horvath, 1995) can not explain the rise of the labor share moving averages (or the HP trend terms) during some recessions. To further confirm that the correlations documented in Table 2.1 are not driven by recessions. we run a regression of growth in LS (quarterly series, Moving averages, and HP trend) against growth of each of other variables, while controlling for a recession dummy in the regression. As shown in Tables 6.2-6.4 in appendix, the sign and significance of correlations after controlling for a recession dummy are essentially the same as those in Table 2.1.

We conclude this section by commenting on the sector composition of labor shares. It has long been noticed that the labor share at the sector level is not as stable as at the aggregate level (Solow, 1958). Recent literature (Boldrin and Zhu, 2021; Hubmer and Respetro, 2021) has documented that the recent labor share decline concentrates in the manufacturing and trade sectors. Young (2010) decomposed movements in labor share from 1958-1996 into a within-sector and between-sector component and found that fluctuations are mainly driven by the within-sector component, while the role of cross-sector reallocation is negligible. This supports our focusing on the aggregate share and abstracting from multi-sectoral considerations in the theoretical analysis, which we turn next.

3 The Model

In this section we develop a theoretical model to reconcile the empirical pattern documented above. As becomes clear below, our model generates long run endogenous growth with medium run cycles. The economy alternates between a phase when the labor income share rises, with rising wage and slow-growing labor productivity and a phase when the labor income share declines with stagnant wage and fast growing labor productivity. We work under the assumptions of a representative agent and of recursively complete markets. For the time being we also assume labor supply is exogenous and fixed at one, endogenous work-leisure choice will be incorporated later in Section 3.3. The model with exogenous labor supply is clean and used to deliver the main intuition, while adding endogenous labor supply generates correlation between labor share and employment/working hours as observed in data.

Preferences When labor supply is exogenous the representative household maximizes the following utility over the infinite horizon,

$$\max \int_0^\infty e^{-\rho t} \log c(t) dt.$$

where $c(t) = \sum_{j=0}^{\infty} c_j(t)$, with $c_j(t)$ the consumption flow from technology j at instant t;

Production Production takes places in three different sectors denoted by $s = 1, 2, 3$. Each sector is composed by a continuum of identical firms endowed with capital of some vintage[15]. The capital stock $k^s(t)$ evolves endogenously over time, as detailed below. The first sector produces consumption goods, the second capital goods and the third new technologies embodied in new kinds of capital goods.

Technological Vintages There exists a countably infinite number of potential technologies, indexed by the subscript $j = 0, 1, ...$ Technologies are embodied in capital goods, hence $k_j^s(t)$ denotes the stock of capital embodying technology j installed in sector s at time t. We say that a technology j is active in sector s during period t if $k_j^s(t) > 0$[16].

Technological Progress A technology with an index j is better than a technology with index $j' < j$ for two reasons. First, to produce one unit of final consumption, a unit of capital of type j requires less labor than a unit of capital of type j', i.e. technological progress is labor saving. Secondly, technological progress is incremental insofar as capital goods of type $j + 1$ can be obtained, at a cost, only from capital goods of type j and not from any other $j' < j$.

Consumption Sector As mentioned, the first sector uses capital of vintage j to produce consumption, $c_j(t)$, using capital $k_j^1(t)$ and labor, $l(t)$ according to a fixed coefficient

[15] Because firms are identical in each sector, we will talk, indifferently, either of a representative firm with a stock of capital equal to $k^s(t)$ or of a continuum of identical firms, each one with $k^s(t)$ units of capital.

[16] Think of them as plants with constant returns to scale.

production function,

$$c_j(t) = \min\{k_j^1(t), \gamma^j l(t)\}, \quad \gamma > 1.$$

This means that, for every given technology, capital and labor are perfectly complementary inputs.[17] Assuming that γ is greater than one captures the assumption that technological progress is labor saving. As new vintages are adopted the labor-input requirement to produce 1 unit of the final good decreases of a factor $1/\gamma$.

Capital Widening Sector The second sector produces additional capital of type j from capital of the same type according to the widening equation,

$$\dot{k}_j(t) = bk_j^2(t), \quad b > 0.$$

The widening technology allows capital of any given vintage to self-accumulate at the rate b.

Capital Deepening Sector The third sector produces a new type of capital, type $j+1$, from capital of type j according to the deepening equation,

$$k_{j+1}(t) = \frac{k_j^3(t)}{a}, \quad a > 1.$$

Capital used in the deepening sector depreciates instantaneously as it is transformed in $1/a$ units of new vintage capital. Capital $j+1$ can be obtained directly only from capital j but not from any $j', j' < j$. However, capital $j+1$ can be obtained from capital $j', j' < j$ by applying the innovation technology $j - j'$ times. The innovation ratio, in this case, would be equal to $a^{j'-j}$

Because capital of vintage j can be employed in either of the three sectors, at any point in time t the following resource constraint holds

$$k_j(t) = k_j^1(t) + k_j^2(t) + k_j^3(t).$$

The *accumulation equation* for capital j is,

$$dk_j(t) = bk_j^2(t)dt - k_j^3(t) + \frac{k_{j-1}^3(t)}{a}.$$

This equation says that the stock of capital j changes because of (i) self accumulation (first term), (ii) full depreciation of the amount used to innovate (second term) and, (iii) innovation from capital $j-1$ (third term). Note that we allow for discrete conversions (jumps) from any vintage of capital to the next, as captured by the second and third terms.

[17] Perfect complementarity is used to deliver analytical solutions. All qualitative results hold for a production function with gross capital-labor complementarity.

This economy is an ordinary diminishing return economy with three sectors: consumption, widening and deepening. Diminishing returns to capital accumulation derives from the fact that capital and labor are complementary inputs and labor supply has an upper bound. Note also that, as there is perfect competition, the welfare theorems hold. That is, the efficient allocation can be decentralized as a competitive equilibrium and vice versa[18]. Therefore, the competitive equilibrium prices correspond to the co-state variables of the planner's problem, a fact we exploit repeatedly in computing the dynamic equilibrium paths.

3.1 Competitive Equilibrium - Illustration

Three parametric assumptions are crucial for the working of our model and incorporate the underlying main intuitions. (1) $b > \rho$, i.e. the rate of capital self-accumulation is larger than the discount rate, which makes capital accumulation profitable. (2) $a > 1$, i.e. capital deepening is costlier than widening, hence innovation will not take place unless a new capital vintage is necessary. (3) $\gamma > 1$, i.e. capital of the more advanced vintages requires less labor to produce one unit of the final good.

Below we prove that, under these assumptions, the competitive equilibrium of our economy settles into a recurring growth-cycle. This cycle contains a *growth* phase, where capital goods of two consecutive vintages are simultaneously used and labor is being reallocated from the less to the more advanced vintage capital, and a *build-up* phase, where capital is accumulated at the backyard until its price decreases to a level that makes profitable to create a new vintage of capital and use it for production of the final good. The labor income share decreases during the *growth* phase and increases during the *build-up* phase. The endogenous interaction of factor shares and technical progress drives the recurring growth cycles.

Figure 3.1 illustrates the evolution of capital stock in the consumption sector for three consecutive growth-cycles.[19] Start at time $t = 0$, when capital $j + 1$ begins to be used in producing the consumption good. Consumption increases over time as capital j is turned into the more advanced capital $j + 1$ and labor reallocated from the former to the latter. This growth phase ends at time $t = \tau^g$, at which point capital $j + 1$ employs all supplied labor. As shown later, at $t = \tau^g$, capital $j + 2$ will not be introduced immediately into producing the final good. Capital is accumulated in the backyard, and capital $j + 1$ remains employing all labor until its price is low enough for introduction of vintage $j + 2$ into the consumption sector profitable. This takes place at $t = \tau^g + \tau^b$ when firms begin creating and employing capital $j + 2$ in producing consumption good. During this build-up phase, from $t = \tau^g$ to $t = \tau^g + \tau^b$, the output of the consumption sector is constant as labor is already fully employed by capital of type $j + 1$. The recurring growing-then-stagnant evolution of consumption is illustrated in Figure 3.2.

[18]The competitive equilibrium of the economy can be described in the usual way by combining utility and profits maximization.

[19]Note what is ploted here is an illustration of the general pattern. The actual value of capital of a specific vintage is addressed later, which is generally not linear in time.

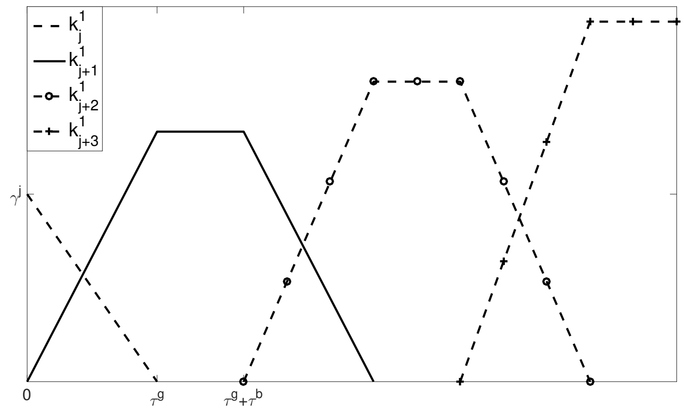


Figure 3.1: Evolution of capital stock in the consumption sector

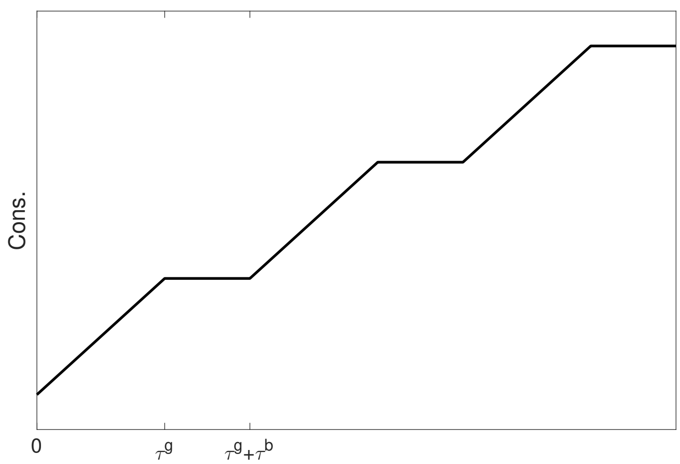


Figure 3.2: Evolution of consumption

Figure 3.3 depicts the evolution of total capital stocks. At time $t = 0$, capital $j + 1$ used in the consumption sector is 0, however, as shown below at that point there is a discrete amount of capital $j + 1$ that is allocated in sector 2 for self accumulation. The total capital of vintage $j + 1$ is therefore strictly positive at $t = 0$. Capital $j + 1$ grows over time as part of k_{j+1} is used for self accumulation, and part of k_j is employed in sector 3 for innovation. At $t = \tau^g$ when the growth phase ends, capital $j + 2$ may or may not be immediately created from capital $j + 1$ and two different - but consumption- and welfare-equivalent - paths are possible. We leave the discussion of the technical details to the appendix and focus here on the path we find more intuitive, represented in the figure. Along this path, for τ^b units of time capital of vintage $j + 1$ is still accumulated, over and above the quantity needed for full employment in the consumption sector. This "over-accumulation" of old capital continues until its price reaches a level low enough for innovation (introduction of vintage $j + 2$) and immediate usage in the consumption sector become profitable. This takes place at $t = \tau^g + \tau^b$, when firms convert a discrete amount of capital $j+1$ into capital $j+2$ and begin employing the latter in the consumption sector. A new growth phase then begins.

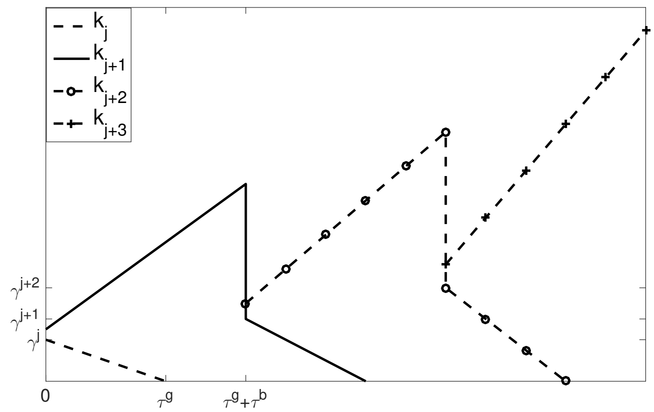


Figure 3.3: Evolution of total capital stock

When capital j and $j+1$ are simultaneously used to produce consumption (growth phase) the labor-saving nature of technological progress changes the factor shares of income. As more labor is employed by capital $j+1$, and less by capital j, the aggregate labor income share decreases. Put differently, the process of labor reallocation from one vintage to the next increases labor productivity but not wages, therefore leading to a declining labor income share. During the build-up phase, the price of capital and its rental rate decline as capital self accumulates and total output remains constant. As a result, during this phase the capital income share decreases and the labor share increases. Figure 3.4 illustrates the recurring cyclical behavior of the labor share.

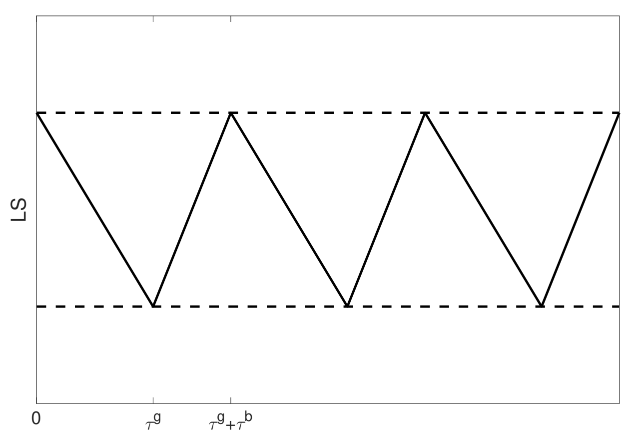


Figure 3.4: Evolution of labor income share

3.2 Competitive Equilibrium - Characterization

We now formally establish the properties of the competitive equilibrium just described. Use marginal utility as the numeraire[20]. The price of period-t consumption is, therefore, $1/c(t)$. Denote with $q_j(t)$ the price of capital j in period t. Capital can be used for widening to create more capital of the same quality. The physical rate of return for capital widening is b. Zero profits for capital widening implies that this return plus the capital gains must equal the subjective discount rate

$$b + \dot{q}_{jt}/q_{jt} = \rho,$$

or, equivalently[21],

$$\dot{q}_j(t)/q_j(t) = -(b-\rho) < 0.$$

As it accumulates, the price of capital decreases over time. Its level is characterized in the following proposition.

