Contagious Bubbles*

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Abstract

This paper proposes a framework to study contagious stock bubbles in a multi-sector production economy with heterogeneous investment technologies. Due to financial frictions, stock bubbles arise endogenously that help inject additional liquidity. Due to financial linkages, the existence of bubbles in different sectors may be interdependent. Theoretically, we characterize the entire set of bubbly equilibria, and provide the condition under which the burst of bubbles in one sector spikes bubbles in other sectors. Quantitatively, we show that due to an unexpected burst of bubbles, it can generate a sizeable recession only when the burst happens in a critical sector and transmits to the rest of the economy.

Keywords: Financial Networks, Financial Frictions, Sectoral Capital Accumulation, Sectoral Bubbles, Financial Contagion.

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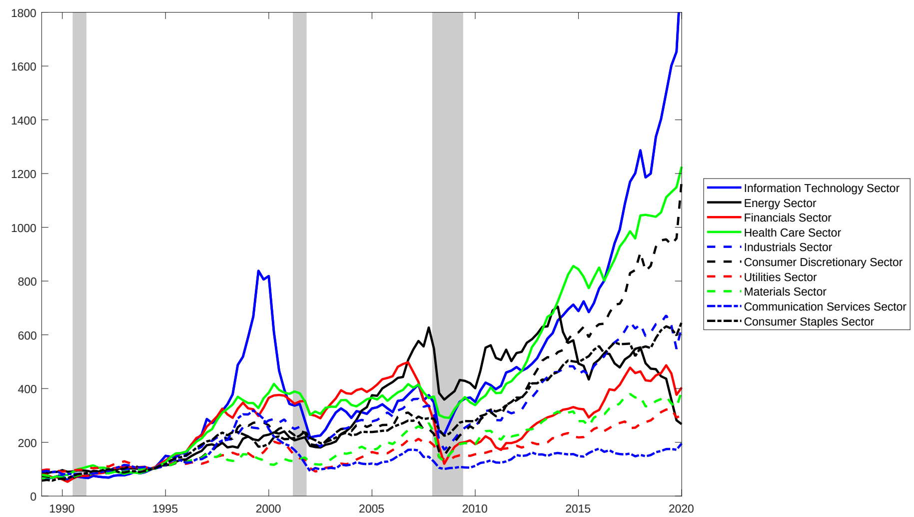
1. Introduction

Not only are stock markets in different parts of the world closely connected such that the burst of a financial bubble in one nation can sometime generate contagious effects across the globe, but individual stocks in all industries are also connected such that the collapse of the price of one asset can sometime drag down the value of other assets in entirely different sectors in the economy.

Figure 1: US Stock Prices By Sectors



For example, it is generally perceived that the burst of the bubble in information technology (IT) around 2000-2001 and that of the housing bubble around 2008 had different effects on the overall economy: The 2008 great recession caused by the financial crisis was far more severe than the 2001 recession caused by the IT bubble bust, even though the IT bubble around 2000 appeared to be far larger than the housing bubble around 2007. Specifically, the IT sector's stock price reached its peak of 838 in the 2000Q1 and then dropped to a trough of 337 in 2001Q4, the total drop in stock price from peak to trough was more than 500 points or 60%. In contrast, the real estate sector's stock price reached its peak of 188 in 2007Q1 and then dropped to a trough of 137 in 2008Q1, the total drop from peak to trough was 51 points or 27%.

Motivated by the different effects of stock price crashes on the rest of the economy, we proposes a framework to study the contagiousness of stock bubbles in a multi-sector economy with heterogeneous agents and financial linkages. We show that bubbles can be contagious in the sense that the bust of bubbles in one sector can trigger the bust of bubbles in other sectors, and we characterize the conditions under which such contagion can happen. The model therefore can potentially rationalize why the IT stock price crash did not lead to a catastrophic outcome as the housing bubble bust.

Bubbles in our model arise due to the presence of financial frictions. In the model economy, firms accumulate capital, but subject to idiosyncratic investment shocks. Without financial frictions, those firms that receive the highest investment shock make all the investment by borrowing from households and other firms. With financial frictions, only limited borrowing is allowed, and firms with their investment shock above certain efficiency cutoff will investment. Compared with the frictionless benchmark, inefficient investment has to be made by some firms. When firms' can use their own equity value as collateral for borrowing, self-fulfilling equity bubbles can arise. This type of bubbles help firms to overcome financial frictions and the economy can achieve a more efficient allocation.

The key to our analysis is that due to the presence of multiple sectors in the economy, the extent to which a firm can borrow not only depends on its own equity value but also equity values of firms in other sectors. The borrowing limit of firms in one sector may be relaxed when firms in other sector perform better. This type of financial linkages naturally make the existence of self-fufilling bubbles in different sectors interdependent, which leaves room for bubble contagion.