Proposition 1: *No more than two vintages of capital are simultaneously used to produce consumption, and these must be consecutive vintages. If j' is used to produce consumption, the price of capital $j, j > j'$ satisfies*

$$q_j(t) \geq v_j(t) \equiv \frac{\gamma^{j-j'}-1}{\gamma^{j-j'}-1/a^{j-j'}} \frac{1}{bc(t)}$$

with equality if j is also used to produce consumption.

Proof: see Appendix.

The key step in the proof is the computation of the price of capital. Without loss of generality, assume both capital j' and j, $j > j'$, are used to produce consumption. Denote w, r_j and $r_{j'}$, respectively, the wage rate, the return to capital j, and that to capital j', all in units of the consumption good.[22] The zero profit condition in the consumption sector implies

$$1 - r_j - \frac{w}{\gamma^j} = 0,$$
$$1 - r_{j'} - \frac{w}{\gamma^{j'}} = 0,$$

and the zero profit condition in the innovation sector leads to,

$$r_j = a^{j-j'} r_{j'}.$$

[20]The price of capital corresponds directly to the co-state variable associated with the law of motion for capital, in the current-value Hamiltonian representation of the planner's problem.

[21]This can also be derived from the Euler equation for the planner's problem.

[22]For computational convenience, factor prices are expressed in units of the consumption good. Multiplying these prices by $1/c(t)$ gives the price in terms marginal utility.

This is a system of three independent equations with three unknowns. It yields

$$w = \gamma^{j'} \frac{a^{j-j'} - 1}{a^{j-j'} - 1/\gamma^{j-j'}},$$

$$r_j = \frac{\gamma^{j-j'} - 1}{\gamma^{j-j'} - 1/a^{j-j'}},$$

and $r_{j'} = r_j / a^{j-j'}$. The rental rate is the flow value of capital. This rate, divided by b, the rate of capital accumulation, gives the value of the stock of capital, which is

$$v_j(t) = \frac{1}{c(t)} \frac{r_j(t)}{b} = \frac{1}{bc(t)} \frac{\gamma^{j'-j} - 1}{\gamma^{j-j'} - 1/a^{j-j'}}.$$

The $1/c(t)$ term converts the price of capital in units of the numeraire, which is marginal utility.

When both capital j and j', $j > j'$, are used in production, 1 extra unit of capital j used in producing the consumption goods demands $1/\gamma^j$ units of labor, which leads to unemployment of $\gamma^{j'-j}$ units of capital j' at a given level of labor supply. The value of 1 extra unit of capital j should therefore reflect the fact that certain units of capital j' become obsolete due to its usage. This is the reason why the coefficient in the formula of Proposition 1, $\frac{\gamma^{j-j'}-1}{\gamma^{j-j'}-1/a^{j-j'}}$ is less than 1. This "replacement effect" also explains why there are at most two consecutive vintages of capital simultaneously used in production. When capital j' produces consumption goods, zero profits in the innovation sector implies that the price of capital $j, j > j' + 1$ increases proportionately by $a^{j-j'}$. However, its value in production, due to the replacement effect, does not increase as much. It is therefore not profitable to adopt any capital $j, j > j' + 1$, into production.

Proposition 2 fully characterizes the cyclical behavior of output growth and of the factor shares.

Proposition 2: *Consumption grows at the rate $b - \rho$ during a growth phase, which lasts for $\tau^g = \frac{\log \gamma}{b-\rho}$ units of time. It is followed by a build-up phase, lasting $\tau^b = \frac{\log a}{b-\rho}$ units of time and during which consumption remains constant. The total length of a cycle is*

$$\tau^* = \frac{\log a + \log \gamma}{b - \rho}$$

The labor income share declines from $\frac{a-1}{a-1/\gamma}$ to $\frac{1}{\gamma}\frac{a-1}{a-1/\gamma}$ in the growth phase, and increases back to $\frac{a-1}{a-1/\gamma}$ in the following build-up phase.

Proof: see Appendix.

Consider a growth phase when capital j and $j+1$ are both used in production. Consumption grows at the rate $b-\rho$ through continual reallocation of labor from capital j to capital $j+1$. At the end of the growth phase, k_{j+1} absorbs all the labor force; the price of k_{j+2} is a times that of k_{j+1} following the zero profit condition for the innovation sector. This price is, however, larger than the value of k_{j+2} being used in producing the consumption good. It is therefore not profitable to immediately introduce k_{j+2} at the end of the growth phase. Instead, firms keep accumulating capital $j+1$[23], and this decreases its price. This build-up phase ends when the (implicit) price of k_{j+2} equals its value in production. Then capital $j+2$ is introduced into the consumption sector, and a new growth phase begins.

In the growth phase, both k_j and k_{j+1} are used in producing consumption good. The latter grows over time as labor is shifted from capital j to $j+1$. The labor income share in firms employing k_j and k_{j+1} is,

$$LS_j = \frac{wl_j}{\gamma^j l_j} = \frac{a-1}{a-1/\gamma}$$
$$LS_{j+1} = \frac{wl_{j+1}}{\gamma^{j+1} l_{j+1}} = \frac{1}{\gamma}\frac{a-1}{a-1/\gamma}$$

As $\gamma > 1$, the labor share in firms using the more advanced technology $j+1$ is smaller than that in firms using capital vintage j. Thus, the reallocation process decreases the labor share (and increases the capital share). The labor income share decreases from $\frac{a-1}{a-1/\gamma}$ at the beginning of a growth phase to $\frac{1}{\gamma}\frac{a-1}{a-1/\gamma}$ at the end.

In the build-up phase the rental rate and capital income share decrease; the wage and labor income share increases, for a period of time equal to τ^b, from $\frac{1}{\gamma}\frac{a-1}{a-1/\gamma}$ to $\frac{a-1}{a-1/\gamma}$. At the end of each build-up phase the factor income shares return to exactly the same level at the beginning of each growth phase. It should be noted that the factor shares we focused upon are only for the consumption sector. We do not explicitly account for the other two sectors because their capital income shares are, trivially, 100%. In the appendix, we compute the factor income shares for the whole economy and confirm that the cyclical pattern

[23]This may be a good moment to discuss the two payoff-equivalent paths mentioned earlier, when illustrating Figure 3.1. We can alternatively assume that firms innovate immediately and create a small amount of capital $j+2$ at the end of the growth phase. Because of the price argument just given, it will still not be profitable to use capital $j+2$ to produce the consumption good. What would be profit maximizing in these circumstances is to self-accumulate this new capital until its price makes it profitable using it in the consumption sector. One can think of this as a small innovative start-up that accumulates its new productive capacity and finances its temporary losses by borrowing against the promise of future revenues. Actually, this argument implies that innovation can take place at any point in time during the build-up phase, the only difference being the "size" of the start-up firms and the length of time during which they self-accumulate productive capacity before using it for production of the consumption good. All such "different" paths are payoff-equivalent in the sense that the time at which vintage $j+2$ capital starts to be used in the consumption sector is the same and the same productive capacity is used. Consumption paths, and utilities, are therefore identical.

holds true for the whole economy.

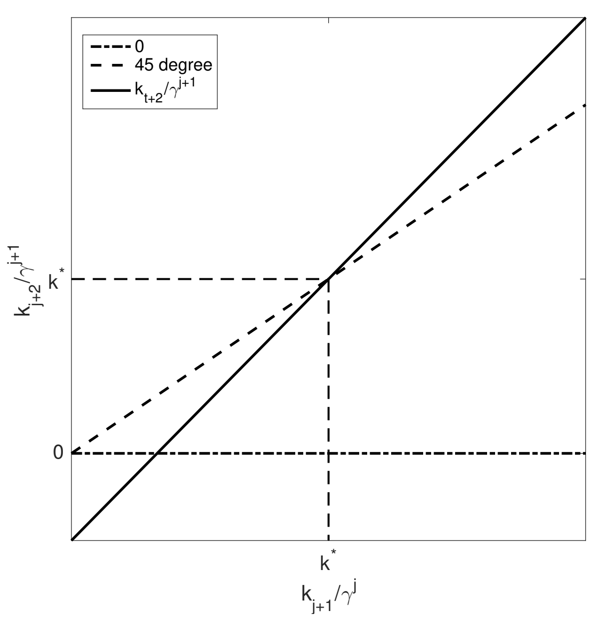
Levels of capital and the initial phase We have shown that our model economy eventually settles into a recurring growth-cycle but, in doing so, we have abstracted from its initial conditions, which we consider next.

Denote $j = 0$ the least advanced capital vintage, and τ_j the time when capital of vintage j is first employed in producing consumption goods. Without loss of generality, start with a growth phase when k_j and k_{j+1} are simultaneously used. Given an initial value of k_{j+1} at $t = \tau_{j+1}$, and the law of motion for capital stock in the following growth and build-up phases, we can calculate k_{j+2} at $t = \tau_{j+2}$, i.e. the beginning of the next growth phase. It is established that $k_{j+2}(\tau_{j+2})$ and $k_{j+1}(\tau_{j+1})$ satisfy the following relation,[24]

$$\frac{k_{j+2}(\tau_{j+2})}{\gamma^{j+1}} = (a\gamma)^{\frac{\rho}{b-\rho}} \frac{k_{j+1}(\tau_{j+1})}{\gamma^j} - x,$$

where $x \equiv a^{\frac{\rho}{b-\rho}}(\gamma^{\frac{\rho}{b-\rho}} - 1)\frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)} > 0$. Figure 3.5 illustrates the normalized capital stock $\frac{k_{j+2}(\tau_{j+2})}{\gamma^{j+1}}$ as a function of $\frac{k_{j+1}(\tau_{j+1})}{\gamma^j}$. As $(a\gamma)^{\frac{\rho}{b-\rho}} > 1$, the function is steeper than the 45-*degree* line. There exists a unique steady state value for the normalized capital stock. An initial value of capital $j+1$ below the steady state eventually leads to a negative capital stock; and any initial capital above the steady state level results in an explosion.[25]

Figure 3.5: Steady state of normalized capital



[24] We refer interested readers to the appendix for the details of this calculation.

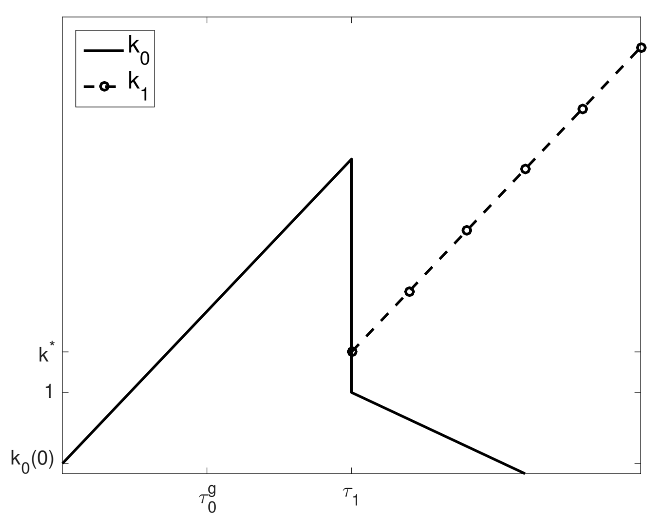
[25] Both violate the Transversality condition, hence are not optimal, hence cannot be equilibria. See Appendix for details.

To investigate the initial capital allocation, we first focus on the case with $0 < k_0(0) < 1$, that is, the initial capital 0 is not large enough to employ all labor at $t = 0$. The first recurring cycle of the economy starts when capital 0 and 1 are simultaneously used in sector 1. $k_1(\tau_1)$ will then equal the steady state value calculated above. During the initial unemployment phase, capital 1 is too expensive to be introduced immediately: there is excess labor, no reason to innovate. The competitive equilibrium allocates $k_0^1(0)$ units of capital 0 in sector 1, and $k_0(0) - k_0^1(0)$ in sector 2. During this initial phase, $k_0^1(t)$ and, consequently, consumption $c(t)$, grow over time until full employment is reached at time $t = \tau_0^g$, when $c(\tau_0^g) = 1$.

The first build-up phase starts at this point, during which capital 0 is accumulated further and consumption remains constant. This build-up phase ends at $t = \tau_1$ when the (implicit) price of capital 1 equals its value in production. At $t = \tau_1$, the economy enters the recurring cycles and behaves as described above.

For the initial cycle, given an initial choice $k_0^1(0), 0 < k_0^1(0) < k_0(1)$, we can calculate the length of this cycle τ_1 and $k_0(t)$ for $0 < t \leq \tau_1$. As shown in Appendix, the equation $k_1(\tau_1) = k^*$ uniquely determines the initial choice, $k_0^1(0)$. Figure 3.6 illustrates the evolution of capital stock during the initial cycle.

Figure 3.6: The initial cycle



Recall that $k_{j+1}(\tau_{j+1})$ and $k_{j+1}(\tau_{j+1} + \tau^g)$ stand for the amount of capital $j+1$ at the beginning and end of the growth phase in which capital j and $j+1$ are simultaneously employed, respectively. The initial capital allocation for $k_0(0) \geq 1$ is determined as: if $k_0(0) \in [1, k_1(\tau_1) * a + 1)$, then 1 unit of capital 0 is used in producing consumption goods and the rest for self accumulation; if $k_0(0) \in [k_{j+1}(\tau_{j+1}) * a^{j+1} + \gamma^j * a^j, k_{j+1}(\tau_{j+1} + \tau^g) * a^{j+1})$ for some j, then $k_0(0)$ is immediately converted into both capital

j and $j+1$; the economy jumps to and starts from the corresponding growth phase;[26] if $k_0(0) \in [k_{j+1}(\tau_{j+1}+\tau^g) * a^{j+1}, k_{j+2}(\tau_{j+2}) * a^{j+2} + \gamma^{j+1} * a^{j+1})$ for some j, then all $k_0(0)$ is converted into capital $j+1$, γ^{j+1} units for producing the consumption good and the rest for self accumulation; the economy starts from the corresponding build-up phase.

Proposition 3 formally summarizes these results.

Proposition 3: *Depending on the quantity of the initial stock of capital, there might be an initial phase when a single vintage of capital is employed and accumulated. After that initial phase, the economy settles into a recurring growth and build-up cycle. The value of capital stock j when it is first introduced at $t = \tau_j$ satisfies $k_j(\tau_j) = \gamma^{j-1}k^*$, where k^* is defined as*

$$k^* = \frac{x}{(a\gamma)^{\frac{\rho}{b-\rho}} - 1},$$

with $x \equiv a^{\frac{\rho}{b-\rho}}(\gamma^{\frac{\rho}{b-\rho}} - 1)\frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}$.

Proof: see Appendix.

3.3 Endogenous labor supply

This subsection relaxes the assumption that labor supply is exogenously fixed at one to study the impact of forces examined in the model on the evolution of employment/working hours. As shown below, incorporating endogenous labor supply does not alter the major qualitative features of the baseline model while generating employment fluctuations as observed in data. In particular, employment decreases during a growth phase when the labor share also decreases, and increases during a build-up phase when the labor share increases too. Labor saving technological progress, therefore, reduces both employment and its share of income. In addition, the model with endogenous labor supply provides new and interesting insights. Among them: (i) During the build-up phase, due to a rising employment, consumption also grows though at a lower rate than during the growth phase. (ii) the lengths of the growth and build-up phase change while the total length of a cycle does not. (iii) As in the baseline model, the factor shares oscillate cyclically, driven by productivity, wages and employment.

We endogenize labor supply in the usual way

$$\int_0^\infty e^{-\rho t}[\log c(t) - \zeta\frac{\eta-1}{\eta}l(t)^{\frac{\eta}{\eta-1}}]dt,$$

[26]Each point of time within a growth phase corresponds to a unique unit of vintage 0 capital. This correspondence is used to determine how much $k_0(0)$ is converted to vintage j capital and how much to vintage $j+1$, as well as the initial allocation across 3 sectors for converted capital of both vintages.

where $\zeta > 0$ and $\eta > 1$. The first order condition w.r.t. working hours $\ell(t)$ is

$$\frac{w(t)}{c(t)} = \zeta * l(t)^{\frac{1}{\eta-1}}$$

The wage is determined by the zero profit conditions in sectors 1 and 2. During the growth phase the wage, in units of current consumption, is constant. The growth rates of consumption and working hours satisfy,

$$-\frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta - 1}\frac{\dot{l}(t)}{l(t)}.$$

As consumption grows at the rate $b - \rho$ during a growth phase, working hours shrink at the rate $(\eta - 1)(b - \rho)$.

In the build-up phase with endogenous labor supply consumption will not remain constant as a rising wage encourages workers to supply more labor, which increases production. Without loss of generality, focus on the build-up phase when only capital $j + 1$ is used in production. Substitute the production relation, $c(t) = \gamma^{j+1} l(t)$, into the first order condition for working hours to obtain

$$w(t) = \zeta \gamma^j l(t)^{\frac{\eta}{\eta-1}}.$$

Therefore, working hours during the build-up phase grow at the rate $(\eta - 1)/\eta$ times the growth rate of wage $w(t)$. As wage grows in the build-up phase, working hours and consumption increase over time. Adding endogenous labor supply also alters the relative length of the growth and build-up phase, while keeping the total length of cycle unchanged. Formally, we have the following proposition[27]

Proposition 4: *The economy with endogenous labor supply settles into a recurring cycle, consisting of a growth phase, when consumption grows at the rate $b - \rho$, and a build-up phase, when consumption grows at the rate $\frac{\frac{\eta-1}{\eta}\log\gamma}{\log a + \frac{\eta-1}{\eta}\log\gamma}(b - \rho)$. A growth phase lasts for $\tilde{\tau}^g = \frac{\log\gamma}{\eta(b-\rho)}$, which is followed by a build-up phase lasting $\tilde{\tau}^g = \frac{\log a + \frac{\eta-1}{\eta}\log\gamma}{b-\rho}$. The total length of a cycle is*

$$\tilde{\tau}^* = \frac{\log a + \log\gamma}{b - \rho}.$$

Further, both the labor income share and the level of employment decline in the growth phase while they increase during the build-up phase.

Proof: see Appendix.

[27] Again, we refer interested readers to Appendix for the details of the proof.

4 Discussion

In this section, we discuss two related issues: irregular cycles in factor shares, and output and population growth.

Irregular cycles In the model, the factor share cycle is always regular in the sense that the length of all growth phases is a constant, and all build-up phases last the same period of time as well. Cycles in the data are of varied lengths. Our model can be extended in a straightforward manner to allow for this variation. For example, if the magnitude of technical progress, captured by the value of γ_j, changes with j, then the length of a build-cycle also varies over time. Our theoretical model can also generate a possible scenario: the labor share fluctuates at a high level for a sustained period of time, then declines, and settles into cycles at a low level.[28]

Output and population growth Throughout the paper, we focus on the consumption sector, which is justified as production in the capital deepening and widening sectors requires one single input by design and the labor share is 0 in both sectors by definition. The total output in the model economy contains consumption and investment. As we show in appendix, the growth rate of real output varies within a growth phase, and one cannot conclude if the growth rate of real output in the growth phase is larger than that in the build-up phase. This implication is different from that for labor productivity (in the final consumption good producing sector), wage and employment/working hours, for which the behaviors between the two phases are clearly distinguished. In the data, we do not find a significant correlation between growth in value added and labor share, both for 2-4 years moving averages and the HP trend, though growth in quarterly value added and quarterly labor share are significantly negatively correlated.[29] This lack of correlation is therefore not in odds with our theoretical model. It should also be mentioned that if we normalize value added by labor force, a significantly negative correlation between growth in value added and labor share, for both moving averages and the HP trend, is obtained.

In the baseline model with exogenous labor supply, population is normalized as 1. We can add population growth into the model and still maintain tractability. For example, one can allow population to grow at an exogenous rate n, assuming $n < b - \rho$. Under this assumption, consumption still grows at the rate $b - \rho$ in a growth phase; however, a growth phase now must last for longer to realize a growth of consumption that is more than γ times (γ multiplied by population growth rate) from the beginning to the end of the growth phase. In the build-up phase, consumption will then grows at the rate of n. If we allow both population growth and endogenous labor supply, it is possible that

[28]One technical requirement is that the model parameters eventually stabilize, a condition needed to guarantee uniqueness of equilibrium.

[29]The result that growth in value added and labor share are not significantly correlated over the longer run is also documented in Leon-Ledesma and Satchi (2019), which takes such a lack of a significant correlation as evidence that the labor share is constant in the long run.

growth rate of of the final consumption good is the same in the growth and build-up phase. That is, a balanced growth can be achieved at the aggregate, though the economy alternates between phases in which the labor share, labor productivity, wage and employment/working hours progress at varied rates.

5 Conclusion

The factor shares are characterized with recursive medium-run cycles: a phase when the labor share declines is always followed by a labor share increasing phase. In periods of declining labor share, labor productivity increases at a fast rate while wage and employment grow slower. The opposite is observed for labor share rising phases. We have examined in the paper a dynamic general equilibrium model in which interaction of factor prices and labor saving technological progress endogenously generates long run growth and permanent and recurring medium run factor share cycles.

Our model contains a growth phase in which two vintages of capital are employed in production and a build-up phase when there is only one vintage of capital. Of course, at any point of time in the real world, various methods of production, embodied in different vintages of capital goods, coexist. The alternation between a phase of two technologies and a phase of one is a theoretical simplification. Therefore one caveat for interpreting our framework is that the growth and build-up phases capture the fact that there are periods in data which are dominated by technical upgrading from old to new ones, and there are others when capital accumulation is mainly in the form of expansion in existing technologies.

In the model, there is no labor in the capital accumulation or innovation sector. On the one hand, we doubt that the type of labor used in knowledge accumulation or creation is a particularly good substitute for the labor used in producing consumption goods, so we do not view the alternative assumption as especially realistic either. On the other hand, what happens if we require some sort of labor in the capital accumulation or innovation process? The welfare theorems still hold regardless of the details of the production process. The lengths of the growth phase and build-up phase will be altered as the endogenous wage now becomes a factor in determining the cost of innovation. However, this should not qualitatively affect the endogenous fluctuations in factor income shares around recurring growth cycles. During the growth phase factor prices are still determined by zero profit conditions, and the capital income share increases in the reallocation process from lower to advanced vintages of capitals. In the build-up phase, accumulation of capital decreases its price and share in total income, as in our baseline model.

The model developed in the paper focus on trend rather than deviation from trend. Therefore we abstract from frictions that is typically at business cycle frequencies, such as wage stickiness. A combination of the technology structure in our model and business cycles frictions have the potential to generate a movement of factor shares that is more in line with data. A second simplification we made is that we assume one single sector in producing the final consumption good. Technology is expected to progress at varied rates across industries in the real economy. Recent literature documents that there are substantial heterogeneity in labor share trend across industries as well. We leave both business cycle and multi-sector considerations to future research.

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6 Appendix

Proof of proposition 1 We have already calculated wage and interest rates when there are two vintages of capital employed in production. Here we show how to derive the price formula, and prove why there are at most two consecutive qualities of capital employed in producing the consumption good. Recall $q_j(t)$ the price of capital j in units of period-t marginal utility. Note the price of capital j in units of period-t consumption good is $q_j(t)c(t)$, and the relative price of period-s consumption good to period-t consumption good is $e^{-\rho(s-t)}\frac{c(t)}{c(s)}$. The following condition holds,

$$q_j(t)c(t) = r_j(t)\Delta + e^{-\rho\Delta}\frac{c(t)}{c(t+\Delta)} * q_j(t+\Delta)c(t+\Delta), \quad \text{as } \Delta \to 0$$

It follows that

$$\begin{aligned} r_j(t) &= -\frac{1}{\Delta}\left\{e^{-\rho\Delta}[q_j(t) + \dot{q}_j(t)\Delta]c(t) - q_j(t)c(t)\right\} \\ &= -\dot{q}_j(t)c(t) + \rho q_j(t)c(t), \quad \text{as } \Delta \to 0 \\ &= q_j(t)c(t)[-\frac{\dot{q}_j(t)}{q_j(t)} + \rho] \\ &= q_j(t)c(t)b \end{aligned}$$

The last equality follows as we know that the price of capital, in units of period-t marginal utility, decreases at the rate of $b - \rho$. Therefore

$$q_j(t) = \frac{1}{bc(t)}r_j(t) = \frac{1}{bc(t)}\frac{\gamma^{j'-j} - 1}{\gamma^{j-j'} - 1/a^{j-j'}}$$

Alternatively, we can calculate capital price using the Hamiltonian. Recall that the zero profit condition in the consumption sector implies that[30]

$$\begin{aligned} c(t) &= r_j(t)k_j^1(t) + r_{j'}(t)k_{j'}^1(t) + w(t) \\ &= r_j(t)[k_j^1(t) + \frac{1}{a^{j-j'}}k_{j'}^1(t)] + w(t) \end{aligned}$$

The corresponding Hamiltonian is[31]

$$\begin{aligned} \mathcal{H} &= \log c(t) + \lambda_j(t)b[k_j(t) - k_j^1(t)] + \lambda_{j'}(t)b[k_{j'}(t) - k_{j'}^1(t)] \\ &= \log c(t) + \lambda_j(t)b[k_j(t) - k_j^1(t)] + \lambda_j(t)\frac{1}{a^{j-j'}}b[k_{j'}(t) - k_{j'}^1(t)] \\ &= \log c(t) + \lambda_j(t)b[k_j(t) + k_{j'}(t)\frac{1}{a^{j-j'}} - k_j^1(t) - k_{j'}^1(t)\frac{1}{a^{j-j'}}] \\ &= \log c(t) + \lambda_j(t)b[k_j(t) + k_{j'}(t)\frac{1}{a^{j-j'}} - \frac{1}{r_j(t)}c(t) - \frac{1}{r_j(t)}w(t)] \end{aligned}$$

[30]Here again we exploit the equivalence of competitive equilibrium and Pareto efficiency. There is no actual prices in planner's problem. However, a relation between $c(t)$,$k_j^1(t)$ and $k_{j'}^1(t)$ as described by the formula always holds.

[31]Note that any positive amount of capital j in sector 3, $k_{j'}^3$, appears as a negative term in the law of motion for $k_{j'}$ and a positive term in that for k_j. These two terms exactly cancel each other.

The first order condition *w.r.t.* c_t gives

$$\lambda_j(t) = \frac{1}{bc(t)} r_j(t) = \frac{1}{bc(t)} \frac{\gamma^{j-j'} - 1}{\gamma^{j-j'} - 1/a^{j-j'}}.$$

To see why there are at most two vintages of capital simultaneously used in production and they must be of consecutive quality, consider the case where capital j' is used in production. Note the price of capital $j'+1$ is $q_{j'+1}(t) = \frac{\gamma-1}{\gamma-1/a} \frac{1}{bc(t)}$. The zero profit condition of innovation implies that for any capital $j, j > j'+1$, its price equals $a^{j-j'-1} q_{j'+1}(t) = a^{j-j'-1} \frac{\gamma-1}{\gamma-1/a} \frac{1}{bc(t)}$, which is strictly larger than its value in production, $\frac{\gamma^{j-j'}-1}{\gamma^{j-j'}-1/a^{j-j'}} \frac{1}{bc(t)}$. Therefore, any capital of vintage larger than $j'+1$ will not be employed in production.

Proof of proposition 2 Without loss of generality, consider a growth phase when capital j and $j+1$ are both used in producing the consumption good. Efficiency dictates that capital $j+1$ self accumulates over time. From Proposition 1, its price $q_{j+1}(t)$ therefore decreases at the rate of $b-\rho$, which implies that consumption grows at the rate of $b-\rho$. Labor is fully employed in capital j at the beginning of the growth phase, and in capital $j+1$ at the end of it. Output therefore increases from γ^j at the beginning to γ^{j+1} at the end. As the rate of increase in consumption is $b-\rho$, the length of the growth phase is $\frac{\log\gamma}{b-\rho}$.

At the end of the growth phase, capital $j+1$ absorbs all labor force, and the price of capital $j+1$ according to proposition 1 is, $q_{j+1}(t) = \frac{\gamma-1}{\gamma/a} \frac{1}{bc(t)}$. At this point, zero profitability of innovation implies that the price of capital $j+2$ satisfies $q_{j+2}(t) = aq_{j+1}(t) = a\frac{\gamma-1}{\gamma-1/a} \frac{1}{bc(t)}$. However, the value of employing capital $j+2$ in producing the consumption good is $v_{j+2}(t) = \frac{\gamma-1}{\gamma-1/a} \frac{1}{bc(t)}$. As

$$q_{j+2}(t) = a\frac{\gamma-1}{\gamma-1/a} \frac{1}{bc(t)} > v_{j+2}(t) = \frac{\gamma-1}{\gamma-1/a} \frac{1}{bc(t)},$$

it is not profitable to introduce capital $j+2$ into production at the end of the growth phase. Capital $j+1$ will further accumulates, which decreases price of capital (of vintage $j+1$ as well as $j+2$). The left hand side in the above inequality decreases at the rate of $b-\rho$ while its right hand side remains constant. Capital $j+2$ will be introduced into production when the LHS decreases and equals RHS.[32]. The price of capital decreases at the

[32] A different, and more technical, interpretation of this (in-)equality is, the original optimal control problems can be divided into a series of sub-problems, each dealing with the optimization problem for the length of period when two consecutive capital goods are used. Denote $\lambda_j(J)$ and $\lambda_{j+1}(J)$ the co-state variables for the dynamics of capital j and $j+1$, respectively, in the sub-problem J when capital j and $j+1$ are simultaneously used in production. A necessary condition for equivalence of the original problem and the series of sub-problems is that $\lambda_{j+1}^{end}(J) = \lambda_{j+1}^{0}(J+1)$. That is, the multiplier for capital $j+1$ at the end of subproblem J should equal that at the beginning of subproblem $J+1$. This is essentially the price condition here.

rate of $b-\rho$, this build-up phase therefore lasts for $\frac{\log a}{b-\rho}$. Note that consumption remains stagnant in the build-up phase, as all labor is employed in capital $j+1$, and there is no new technology introduced into production.