The main contribution of the paper is to provide a complete characterization of the entire set of bubbly equilibria, and provide the conditions under which some sector's bubble burst will be contagious to other sectors. Though the multi-sector economy is complicated, it turns out that the condition in determining whether a bubbly equilibrium exists or not boils down to a simple condition on the efficiency cutoff–a bubbly equilibrium only exists if the efficiency cutoff is higher than that in a bubble free equilibrium. The efficiency cutoff hinges on the financial linkages across sectors and the severity of financial frictions, and we show how the interaction between these factors jointly shape the existence of a bubbly equilibrium. Roughly speaking, sectoral bubbles are more likely to take place when financial linkages are strong and when equity pledgeability is high.

We are also able to identify critical sectors such that a bubble burst in such sectors can have maximum impact on the economy as it will destroy bubbles in some other sectors at the same time. We find that these critical sectors need to have strong connections with other sectors through financial linkages, but they do not need to have a large value added in total output.

Given that the U.S. mergers and acquisitions transaction data exhibit "financial linkages," we explore this linkage to formulate our model and show that firms' stock values are inter-connected in a way to generate asymmetric contagiousness after a bubble burst. In particular, since a severe recession can follow a stock market crash only if the vital sector's asset prices collapse, our model predicts that the 2008 housing bubble burst would generate a recession much deeper than the 2001 IT bubble burst, because the IT sector is not as vital as the housing sector in the financial network. Our study has important policy implications: The US monetary policy may have over-reacted to 2001 IT bubble burst but under-reacted to 2008 housing bubble burst.

Our paper contributes to this bubble literature by introducing financial networks. The early works on asset bubbles include Samuelson (1958) and Tirole (1985) in the overlapping-generations (OLG) setup. See also Weil (1987) for an introduction to stochastic bubbles in the same framework. The recent progress made on bubble theory consists of two branches of research. The works based on the OLG framework include Arce and López-Salido (2011), Martin and Ventura (2012), Farhi and Tirole (2011), Martin and Ventura (2016), Ikeda and Phan (2016), Chen and Wen (2017), Bengui and Phan (2018), Biswas, Hanson, and Phan (2018) and Ikeda and Phan (2019), among many others. On the other hand, the research based on the infinite-horizon framework includes Kocherlakota (2009), Wang and Wen (2012b), Hirano, Inaba, and Yanagawa (2015), Miao, Wang, and Xu (2015), Hirano and Yanagawa (2016), and Miao and Wang (2018) for firm bubbles; Dong et al. (2019) and Liu, Wang, and Zha (2019) for household housing bubbles; and Aoki and Nikolov (2015) and Miao and Wang (2015) for banking bubbles. Also see Kocherlakota (2008) and Kocherlakota (1992) among others for the purely theoretical observation that the frictions on debt accumulation can cause bubbles. Meanwhile, there has been a growing literature on the interaction of asset bubbles and monetary policy; see Galí (2014); Galí and Gambetti (2015), Asriyan et al. (2016), and Dong, Miao, and Wang (2020). Moreover, Miao (2014) and Martin and Ventura (2018) provide comprehensive surveys on rational bubbles.

In addition to macroeconomic theory on asset bubbles, there has been a burgeoning literature on the empirical and quantitative analysis of bubbles. Galí and Gambetti (2015) investigate the effects of monetary policy on stock market bubbles. Xiong and Yu (2011) use Chinese warrant bubbles as a laboratory to highlight the joint effects of short-sale constraints and heterogeneous beliefs in driving bubbles. Miao, Wang, and Xu (2015) analyze a Bayesian DSGE model of stock market bubbles and business cycles. Giglio, Maggiori, and Stroebel (2016) test whether bubbles exist in the UK and Singapore housing markets. Chen and Wen (2017) use an OLG model to quantitatively account for the housing boom in China during the country's economic transition. Domeij and Ellingsen (2018) offers a quantitative analysis of rational bubbles and Ponzi debts. Martin, Moral-Benito, and Schmitz (2018) study the financial transmission channel of housing bubbles in Spain.

The rest of the paper proceeds as below. Section 2 and describes and characterizes the baseline model respectively. Section 6 concludes. All the proofs, alongside with the notation table and the dynamical system are left in the Appendix.

2. Environment

Time is discrete. There are two types of infinitely lived agents in the economy: firms and households. There are two types of firms: intermediate-goods firms and final-goods firms. All firms are owned by households. There are S sectors for intermediate goods production, and each sector $j \in \{1, 2, \ldots, S\}$ has a unit measure of firms indexed ι.