Levels of capital stock We now investigate the evolution of capital stock. First calculate how much capital is transformed into that of a more advanced vintage at the beginning of a growth phase once the economy enters the recurring cycles. Consider a growth phase when capital j and $j+1$ are simultaneously employed in production. At the beginning of that phase, $a * k_j(t_0)$ units of capital j is converted into $k_{j+1}(t_0)$ units of capital $j+1$. Note that to guarantee the continuity of consumption, the remaining capital j used in producing consumption goods, $k_j^1(t_0) = \gamma^j$, and total consumption is $c(t_0) = \gamma^j$. Without loss of generality, normalize $t_0 = 0$[33].

During the growth phase, denote $\sigma_j(t)$ the fraction of labor employed by capital j,

$$\gamma^j \sigma_j(t) + \gamma^{j+1}(1-\sigma_j(t)) = c(t)$$

It follows that $\sigma_j(t) = \frac{\gamma^{j+1}-c(t)}{\gamma^{j+1}-\gamma^j} = \frac{\gamma^{j+1}-\gamma^j e^{(b-\rho)t}}{\gamma^{j+1}-\gamma^j} = \frac{\gamma-e^{(b-\rho)t}}{\gamma-1}$, where the second equality holds as consumption in the growth phase increases at the rate of $b-\rho$. Note that when $t = \frac{\log \gamma}{b-\rho}$, $\sigma_j(t) = 0$. That is, at the end of growth phase, all labor reallocates from capital j to $j+1$.

Assume that capital j is converted to capital $j+1$ as soon as it is freed from use in producing consumption goods during the growth phase[34]. That is,

$$\begin{aligned} k_j^3(t) &= -dk_j(t) = -\gamma^j * d\sigma_j(t) \\ &= \gamma^j \frac{b-\rho}{\gamma-1} e^{(b-\rho)t} * dt \end{aligned}$$

[33]Without normalization, one can simply add all time variables in this subsection by the initial value, and all results here remain.

[34]Alternatively, we can assume that capital j released from production is first self-accumulated from time t for a period of positive length Δ_t, and converted to capital $j+1$ altogether at Δ_t. These two assumptions are equivalent in the sense that they deliver exactly the same amount of vintage $j+1$ capital goods at time $t+\Delta_t$.

Therefore, during the growth phase, the law of motion for capital $j+1$ is[35]

$$\begin{aligned} dk_{j+1}(t) &= bk^2_{j+1}(t)dt - k^3_{j+1}(t) + \frac{k^3_j(t)}{a} \\ &= b[k_{j+1}(t) - \gamma^{j+1}(1-\sigma_j(t))]dt - 0 + \frac{1}{a} * \gamma^j \frac{b-\rho}{\gamma-1} e^{(b-\rho)t} dt \end{aligned}$$

Equivalently,

$$\begin{aligned} \dot{k}_{j+1}(t) &= b[k_{j+1}(t) - \gamma^{j+1}(1-\sigma_j(t))] + \gamma^j \frac{b-\rho}{a(\gamma-1)} e^{(b-\rho)t} \\ &= b[k_{j+1}(t) + \frac{\gamma^{j+1}}{\gamma-1}] + \gamma^j \frac{b(1-a\gamma)-\rho}{a(\gamma-1)} e^{(b-\rho)t} \end{aligned}$$

until $t = \tau^g \equiv \frac{\log\gamma}{b-\rho}$ when the growth phase ends. The solution to this ordinary differential equation has the following form: $k_{j+1}(t) = \theta_0 + \theta_1 e^{bt} + \theta_2 e^{(b-\rho)t}$. Differentiating both sides w.r.t. time t and matching coefficients in common terms, we have

$$\theta_0 = -\frac{\gamma^{j+1}}{\gamma-1},$$

and

$$\theta_2 = \gamma^j \frac{(a\gamma-1)b+\rho}{\rho a(\gamma-1)}.$$

Substituting these back into the formula for $k_{j+1}(t)$,

$$k_{j+1}(t) = -\frac{\gamma^{j+1}}{\gamma-1} + \theta_1 e^{bt} + \gamma^j \frac{(a\gamma-1)b+\rho}{\rho a(\gamma-1)} e^{(b-\rho)t}.$$

Using the initial condition at time $t=0$,

$$k_{j+1}(0) = -\frac{\gamma^{j+1}}{\gamma-1} + \theta_1 + \gamma^j \frac{(a\gamma-1)b+\rho}{\rho a(\gamma-1)},$$

we have,

$$\theta_1 = k_{j+1}(0) - \gamma^j \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}.$$

[35]Note that here we assume that before capital $j+2$ is used in producing consumption goods, say at $t=\bar{t}$, capital $j+1$ will only be used in (producing consumption goods and) replicating itself, and not be used in creating capital $j+2$. Alternatively, we can assume that any capital $j+1$ beyond the necessary amount in producing consumption goods is converted immediately to capital $j+2$. The amount of capital $j+2$ obtained at $t=\bar{t}$ under two assumptions would be the same. In addition, as capital $j+2$ will not be used in producing consumption goods before $t=\bar{t}$, the price of capital $j+1$ is determined as before, and the (implied) price of capital $j+2$ is also not altered.

At time $t = \tau^g \equiv \frac{\log \gamma}{b-\rho}$,

$$\begin{aligned} k_{j+1}(\tau^g) &= -\frac{\gamma^{j+1}}{\gamma-1} + \theta_1 \gamma^{\frac{b}{b-\rho}} + \gamma^j \frac{(a\gamma-1)b+\rho}{\rho a(\gamma-1)}\gamma \\ &= \gamma^{j+1} + (k_{j+1}(0) - \gamma^j \tilde{x})\gamma^{\frac{b}{b-\rho}} + \gamma^{j+1}\tilde{x} \end{aligned}$$

where $\tilde{x} \equiv \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}$.

The build-up phase comes next and lasts until $t = \frac{\log\gamma + \log a}{b-\rho}$. During the build-up phase, $k_j^3(t) = 0$ as capital j has been used up; and $k_{j+1}^1 = \gamma^{j+1}$ as labor is all and only employed by capital $j+1$. Assume $k_{j+1}^3(t) = 0$.[36] The dynamics for $k_{j+1}(t)$ is

$$dk_{j+1}(t) = b[k_{j+1}(t) - \gamma^{j+1}]dt.$$

Solve this differential equation, and capital $j+1$ satisfies

$$k_{j+1}(t) = \gamma^{j+1} + e^{b(t-\tau^g)}[k_{j+1}(\tau^g) - \gamma^{j+1}], \quad \text{for } \tau^g \le t \le \tau^g + \tau^b,$$

where $k_{j+1}(\tau^g)$ is the amount of capital $j+1$ at $t = \tau^g$. When $t = \tau^g + \tau^b = \frac{\log\gamma+\log a}{b-\rho}$, capital $j+1$ is

$$k_{j+1}(\tau^g + \tau^b) = \gamma^{j+1} + a^{\frac{b}{b-\rho}}[k_{j+1}(\tau^g) - \gamma^{j+1}].$$

At time $t = \tau^g + \tau^b$, γ^{j+1} units of capital are employed in producing consumption goods, and the remaining capital of vintage $j+1$, $k_{j+1}(\tau^g + \tau^b) - \gamma^{j+1}$, is converted to capital of vintage $j+2$. It follows that,

$$\begin{aligned} k_{j+2}(\tau^g + \tau^b) &= \frac{1}{a}[k_{j+1}(\tau^g + \tau^b) - \gamma^{j+1}] \\ &= \frac{a^{\frac{b}{b-\rho}}}{a}[k_{j+1}(\tau^g) - \gamma^{j+1}] \\ &= a^{\frac{\rho}{b-\rho}}\{[k_{j+1}(0) - \gamma^j\tilde{x}]\gamma^{\frac{b}{b-\rho}} + \gamma^{j+1}\tilde{x}\} \end{aligned}$$

again $\tilde{x} \equiv \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}$. Equivalently,

$$\begin{aligned} \frac{k_{j+2}(\tau^g+\tau^b)}{\gamma^{j+1}} &= a^{\frac{\rho}{b-\rho}}\left\{[\frac{k_{j+1}(0)}{\gamma^j} - \tilde{x}]\gamma^{\frac{\rho}{b-\rho}} + \tilde{x}\right\}, \\ &= (a\gamma)^{\frac{\rho}{b-\rho}}\frac{k_{j+1}(0)}{\gamma^j} - a^{\frac{\rho}{b-\rho}}(\gamma^{\frac{\rho}{b-\rho}} - 1)\tilde{x}, \\ &\equiv (a\gamma)^{\frac{\rho}{b-\rho}}\frac{k_{j+1}(0)}{\gamma^j} - x. \end{aligned}$$

[36] Note that here we made the assumption that, before $t = \frac{\log\gamma+\log a}{b-\rho}$, capital $j+1$ is only used in replicating itself and not used in creating $j+2$. Both activities satisfy zero profit conditions. In essence, between $t = \frac{\log\gamma}{b-\rho}$ and $t = \frac{\log\gamma+\log a}{b-\rho}$, various arrangements regarding what percentage of and when non-production capital $j+1$ is converted into capital $j+2$ are equivalent as they generate the same amount of capital $j+2$ at $t = \frac{\log\gamma+\log a}{b-\rho}$.

with $x \equiv a^{\frac{\rho}{b-\rho}}(\gamma^{\frac{\rho}{b-\rho}} - 1)\tilde{x} = a^{\frac{\rho}{b-\rho}}(\gamma^{\frac{\rho}{b-\rho}} - 1)\frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}$. This is the formula we showed in text. From this equation, there is a unique steady state value of the normalized capital stock, $k^* \equiv \frac{k_{j+1}(\tau_{j+1})}{\gamma^j}$, with τ_{j+1} the first time capital $j+1$ used in production, which satisfies,[37]

$$k^* = \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)} \frac{(a\gamma)^{\frac{\rho}{b-\rho}} - a^{\frac{\rho}{b-\rho}}}{(a\gamma)^{\frac{\rho}{b-\rho}} - 1}.$$

Denote $j = 0$ the least advanced capital. The economy enters a recurring cycle when capital of vintage 1 is created and employed in production. Denote τ_1 the first time capital 1 is used in production. The fact that $k_1(\tau_1) = k^*$ is established from the Transversality condition. Note that each vintage of capital has a finite 'life cycle' in our model, it is therefore sufficient to show that the Transversality condition hods at a certain point in the 'life cycle' for each vintage of capital. Without loss of generality, we choose the time when a vintage of capital is firstly introduced, i.e. τ_{j+1} for capital $j+1$. The Transversality condition reads,

$$0 = \lim_{j\to\infty} e^{-\rho\tau_{j+1}} \frac{k_{j+1}(\tau_{j+1})}{c(\tau_{j+1})} = \lim_{j\to\infty} e^{-\rho\tau_1}(a\gamma)^{\frac{-\rho}{b-\rho}j} * \frac{k_{j+1}(\tau_{j+1})}{\gamma^j}$$

where the last equality follows from the fact $\tau_{j+1} = j * \frac{\log a + \log\gamma}{b-\rho} + \tau_1$ and $c(\tau_{j+1}) = \gamma^j$. From the law of motion for $\frac{k_{j+1}(\tau_{j+1})}{\gamma^j}$ above, we have that $\frac{k_{j+1}(\tau_{j+1})}{\gamma^j} - k^* = (a\gamma)^{\frac{\rho}{b-\rho}}(\frac{k_j(\tau_j)}{\gamma^{j-1}} - k^*) = (a\gamma)^{\frac{\rho}{b-\rho}j}(k_1(\tau_1) - k^*)$. That is, $\frac{k_{j+1}(\tau_{j+1})}{\gamma^j} = k^* + (a\gamma)^{\frac{\rho}{b-\rho}j}(k_1(\tau_1) - k^*)$. Substitute this formula into the Transversality condition,

$$\begin{aligned} 0 &= \lim_{j\to\infty} e^{-\rho\tau_1}(a\gamma)^{\frac{-\rho}{b-\rho}j} * [k^* + (a\gamma)^{\frac{\rho}{b-\rho}j}(k_1(\tau_1) - k^*)], \\ &= \lim_{j\to\infty} e^{-\rho\tau_1}[k_1(\tau_1) - k^*]. \end{aligned}$$

It follows that $k_1(\tau_1) = k^*$. The value of $k_1(\tau_1)$ is endogenously determined in the initial growth and build-up cycle, which we turn to next.

The initial cycle Denote $k_0(0)$ the initial value of capital of vintage 0. Start with the case $0 < k_0(0) < 1$. That is, there is not enough initial capital to employ all labor force. We need to determine how to allocate initial capital between producing consumption goods and self-accumulation at $t = 0$. Denote $k_0^1(0)$ units of capital allocated in producing consumption goods.

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[37]Substitute k^* into the formula for $k_{j+1}(\tau^g)$, and we have $k_{j+1}(\tau^g) = \gamma^{j+1}\left[1 + \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)} \frac{(\gamma^{\frac{\rho}{b-\rho}} - 1)}{(a\gamma)^{\frac{\rho}{b-\rho}} - 1}\right]$. $k_{j+1}(\tau^g) > k_{j+1}(0)$ requires $\gamma > \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)} \frac{\gamma^{\frac{\rho}{b-\rho}} - 1}{(a\gamma)^{\frac{\rho}{b-\rho}} - 1}(a^{\frac{\rho}{b-\rho}} - \gamma)$.

Note that during this phase, the price of capital in terms of current marginal consumption is $q_0(t) = \frac{1}{bc(t)}$[38]. The rental price of capital is 1, and wage is 0, both in units of current consumption goods. The production function is $c(t) = \min\{k_0^1(t), l_0(t)\}$. During this stage, consumption grows at the rate of $b - \rho$. The dynamics of $k_0(t)$ is

$$\begin{aligned}\dot{k}_0(t) &= b[k_0(t) - k_0^1(t)] \\ &= b[k_0(t) - k_0^1(0)e^{(b-\rho)t}]\end{aligned}$$

The solution to this ODE is of the form: $k_0(t) = \phi_0 + \phi_1 e^{bt} + \phi_2 e^{(b-\rho)t}$. Differentiating this ODE and matching coefficients with the formula above gives

$$\phi_0 = 0, \quad \phi_2 = k_0^1(0)\frac{\rho}{b}$$

Further use the initial condition to obtain $\phi_1 = k_0(0) - \frac{b}{\rho}k_0^1(0)$. This initial growth phase stops at $c(\tau_0^g) = k_0^1(0)e^{(b-\rho)\tau_0^g} = 1$, that is, at $\tau_0^g = \frac{1}{b-\rho}\log\frac{1}{k_0^1(0)}$. The capital stock at $t = \tau_0^g$ is

$$k_0(\tau_0^g) = k_0(0) * k_0^1(0)^{\frac{-b}{b-\rho}} - \frac{b}{\rho}k_0^1(0)^{\frac{-\rho}{b-\rho}} + \frac{b}{\rho}.$$

The economy then enters the initial build-up phase where the dynamics of capital is given by

$$\dot{k}(t) = b[k(t) - 1].$$

The solution to this ODE is

$$k_0(t) = 1 + e^{b(t-\tau_0^g)}[k_0(\tau_0^g) - 1]$$

To determine the length of the initial build-up phase, note that the price of capital 0 at $t = \tau_0^g$ is $q_0(\tau_0^g) = \frac{1}{bc(\tau_0^g)} = \frac{1}{b}$. From the zero profit condition of innovation, the (implicit) price of capital 1 is $q_1^*(\tau_0^g) = aq_0(\tau_0^g) = \frac{a}{b}$. Denote τ_1 the first time when capital 1 is created and employed in production. The length of this build-up phase is therefore $\tau_1 - \tau_0^g$. The price of capital 1 at $t = \tau_1$ is $q_1(\tau_1) = \frac{\gamma-1}{\gamma-1/a}\frac{1}{b}$. As the capital price decreases at the rate of $b - \rho$ during the build-up phase, we have

$$q_1^*(\tau_0^g)e^{-(b-\rho)(\tau_1-\tau_0^g)} = q_1(\tau_1)$$

The length of this build-up phase is

$$\tau_1 - \tau_0^g = \frac{1}{b-\rho}\log(\frac{a\gamma-1}{\gamma-1})$$

Note that this is different from the length of a build-up phase after the economy enters the recurring cycles. At $t = \tau_1$, the value of capital 0 is

$$k_0(\tau_1) = 1 + (\frac{a\gamma-1}{\gamma-1})^{\frac{b}{b-\rho}}[k_0(\tau_0^g) - 1],$$

[38]This is obtained from the Euler equation.

Of which 1 unit is used in producing consumption goods, and the remaining $k_0(t_1)-1$ units converted to capital of vintage 1. Therefore the amount of vintage-1 capital at $t=\tau_1$ is

$$\begin{aligned}
k_1(\tau_1) &= \frac{1}{a}(\frac{a\gamma-1}{\gamma-1})^{\frac{b}{b-\rho}}[k_0(\tau_0^g)-1] \\
&= \frac{1}{a}(\frac{a\gamma-1}{\gamma-1})^{\frac{b}{b-\rho}}[k_0(0)*k_0^1(0)^{\frac{-b}{b-\rho}}-\frac{b}{\rho}k_0^1(0)^{\frac{-\rho}{b-\rho}}+\frac{b}{\rho}-1] \\
&= \frac{1}{a}(\frac{a\gamma-1}{\gamma-1})^{\frac{b}{b-\rho}}\left\{k_0(0)^{\frac{-\rho}{b-\rho}}\chi^{\frac{-b}{b-\rho}}[1-\frac{b}{\rho}\chi]+\frac{b}{\rho}-1\right\}
\end{aligned}$$

where $\chi \equiv \frac{k_0^1(0)}{k_0(0)}$ is the fraction of initial capital that is used in producing consumption goods. The steady state condition we derived before requires $k_1(\tau_1)=k^*$. Note that $k_1(\tau_1)$ is a strictly decreasing function of χ. As $\chi\to 0$, $k_1(t_1)\to\infty$. On the other hand, as $\chi\to 1$, $k_1(t_1)\to\frac{1}{a}(\frac{a\gamma-1}{\gamma-1})^{\frac{b}{b-\rho}}(\frac{b}{\rho}-1)(1-k_0(0)^{\frac{-\rho}{b-\rho}})<0$. The monotonicity of $k_1(\tau_1)$ guarantees existence and uniqueness of a $k_0^1(0)$ that satisfies the steady state condition.

Factor shares in the whole economy Note that so far we focus on factor income shares in the consumption sector, instead of the whole economy. This is justified by the fact that the investment sector has a zero labor income share, or equivalently 100% capital income share. Adjusting the factor income share accordingly change its *levels*, but does not affect *trend*. To see this point, consider a growth phase where capital j and $j+1$ are simultaneously used in production. Denote $p^i(t)$ the relative price of investment goods (i.e. capital of vintage $j+1$) to consumption good. Recall the price of capital good $j+1$ in units of current marginal utility is $q_{j+1}(t)$, it follows that $p^i(t)=q_{j+1}(t)c(t)$. Denote τ_{j+1} the first time capital $j+1$ is created and employed in production. At $t=\tau_{j+1}$, the gross labor income share[39] in the whole economy is

$$\begin{aligned}
\tilde{LS}(\tau_{j+1}) &= \frac{w(\tau_{j+1})}{c(\tau_{j+1})+p^i(\tau_{j+1})\dot{k}_{j+1}(\tau_{j+1})}, \\
&= \frac{w(\tau_{j+1})/c(\tau_{j+1})}{1+\frac{\gamma-1}{\gamma-1/a}\frac{\dot{k}_{j+1}(\tau_{j+1})}{bc(\tau_{j+1})}}.
\end{aligned}$$

where $\frac{w(\tau_{j+1})}{c(\tau_{j+1})}$ is the labor share in the consumption goods producing sector. As $\frac{\dot{k}_{j+1}(\tau_{j+1})}{c(\tau_{j+1})}$ is a constant for all j in the normalized steady state, the aggregate labor share in the whole economy declines relative to the labor share in the consumption sector at $t=\tau_{j+1}$, and the decline is independent of capital vintages.

[39]The same property holds for net labor share. In the model, the net value added is obtained by subtracting the value of depreciated capital in the innovation sector from gross value added. It is straightforward to show that the net labor income share displays the same cycles as the gross labor share. Actually the net and gross labor share equals each other during a build-up phase.

For $t > \tau_{j+1}$,

$$\begin{aligned}
\tilde{LS}(t) &= \frac{w(t)}{c(t) + p^i(t)\dot{k}_{j+1}(t)} \\
&= \frac{w(t)/c(t)}{1 + q_{j+1}(t)\dot{k}_{j+1}(t)} \\
&= \frac{w(t)/c(t)}{1 + \frac{\gamma-1}{\gamma-1/a}\frac{\gamma^j}{bc(t)}[b(\frac{k_{j+1}(t_{j+1})}{\gamma^j} - \bar{x})e^{b(t-\tau_{j+1})} + (b-\rho)\bar{x}e^{(b-\rho)(t-\tau_{j+1})}]}
\end{aligned}$$

where $\bar{x} \equiv \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}$ is a constant. Note $c(t) = \gamma^j e^{(b-\rho)(t-\tau_{j+1})}$. $\frac{k_{j+1}(\tau_{j+1})}{\gamma^j}$ is a constant in steady state. Therefore the denominator is a function of $t - \tau_{j+1}$, and independent of j itself. We have already shown that the numerator, $w(t)/c(t)$, which is the labor share in the consumption production sector, does not depend on j. Therefore, the aggregate labor share is also independent of capital vintages.

During the build-up phase, the price of vintage $j+1$ capital/investment goods, in terms of consumption good, decreases at the rate of $b - \rho$. Consumption remains stagnant at γ^{j+1}. The aggregate labor share is

$$\begin{aligned}
\tilde{LS}(t) &= \frac{w(t)}{c(t) + p^i(t)\dot{k}_{j+1}(t)} \\
&= \frac{w(t)/c(t)}{1 + q_{j+1}(t)\dot{k}_{j+1}(t)} \\
&= \frac{w(t)/c(t)}{1 + \frac{\gamma-1}{\gamma-1/a}\frac{\gamma^j}{bc(t)}e^{-(b-\rho)(t-\tau^g_{j+1})}[be^{b(t-\tau^g_{j+1})}(\frac{k_{j+1}(\tau^g_{j+1})}{\gamma^j} - \gamma)]} \\
&= \frac{w(t)/c(t)}{1 + \frac{\gamma-1}{\gamma-1/a}\frac{\gamma^{j+1}}{bc(t)}[be^{\rho(t-t^g_{j+1})}(\frac{k_{j+1}(\tau^g_{j+1})}{\gamma^j} - \gamma)]}
\end{aligned}$$

where τ^g_{j+1} denotes the ending (beginning) time of the growth (build-up) phase. Same as in the growth stage, both numerator and denominator are independent of capital vintages, so do the aggregate labor share.

It is straightforward to see that the aggregate labor share still declines in the growth phase: total output grows at an even faster rate than consumption during the growth phase. During a build-up phase, wage grows at the rate of $b - \rho$ and wage at the end of the phase if γ times higher than that at the beginning. The nominal value of investment (in units of current consumption good) grows at the rate ρ, and consumption remains constant. To retain the results that labor income share rises during the build-up phase, we need that total output grow less than γ times during the phase, which is equivalent to

$$\frac{1}{\gamma - 1/a}[\frac{k_{j+1}(\tau^g_{j+1})}{\gamma^j} - \gamma](\gamma^{\frac{\rho}{b-\rho}} - \gamma) < 1.$$

A sufficient condition is that $\rho < b - \rho$.

Real outpur growth rate Total output in the model economy contains consumption and investment. Again denote $p^i(t)$ the relative price of capital/investment goods to the consumption good. In a growth phase, $p^i(t)$ is a constant at $\frac{\gamma-1}{\gamma-1/a}\frac{1}{b}$. During a build-up phase, this relative price declines at the rate $b-\rho$. The growth rate of real output $(c(t)+p^i(t)i(t))$ is

$$\frac{c(t)}{c(t)+p^i(t)i(t)}\frac{\dot{c}(t)}{c(t)} + \frac{p^i(t)i(t)}{c(t)+p^i(t)i(t)}\frac{\dot{i}(t)}{i(t)}$$

Without loss of generality, focus again on the cycle which contains the growth phase using capital j and $j+1$ and the following build-up phase, and normalize the beginning of the cycle as $t=0$. During the growth phase,

$$i(t) = \dot{k}(t) = b[k_{j+1}(t) + \frac{\gamma^{j+1}}{\gamma-1}] - \underbrace{\gamma^j \frac{b(a\gamma-1)+\rho}{a(\gamma-1)} e^{(b-\rho)t}}_{\iota(t)}$$

It follows that during the growth phase

$$\frac{\dot{i}(t)}{i(t)} = b - \frac{\iota(t)}{\dot{k}(t)},$$

i.e. $i(t)$ grows at a rate smaller than b. We have already known that consumption during the growth phase grows at the rate of $b-\rho$. The growth rate of GDP is weighted average of those for consumption and investment. As investment grows at a varying rate, so does the real output in the growth phase.

During the following build-up phase, the growth rate of consumption is zero. Real investment satisfies

$$i(t) = \dot{k}_{j+1}(t) = b[k_{j+1}(t) - \gamma^{j+1}]$$

Therefore real investment grows at the rate b. Further, during the build-up phase, $p^i(t)$ declines at the rate of $b-\rho$, so it matters which period are used as the benchmark in calculating growth rate of real GDP during the build-up phase. At time t in the phase, growth rate of real GDP is higher if τ^g is benchmark than if relative prices at time t is used.

The general message is that, different from labor productivity (in the consumption sector), wage rate, and employment/working hours for which our model has a clear prediction, there is no such clean distinction in the growth rate of real output between the growth and build-up phases.

Endogenous labor supply Consider a growth phase in the recurring cycles when capital j and $j+1$ are simultaneously employed in production. For simplicity of notation, we normalize the beginning of that phase, as $t = 0$. To determine the length of the growth phase, denote $\sigma_j(t)$ the fraction of labor employed by capital j, total production is

$$\gamma^j l(t)\sigma_j(t) + \gamma^{j+1} l(t)(1 - \sigma_j(t)) = c(t)$$

It follows that $\sigma_j(t) = \frac{\gamma^{j+1} - c(t)/l(t)}{\gamma^{j+1} - \gamma^j} = \frac{\gamma^{j+1} - \frac{c(0)}{l(0)} e^{\eta(b-\rho)t}}{\gamma^{j+1} - \gamma^j} = \frac{\gamma - e^{\eta(b-\rho)t}}{\gamma - 1}$, where the second equality holds as consumption in the growth phase increases at the rate of $b - \rho$, and hours decrease at the rate of $(\eta - 1)(b - \rho)$. Note that when $t = \frac{\log \gamma}{\eta(b-\rho)}$, $\sigma_j(t) = 0$. That is, with endogenous labor supply, the length of the growth phase shrinks from $\frac{\log \gamma}{b-\rho}$ to $\frac{\log \gamma}{\eta(b-\rho)}$.

At $t = 0$, the following three conditions hold

$$c(0) = \gamma^j l(0); \quad \frac{w(0)}{c(0)} = \zeta l(0)^{\frac{1}{\eta-1}}; \quad w(0) = \gamma^j \frac{a-1}{a - 1/\gamma}.$$

Therefore, we have[40]

$$l(0) = [\frac{a-1}{a-1/\gamma}\frac{1}{\zeta}]^{\frac{\eta-1}{\eta}}, \qquad c(0) = \gamma^j l(0).$$

At $t = 0$, $a * k_j(0)$ units of capital j is converted into $k_{j+1}(0)$ units of capital $j + 1$. The remaining capital j that is used in producing consumption goods, $k_j^1(t_0) = \gamma^j l(0)$, and total consumption is $c(0) = \gamma^j l(0)$.

At $t = \tau^g$, the labor supply, $l(\tau^g)$, is

$$l(\tau^g) = l(0) * e^{-(\eta-1)(b-\rho)\frac{\log\gamma}{\eta(b-\rho)}} = l(0)\gamma^{-\frac{\eta-1}{\eta}}.$$

The price of capital $j + 1$ in units of current marginal utility, $q_{j+1}(\tau_g)$, is[41],

$$q_{j+1}(\tau^g) = \frac{\gamma - 1}{\gamma - 1/a}\frac{1}{bc(\tau^g)},$$

Then comes the build-up phase, during which $q_{j+1}(t)$ still declines at the rate of $b - \rho$. However, with endogenous labor supply, $c(t)$ now increases over time in the build-up phase. The build-up phase ends at $t = \tau^g + \tau^b$ when the capital price satisfies

$$q_{j+1}(\tau^g + \tau^b) = \frac{1}{a}\frac{\gamma - 1}{\gamma - 1/a}\frac{1}{bc(\tau^g + \tau^b)}.$$

The length of the build-up phase should therefore satisfy

$$e^{(b-\rho)\tau^b} = a\frac{c(\tau^g + \tau^b)}{c(\tau^g)} = a\frac{l(\tau^g + \tau^b)}{l(\tau^g)}.$$

[40] assuming that $\frac{a-1}{a-1/\gamma}\frac{1}{\zeta} < 1$.