2.1 Household

We assume there are a large number of identical households in the economy. The problem of a representative household is given by

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(C_t\right), \tag{2.1}$$

subject to

$$C_t + \sum_{j=1}^{S} \int_0^1 \zeta_{j,t+1}\left(\iota\right)\left(V_{jt}\left(\iota\right) - d_{jt}\left(\iota\right)\right) d\iota + \frac{D_{t+1}}{1+r_t} = \sum_{j=1}^{S} \int_0^1 \zeta_{jt}\left(\iota\right) V_{jt}\left(\iota\right) d\iota + W_t + D_t,$$

and

$$D_{t+1} \geq 0.$$

where β is the discount factor, C_t is consumption of the final goods, D_t is the deposit, and r_t is the interest rate. We assume that labor is perfectly mobile across sectors and normalize total labor supply to be 1.[1] In terms of the portfolio choice, $\zeta_{jtjt}\left(\iota\right)$ denotes the share holdings of firm ι in sector j at period t, $V_{jt}\left(\iota\right)$ the date-t ex-dividend stock price, and $d_{jt}\left(\iota\right)$ the associated dividend. For later use, we define $\Lambda_t \equiv u'\left(C_t\right)$ the marginal utility of consumption.

2.2 Final Goods Firms

A representative firm transforms intermediate goods X_{jt} from different sectors into final goods Y_t, with a Cobb-Douglas aggregator

$$Y_t = \prod_{j=1}^{S} X_{jt}^{\varphi_j}.$$

We normalize the final goods price to be one, and it follows that firms solve the following problem

$$\max_{\left\{X_{jt}\right\}} \; Y_t - \sum_{j=1}^{S} P_{jt} X_{jt}. \tag{2.2}$$

Optimal demand implies that sector j's output to total GDP ratio is constant over time, i.e., $P_{jt}X_{jt}/Y_t = \varphi_j$.

[1] The inelastic labor supply assumption helps to simplify the characterization of steady-state allocation, and we can endogenize labor supply without changing the main results.

2.3 Intermediate Goods Firms

In the baseline analysis, we assume that intermediate goods production only requires primary inputs. The production function of firm ι in sector j is given by

$$o_{jt}(\iota) = k_{jt}^{\alpha}(\iota)\, n_{jt}^{1-\alpha}(\iota), \tag{2.3}$$

where $k_{it}(\iota)$ is the predetermined capital stock and $n_{it}(\iota)$ is the labor input.[2] Given the amount capital stock, firm ι chooses its optimal employment level, and the amount revenue is proportional to its capital stock

$$R_{jt}k_{jt}(\iota) \equiv \max_{n_{jt}(\iota)} P_{jt}o_{jt}(\iota) - W_t n_{jt}(\iota). \tag{2.4}$$

Firm ι chooses its investment $i_{jt}(\iota)$ in terms of final goods every period to accumulate capital, subject to the irreversible constraint[3]

$$i_{jt}(\iota) \geq 0. \tag{2.5}$$

As in Wang and Wen (2012b), the capital stock evolution is subject to idiosyncratic investment efficiency shock

$$k_{j,t+1}(\iota) = (1-\delta)\, k_{jt}(\iota) + \epsilon_{jt}(\iota)\, i_{jt}(\iota), \tag{2.6}$$

where δ represents the depreciation rate and $\epsilon_{jt}(\iota)$ the investment efficiency of firm ι in sector j. We assume $\epsilon_{jt}(\iota)$ is i.i.d across firms and sectors and over time, with CDF $F(\epsilon)$ and support $(\underline{\epsilon}, \overline{\epsilon})$.[4] In addition, we normalize such that $\mathbb{E}(\epsilon) = 1$. Firms that receive a good investment shock would like to borrow from households and other firms to accumulate more capital, but their borrowing capacities will depend on the severity of financial frictions.

The firms' problem is to maximize the present value of dividends,

$$V_{jt}\left(k_{jt}(\iota), l_{jt}(\iota), \epsilon_{jt}(\iota)\right) = \max_{k_{j,t+1}(\iota), l_{j,t+1}(\iota)} d_{jt}(\iota) + \mathbb{E}_t \frac{\beta\Lambda_{t+1}}{\Lambda_t} V_{j,t+1}\left(k_{j,t+1}(\iota), l_{j,t+1}(\iota), \epsilon_{j,t+1}(\iota)\right), \tag{2.7}$$

and the current dividend $d_{jt}(\iota)$ is given by

$$d_{jt}(\iota) = R_{jt}k_{jt}(\iota) + \frac{1}{1+r_t} l_{j,t+1}(\iota) - l_{jt}(\iota) - i_{jt}(\iota), \tag{2.8}$$

where $l_{j,t}(\iota)$ is the firm's existing debt and $l_{j,t+1}(\iota)$ is the amount of debt that is due next period. Firms are also subject to financial frictions that we now discuss in details.