[41] The (implied) price of capital $j + 2$, $q_{j+2}(\tau^g)$, is, $q_{j+2}(\tau^g) = a * q_{j+1}(\tau^g) = a\frac{\gamma-1}{\gamma-1/a}\frac{1}{bc(\tau^g)}$.

According to results established in the baseline model, the wage rate, in units of the current consumption good, grows from $w(\tau^g) = \gamma^j \frac{a-1}{a-1/\gamma}$ at the beginning of a build-up phase, to $w(\tau^g + \tau^b) = \gamma^{j+1} \frac{a-1}{a-1/\gamma}$ at the end of the phase. On the other hand, during the build-up phase, a combination of the production function and the first order condition w.r.t. labor supply implies that $w(t) = \zeta \gamma^j l(t)^{\frac{\eta}{\eta-1}}$. It then follows

$$\frac{l(\tau^b + \tau^b)}{l(\tau^g)} = \gamma^{\frac{\eta-1}{\eta}}.$$

Notice the recurring nature of the problem, as in $l(\tau^g + \tau^b) = l(0)$. That is, labor supply is the same at the beginning of different growth phases. For later reference, denote $l^H \equiv l(0)$ and $l^L \equiv l(\tau^g)$. The length of the build-up phase then is

$$\tau^b = \frac{\log a + \frac{\eta-1}{\eta} \log \gamma}{b - \rho},$$

which is longer than the case with exogenous labor supply. Recall that $\tau^g = \frac{\log \gamma}{\eta(b-\rho)}$, which is shorter with endogenous labor supply. The length of a whole cycle remains unchanged,

$$\tau^g + \tau^b = \frac{\log \gamma}{\eta(b-\rho)} + \frac{\log a + \frac{\eta-1}{\eta} \log \gamma}{b - \rho} = \frac{\log a + \log \gamma}{b - \rho},$$

The growth rate of consumption, as well as working hours, in the build-up phase satisfies

$$g^b = \frac{\frac{\eta-1}{\eta} \log \gamma}{\log a + \frac{\eta-1}{\eta} \log \gamma}(b - \rho).$$

This growth rate is smaller than the growth rate of consumption in the growth phase, which is $g^g = b - \rho$. On the other hand, the labor income share behaves the same as in the exogenous labor supply case. That is, it decreases in the growth phase, and increases in the build-up phase.

Next determine the capital stock. Consider again the growth phase when capital j and $j+1$ are simultaneously employed in production, and normalize the beginning time of the phase as $t(0) = 0$ for notational convenience. Assume that, during the growth phase, capital j is converted to capital $j+1$ as soon as it is freed from use in producing consumption goods. Assume that, during the growth phase, capital j is converted to capital $j+1$ as soon as it is freed from use in producing the consumption good.[42] That is,

$$\begin{aligned} k_j^3(t) &= -dk_j(t) = -\gamma^j * d[l(t)\sigma_j(t)] \\ &= \gamma^j l(0) \frac{b-\rho}{\gamma-1} [\gamma(\eta-1)e^{-(\eta-1)(b-\rho)t} + e^{(b-\rho)t}] * dt \end{aligned}$$

[42] Alternatively, we can assume that capital j released from production is first self-accumulated from time t for a period of positive length Δ_t, and converted to capital $j+1$ altogether at Δ_t. These two assumptions are equivalent in the sense that they deliver exactly the same amount of vintage $j+1$ capital goods at time $t + \Delta_t$.

where the second equation follows from the dynamics for $l(t)$ and $\sigma_j(t)$ during the growth phase calculated in proof of Proposition 2.

The law of motion for capital $j+1$ in the growth phase is

$$\begin{aligned} dk_{j+1}(t) =& bk^2_{j+1}(t)dt - k^3_{j+1}(t) + \frac{k^3_j(t)}{a} \\ =& b[k_{j+1}(t) - \gamma^{j+1}(1-\sigma_j(t))]dt - 0 \\ &+ \frac{1}{a} * \gamma^j l^H \frac{b-\rho}{\gamma-1}[\gamma(\eta-1)e^{-(\eta-1)(b-\rho)t} + e^{(b-\rho)t}] * dt \end{aligned}$$

Equivalently,

$$\begin{aligned} \dot{k}_{j+1}(t) &= b[k_{j+1}(t) - \gamma^{j+1}(1-\sigma_j(t))] + \gamma^j l(0)\frac{b-\rho}{a(\gamma-1)}[\gamma(\eta-1)e^{-(\eta-1)(b-\rho)t} + e^{(b-\rho)t}] \\ &= bk_{j+1}(t) + \frac{\gamma^j l^H}{a(\gamma-1)}\left\{[b(1-a\gamma)-\rho]e^{(b-\rho)t} + \gamma[b(a+\eta-1)-\rho(\eta-1)]e^{-(\eta-1)(b-\rho)t}\right\} \end{aligned}$$

The solution to this ordinary differential equation is

$$k_{j+1}(t) = \theta_1 e^{bt} + \theta_2 e^{(b-\rho)t} + \theta_3 e^{-(\eta-1)(b-\rho)t},$$

with

$$\begin{aligned} \theta_1 &= k_{j+1}(0) - \theta_2 - \theta_3; \\ \theta_2 &= \frac{\gamma^j l^H}{a\rho(\gamma-1)}[b(a\gamma-1)+\rho]; \\ \theta_3 &= \frac{\gamma^j l^H}{a(\gamma-1)}\gamma\frac{ab+(b-\rho)(\eta-1)}{b-(\eta-1)(b-\rho)}. \end{aligned}$$

At time $t = \tau^g \equiv \frac{\log\gamma}{\eta(b-\rho)}$,

$$k_{j+1}(\tau^g) = \theta_1\gamma^{\frac{b}{\eta(b-\rho)}} + \theta_2\gamma^{\frac{1}{\eta}} + \theta_3\gamma^{-\frac{\eta-1}{\eta}}.$$

The build-up phase comes next and lasts until $t = \frac{\log\gamma+\log a}{b-\rho}$. During the build-up phase, $k^3_j(t) = 0$ as capital j has been used up; and $k^1_{j+1}(t) = \gamma^{j+1}l(t)$ as labor is all and only employed by capital $j+1$. Assume $k^3_{j+1}(t) = 0$.[43] The dynamics for $k_{j+1}(t)$ is

$$\begin{aligned} dk_{j+1}(t) &= b[k_{j+1}(t) - \gamma^{j+1}l(t)]dt, \\ &= b[k_{j+1}(t) - \gamma^{j+1}l^L e^{g^b t}]dt. \end{aligned}$$

[43] Note that here we made the assumption that, during the build-up phase and before $t = \tau^g + \tau^b$, capital $j+1$ is only used in replicating itself and not used in creating $j+2$. As mentioned earlier, both activities satisfy zero profit conditions. In essence, between $t = \tau^g$ and $t = \tau^g + \tau^b$, various arrangements regarding what percentage of and when non-production capital $j+1$ is converted into capital $j+2$ are equivalent as they generate the same amount of capital $j+2$ at $t = \tau^g + \tau^b$.

The solution to this differential equation is

$$k_{j+1}(t) = [k_{j+1}(\tau^g) - \theta]e^{b(t-\tau^g)} + \theta e^{g^b(t-\tau^g)}, \quad \text{for } \tau^g \le t \le \tau^g + \tau^b,$$

with

$$\theta \equiv \frac{b}{b-g^b}\gamma^{j+1}l^L.$$

At $t = \tau^g + \tau^b = \frac{\log\gamma + \log a}{b-\rho}$, capital $j+1$ is

$$k_{j+1}(\tau^g+\tau^b) = k_{j+1}(\tau^g)[a\gamma^{\frac{\eta-1}{\eta}}]^{\frac{b}{b-\rho}} - \theta a\gamma^{\frac{\eta-1}{\eta}}[(a\gamma^{\frac{\eta-1}{\eta}})^{\frac{\rho}{b-\rho}} - \frac{1}{a}].$$

At time $t = \tau^g + \tau^b$, $\gamma^{j+1}l^H$ units of capital $j+1$ are employed in producing the consumption good. The remaining capital $j+1$, $k_{j+1}(\tau^g+\tau^b) - \gamma^{j+1}l^H$, is converted to capital $j+2$. It follows that,

$$\begin{aligned}
k_{j+2}(\tau^g+\tau^b) =& \frac{1}{a}[k_{j+1}(\tau^g+\tau^b) - \gamma^{j+1}l^H] \\
=& \frac{1}{a}\{[(k_{j+1}(0) - \theta_2 - \theta_3)\gamma^{\frac{b}{\eta(b-\rho)}} + \theta_2\gamma^{\frac{1}{\eta}} + \theta_3\gamma^{-\frac{\eta-1}{\eta}}](a\gamma^{\frac{\eta}{\eta-1}})^{\frac{b}{b-\rho}} \\
&- \theta a\gamma^{\frac{\eta-1}{\eta}}[(a\gamma^{\frac{\eta-1}{\eta}})^{\frac{\rho}{b-\rho}} - \frac{1}{a}] - \gamma^{j+1}l^H\} \\
=& k_{j+1}(0)a^{\frac{\rho}{b-\rho}}\gamma^{\frac{b}{b-\rho}} - a^{\frac{\rho}{b-\rho}}\gamma^{\frac{b}{b-\rho}}[\theta_2(1-\gamma^{-\frac{\rho/\eta}{b-\rho}}) + \theta_3(1-\gamma^{-\frac{\eta b-(\eta-1)\rho}{\eta(b-\rho)}})] \\
&- \theta a\gamma^{\frac{\eta-1}{\eta}}[(a\gamma^{\frac{\eta-1}{\eta}})^{\frac{\rho}{b-\rho}} - \frac{1}{a}] - \gamma^{j+1}l^H
\end{aligned}$$

Equivalently,

$$\frac{k_{j+2}(\tau^g+\tau^b)}{\gamma^{j+1}} = (a\gamma)^{\frac{\rho}{b-\rho}}\frac{k_{j+1}(0)}{\gamma^j} - \underline{x},$$

with $\underline{x} > 0$ defined as

$$\underline{x} \equiv (a\gamma)^{\frac{\rho}{b-\rho}}[\frac{\theta_2}{\gamma^j}(1-\gamma^{-\frac{\rho/\eta}{b-\rho}}) + \frac{\theta_3}{\gamma^j}(1-\gamma^{-\frac{\eta b-(\eta-1)\rho}{\eta(b-\rho)}})] + \frac{\theta}{\gamma^j}a\gamma^{-\frac{1}{\eta}}[(a\gamma^{\frac{\eta-1}{\eta}})^{\frac{\rho}{b-\rho}} - \frac{1}{a}] + \gamma l^H.$$

Note that as there is a γ^j term in all θ, θ_2 and θ_3, $\underline{x}$ is independent of j. The relation between $\frac{k_{j+2}(t_{j+2})}{\gamma^{j+1}}$ and $\frac{k_{j+1}(t_{j+1})}{\gamma^j}$ is essentially the same as in the exogenous labor supply case, as depicted in Figure 3.5. As $(a\gamma)^{\frac{\rho}{b-\rho}} > 1$, there exists a unique steady state value of capital. Further, any initial capital below the steady state (consuming too much at the beginning) will eventually leads to a negative capital stock; and any initial capital above the steady state level (consuming too little at the beginning) leads to an explosion of the capital stock.

We now move to the initial growth phase. Denote $k_0(0)$ the initial value of capital of vintage 0, and assume that $0 < k_0(0) < k_0^* \equiv ak^* + l^H$. At $t = 0$, we need to determine how to allocate the initial capital between production of the consumption good and self-accumulation. Denote $k_0^1(0)$ the units of capital 0 in producing the consumption good. Recall the production function is $c(t) = \min\{k_0^1(t), l_0(t)\}$. The following equilibrium conditions hold,

$$c = k_0^1 = l_0; \quad 1 = r + w; \quad w = \zeta l_0^{\frac{\eta}{\eta-1}}.$$

Given $k_0^1(0)$, the wage rate and rental price are $w(0) = \zeta k_0^1(0)^{\frac{\eta}{\eta-1}}$, and $r(0) = 1 - w(0)$. The implied price of capital 0 is[44]

$$q_0(0) = [1 - \zeta k_0^1(0)^{\frac{\eta}{\eta-1}}]\frac{1}{bc(0)}.$$

Denote τ_1 the first time when capital 1 is introduced to producing the consumption good. The price of capital 0 for $t \in [0, \tau_1]$ satisfy

$$q_0(t) = [1 - \zeta k_0^1(t)^{\frac{\eta}{\eta-1}}]\frac{1}{bc(t)}.$$

On the other hand, we know from Proposition 1 that, at $t = \tau_1$,

$$q_0(\tau_1) = \frac{1}{a}\frac{\gamma - 1}{\gamma - 1/a}\frac{1}{bc(\tau_1)}.$$

It follows that $k_0^1(\tau_1) = [\frac{a-1}{a-1/\gamma}\frac{1}{\zeta}]^{\frac{\eta-1}{\eta}} = l^H$.

During the initial phase, $c(t) = k_0^1(t)$. Capital 0 self accumulates in this phase, which implies that its price decrease over time at the rate $b - \rho$. That is, $q_0(t) = q_0(0)e^{-(b-\rho)t}$. Equivalently,

$$[k_0^1(t)^{-1} - \zeta k_0^1(t)^{\frac{1}{\eta-1}}] = [k_0^1(0)^{-1} - \zeta k_0^1(0)^{\frac{1}{\eta-1}}]e^{-(b-\rho)t}. \tag{1}$$

As $h(y) \equiv y^{-1} - \zeta y^{\frac{1}{\eta-1}}$ is a monotonically decreasing function in y, Equation (1) uniquely determines values of $k_0^1(t)$ for a given $k_0^1(0)$.

The length of the initial growth phase, τ_1, is endogenously determined by the choice of the initial allocation. Specifically, from $q_0(\tau_1) = q_0(0)e^{-(b-\rho)\tau_1}$, τ_1 should satisfy

$$\tau_1 = \frac{1}{b-\rho}\log\left\{a\frac{k_0^1(0)^{-1} - \zeta k_0^1(0)^{\frac{1}{\eta-1}}}{\frac{\gamma-1}{\gamma-1/a}\frac{1}{l^H}}\right\}.$$

The dynamics of $k_0(t)$ in the initial phase is

$$\dot{k}_0(t) = b[k_0(t) - k_0^1(t)], \quad \textit{for } t \in [0, \tau_1]$$

[44] As before, wage rate and rental price are in units of current consumption good for computational convenience; for the price of capital, the numeraire is marginal utility.

with $k_0^1(t)$ satisfying Equation (1). This ordinary differential equation, though not admitting an analytical solution, uniquely determines the values of $k_0(t)$ for $0 < t \leq \tau_1$ given an initial value $k_0^1(0)$. At $t = \tau_1$, the following boundary condition must hold,

$$k_1(\tau_1) = \frac{1}{a}[k_0(\tau_1) - l^H] = k^*.$$

where k^* is the steady state value of the normalized capital stock calculated above.

Note that a larger value of $k_0^1(0)$, on the one hand leads to a larger $k_0^1(t), \forall t \in [0, \tau_1]$, and consequently smaller $k_0(t), \forall t \in [0, \tau_1]$, and on the other hand implies a smaller value of τ_1. As a result, with a larger choice of initial $k_0^1(0)$ the value of $k_0(\tau_1)$ and $k_1(\tau_1)$ would be smaller. Therefore the left hand side of the last equation is a monotonically decreasing function of $k_0^1(0)$. Furthermore, when $k_0^1(0) \to 0$, the value of $\tau_1 \to \infty$ and $k_1(\tau_1) \to \infty > k^*$; When $k_0^1(0) \to k_0(0)$, $k_0(\tau_1) \to k_0(0)$ and the implied value of $k_1(\tau_1) < k^*$. These properties guarantee the existence of a unique value $k_0^1(0)$ that satisfies the boundary condition above.

Technology shocks with a CES production function Consider an aggregate CES production function, $y_t = e^{z_t}[\theta k_t^\rho + (1-\theta) l_t^\rho]^{\frac{1}{\rho}}$, with $\rho < 0$. A positive shock on z_t should immediately increase working hours. Capital, as a stock variable, increases much slower. Therefore, the capital-labor ratio shows a hump-shape after a positive technology shock. Consequently, the labor share displays an U-shape response in the labor share. Zhang (2007) has done a similar exercise. We show here that such a mechanism would produce variation in factor shares that is a too small under reasonable parameter values. To see this, note that wage is $w_t = e^{z_t}[\theta k_t^\rho + (1-\theta) l_t^\rho]^{\frac{1}{\rho}-1}(1-\theta) l^{\rho-1}$, and the labor share is $ls = \frac{wl}{y} = \frac{1-\theta}{1-\theta+\theta(\frac{k}{l})^\rho}$. Denote ls^* the steady state labor share, the standard deviation of ls satisfies

$$\sigma(ls) = \frac{\partial ls}{\partial \frac{k}{l}}(ls^{ss}) * \sigma(\frac{k}{l}) = \frac{1-\theta}{[(1-\theta)/ls^{ss}]^2}\theta\rho(\frac{1-\theta}{\theta})^{\frac{\rho-1}{\rho}}(\frac{1}{ls^{ss}} - 1)^{\frac{\rho-1}{\rho}}\sigma(\frac{k}{l}) \equiv \Delta * \sigma(\frac{k}{l}).$$

Take the following parameter values, $ls^{ss} = 0.64$, $\theta = 0.3$ and $\rho = -0.25$ (to match an elasticity of substitution, $\frac{1}{1-\rho} = 0.8$), It follows that $\Delta = 0.019$. If $\rho = -0.5$, $\Delta = 0.067$. These values are too small to generate enough variations in $\sigma(ls)$ as observed in data. On the other hand, wage is $w = (1-\theta)(e^{z_t})^\rho(\frac{y}{l})^{1-\rho}$, which would generate a volatility of wage that is much larger than observed in data.

Appendix: Tables and Figures

Table 6.1: Corr. btw. growth in LS and other var.

Variable	LP	Wage	Emp.	Hours
Quarterly	-0.61^{***}	0.42^{***}	-0.17^{***}	-0.20^{***}
9-quarter MA	-0.40^{***}	0.30^{***}	0.24^{***}	0.15^{**}
13-quarter MA	-0.36^{***}	0.31^{***}	0.25^{***}	0.17^{***}
17-quarter MA	-0.27^{***}	0.34^{***}	0.28^{***}	0.21^{***}
HP trend	-0.31^{***}	0.23^{***}	0.22^{***}	0.18^{***}

Note: Non-farm business sectors; LS is defined as in text. The results are qualitative the same if we use the BLS defined LS. BLS assume wage for proprietors is the same as the rest of economy (see e.g. Elsby et al. 2013).

Table 6.2: Regression coefficients

	Dep. var.: ΔLS, quarterly			
	(1)	(2)	(3)	(4)
ΔLP	-0.51^{***}			
	(0.05)			
$\Delta Wage$		0.48^{***}		
		(0.06)		
ΔEmp			-0.32^{***}	
			(0.05)	
$\Delta Hours$				-0.34^{***}
				(0.05)
Recession D.	Y	Y	Y	Y
R^2	0.25	0.21	0.13	0.16
Obs.	294			

Table 6.3: Regression coefficients

	Dep. var.: ΔLS, 2-yr MA				Dep. var.: ΔLS, 3-yr MA			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
ΔLP	-0.42^{***}				-0.34^{***}			
	(0.05)				(0.05)			
$\Delta Wage$		0.32^{***}				0.34^{***}		
		(0.05)				(0.05)		
ΔEmp			0.14^{***}				0.15^{***}	
			(0.04)				(0.04)	
$\Delta Hours$				0.05				0.07^{**}
				(0.04)				(0.04)
Recession D.	Y	Y	Y	Y	Y	Y	Y	Y
R^2	0.22	0.13	0.06	0.02	0.16	0.18	0.08	0.04
Obs.	286				282			

Table 6.4: Regression coefficients

	Dep. var.: ΔLS, 4-yr MA				Dep. var.: ΔLS, HP trend			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
ΔLP	-0.24^{***}				-0.25^{***}			
	(0.05)				(0.04)			
$\Delta Wage$		0.36^{***}				0.25^{***}		
		(0.04)				(0.04)		
ΔEmp			0.20^{***}				0.20^{***}	
			(0.03)				(0.03)	
$\Delta Hours$				0.13^{***}				0.16^{**}
				(0.03)				(0.04)
Recession D.	Y	Y	Y	Y	Y	Y	Y	Y
R^2	0.08	0.22	0.11	0.06	0.12	0.13	0.11	0.07
Obs.	278				294			

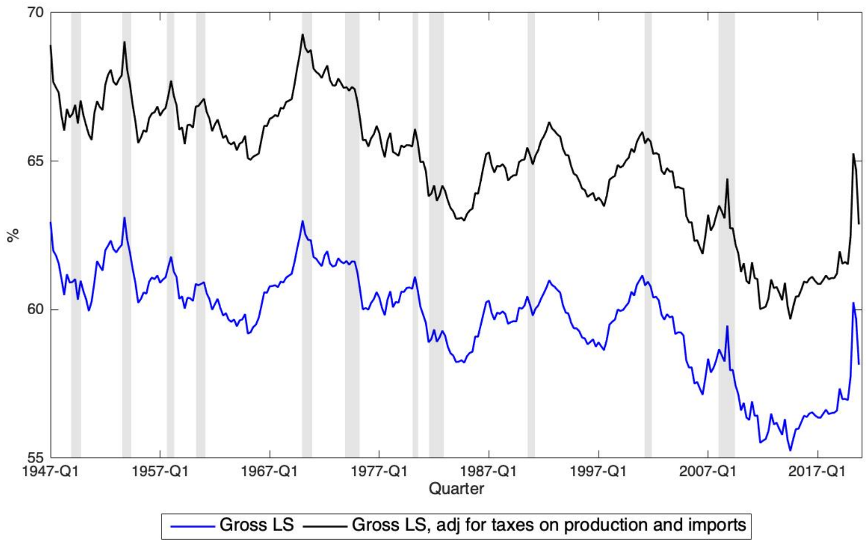

Figure 6.1: Gross labor share, adjusted for taxes on production and imports

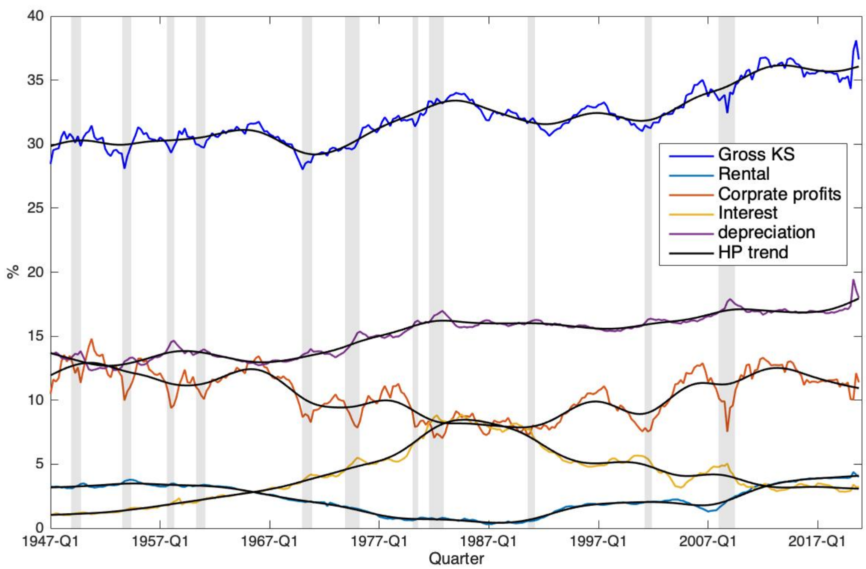

Figure 6.2: Gross capital share and its components

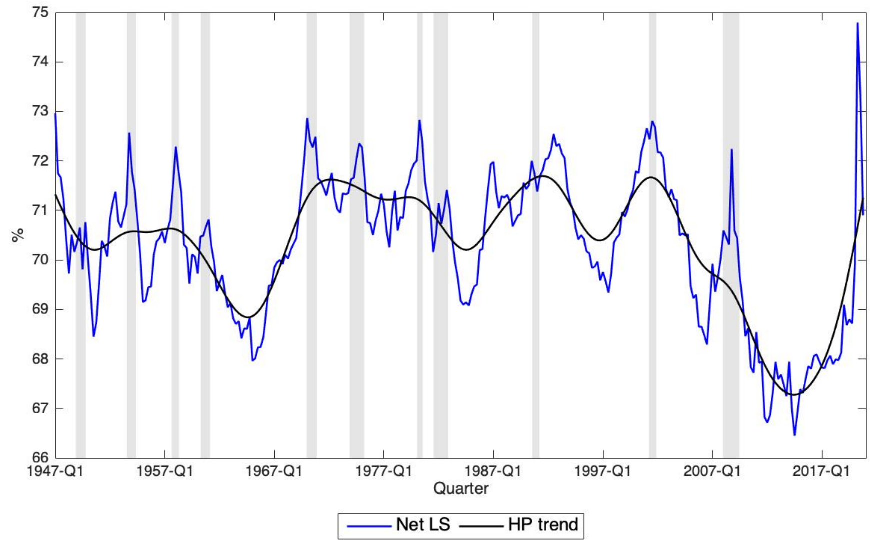

Figure 6.3: Net labor share

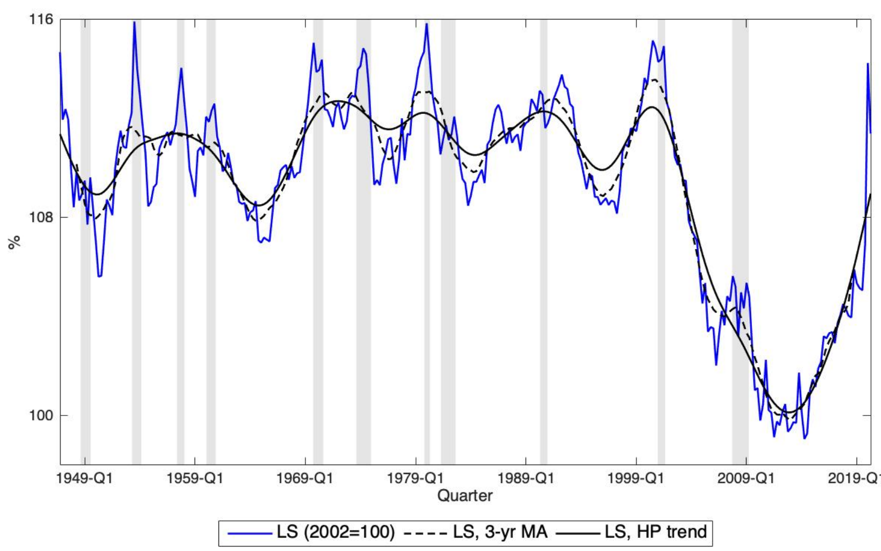

Figure 6.4: LS, 3-year Moving average and HP trend

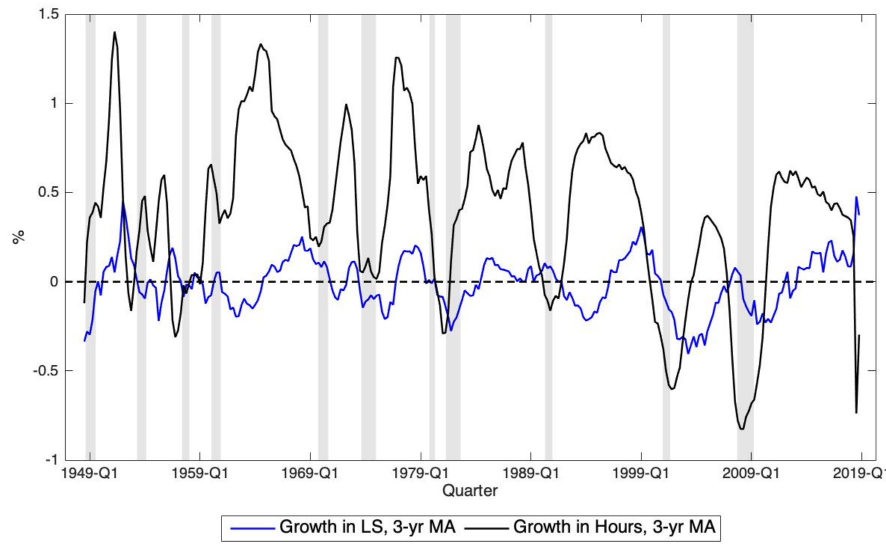

Figure 6.5: Growth in LS and Hours, 3-yr MA

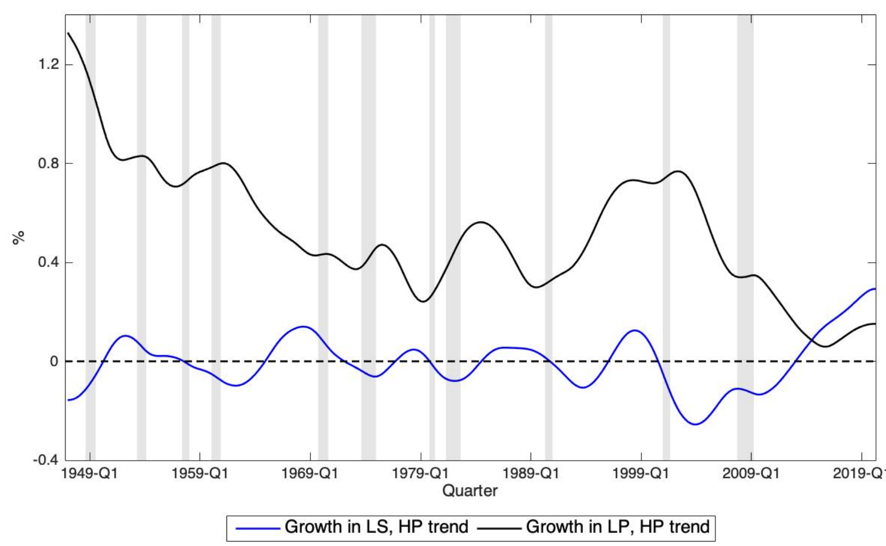

Figure 6.6: Growth in LS and LP, HP trend

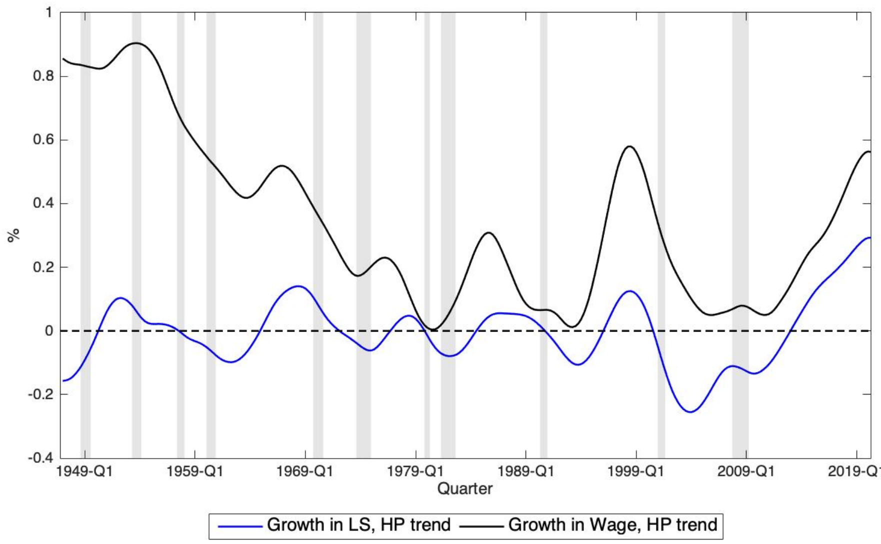

Figure 6.7: Growth in LS and Wage, HP trend

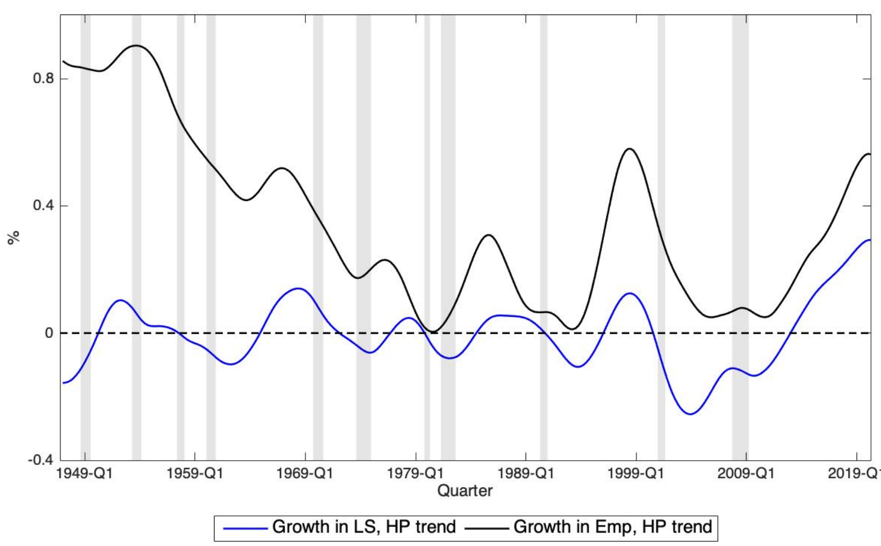

Figure 6.8: Growth in LS and Employment, HP trend

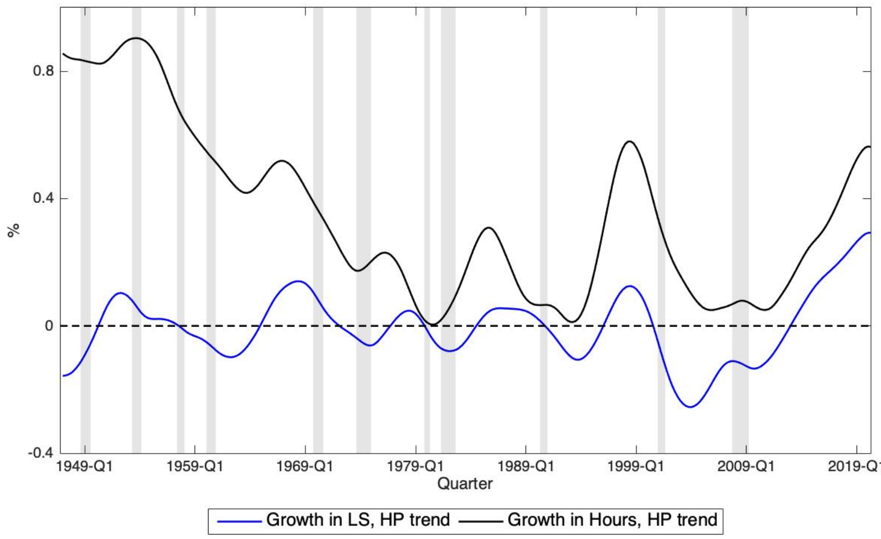

Figure 6.9: Growth in LS and Hours, HP trend

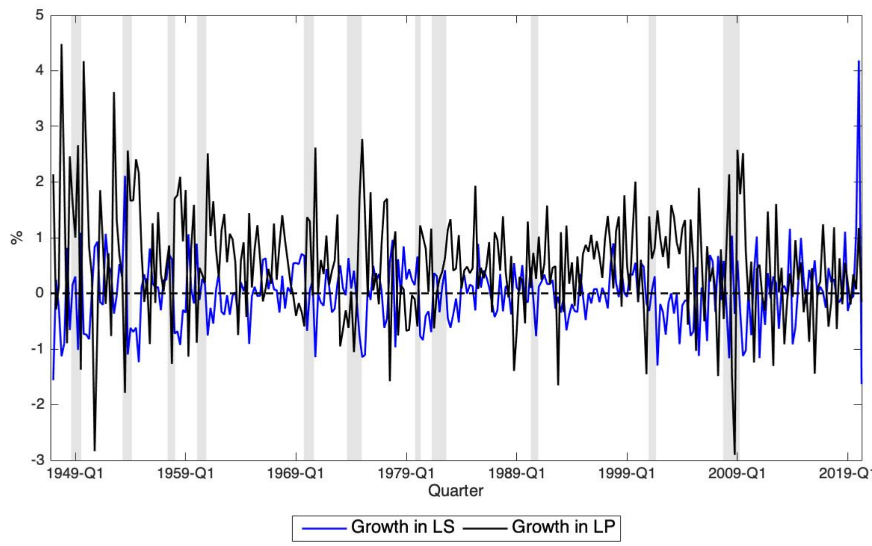

Figure 6.10: Growth in LS and LP

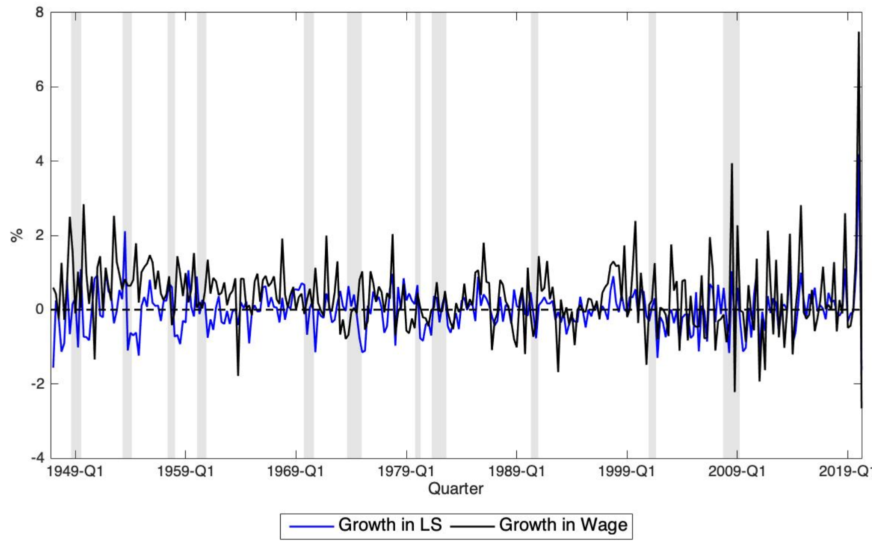

Figure 6.11: Growth in LS and Wage

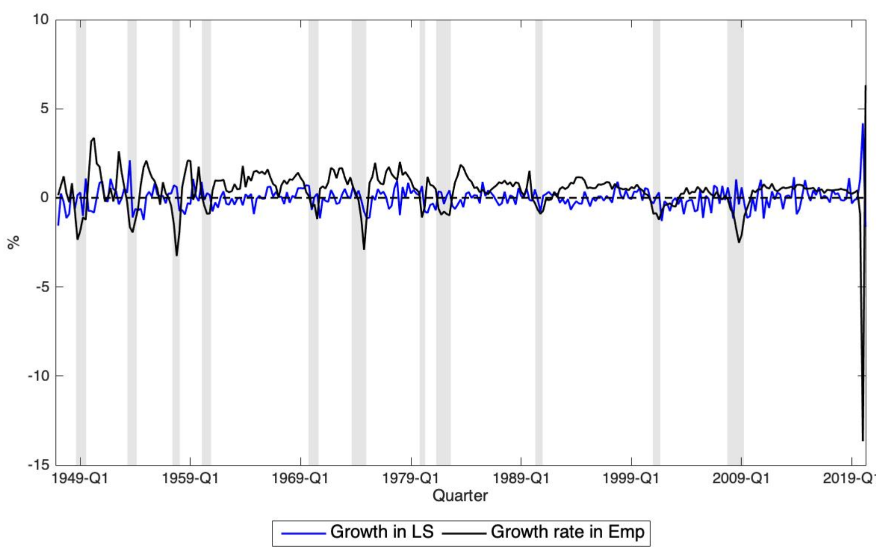

Figure 6.12: Growth in LS and Employment

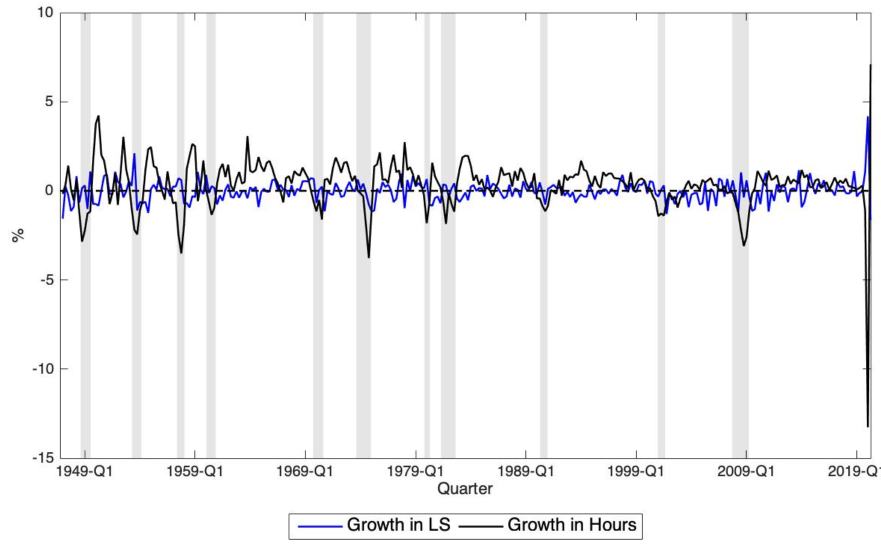

Figure 6.13: Growth in LS and Hours

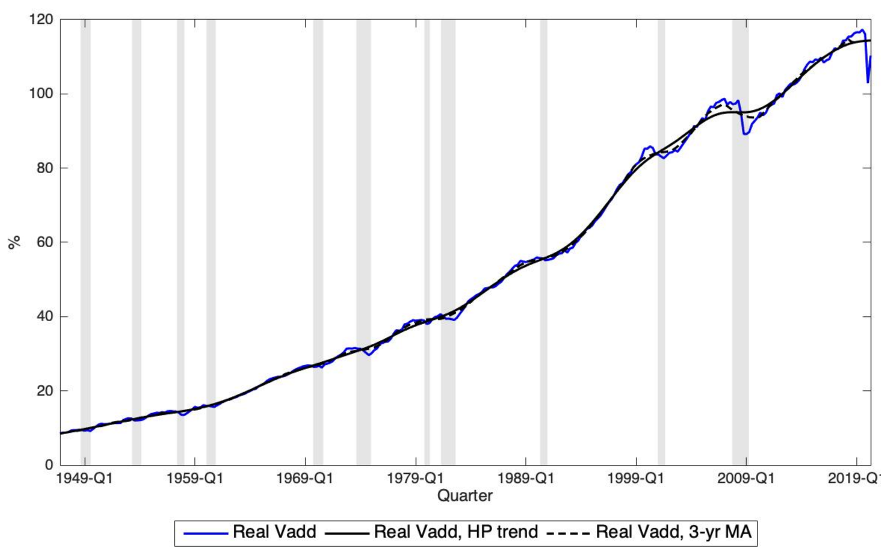

Figure 6.14: Real Valued, HP trend and 3-yr Moving average