[2] In the Appendix, we show that the framework can be extended to include a production network where firms also use other outputs from other industries as intermediate inputs.

[3] A more generalized constraint for investment irreversibility is $i_t(\iota) \geq -\zeta_t k_t(\iota)$ with $\zeta_t \geq 0$. See Wang and Wen (2012b) for more details.

[4] Tractability is well preserved if the distribution is sector specific $F_j(\cdot)$.

The first one is the equity constraint. As commonly assumed in the literature, external financial markets are imperfect so that firms are facing the constraint on equity issuance:[5]

$$d_{jt}(\iota) \geq 0. \tag{2.9}$$

In addition to equity frictions, we extend the one-sector debt constraint à la Miao and Wang (2018) to a multi-sector one:

$$\begin{aligned} \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \overline{V}_{j,t+1}\left(k_{j,t+1}(\iota), l_{j,t+1}(\iota)\right) \quad \geq \quad & \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \overline{V}_{j,t+1}\left(k_{j,t+1}(\iota), 0\right) \\ & -\xi_j \sum_{i=1}^{S} \mathcal{M}_{ji} \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \overline{V}_{i,t+1}\left(\sigma_j k_{jt}(\iota), 0\right), \end{aligned} \tag{2.10}$$

where $\overline{V}_{j,t+1}\left(k_{j,t+1}(\iota), l_{j,t+1}(\iota)\right) \equiv \int V_{t+1}\left(k_{j,t+1}(\iota), l_{j,t+1}(\iota), \epsilon_{j,t+1}(\iota)\right) dF\left(\epsilon_{j,t+1}(\iota)\right)$ denotes the expected value of a firm before the idiosyncratic shock is realized.

Constraint (2.10) is the generalized borrowing constraint in the presence of financial linkage across sectors.[6] It can be interpreted as the incentive constraint in the presence of firm's limited commitment. Firm ι decides whether to default on its debt at the beginning of $t+1$. Without default, the continuation value is simply the expected value of operation on the LHS of condition (2.10). In case of default, debt is renegotiated and the repayment is relived. In this scenario, the creditor is able to recover $\sigma_j k_{jt}$ amount of capital and reestablish a firm in sector i with probability $\mathcal{M}_{ji}$. Eventually, the creditor can sell this new firm at a discounted price $\xi_j \overline{V}_{i,t+1}\left(\sigma_j k_{jt}(\iota), 0\right)$. The parameters σ_j and ξ_j jointly describe the severity of financial frictions, where the former captures depreciation and potential mismatch, and the latter captures the fire sale risk.[7] As in Jermann and Quadrini (2012), firms are assumed to have complete bargaining power. The RHS of condition (2.10) then represents the continuation value in case of default. When constraint (2.10) is satisfied, no equilibrium default will occur.

In addition to the standard risk free bond, the matrix $\mathcal{M}$ captures an additional channel for financial linkages in this economy. Appreciation or depreciation for firms in one sector has spillover effects on other sectors' borrowing limit, which in turn affects the aggregate investment. Furthermore, as we explain in the sequel, the inter-dependence of the borrowing limits also helps transmit or block sectoral bubbles, which is the core of our analysis.

[5] A more generalized setup for (2.9) is $d_t(\iota) \geq -\nu \cdot k_t(\iota)$, where $\nu \geq 0$ governs the severity of equity frictions. Without loss of generality, we set $\nu = 0$ to simplify our analysis. See Wang and Wen (2012a) and Miao, Wang, and Xu (2015), for details of the generalized setup.

[6] See Kiyotaki and Moore (1997), Bernanke, Gertler, and Gilchrist (1999), Jermann and Quadrini (2012) and Moll (2014) etc. for the discussion of borrowing constraints and financial accelerator.

[7] See Shleifer and Vishny (2011) for the literature on fire sale in finance and macroeconomics.

2.4 Sectoral Bubble

We use a guess-and-verify approach to characterize firms' investment decision and their value function, and we will show that sectoral bubble can arise. Intuitively, firms with a better investment technology (higher $\epsilon_{jt}(\iota)$) are willing to invest more. In fact, due to the CRS assumption and homogeneous production technology within a sector, the most efficient arrangement is to let the firm with the highest $\epsilon_{jt}(\iota)$ borrow from other firms and households to make all the investment. However, due to the financial frictions, this allocation is not feasible. In equilibrium, in each sector, firms with technology shock higher than a threshold ϵ^*_{jt} will borrow to their limit and invest, and the remaining firms stay inactive. We define the degree of investment misallocation in sector j as

$$\Gamma\left(\epsilon^*_{jt}\right) \equiv \int_{\epsilon^*_{jt}}^{\overline{\epsilon}} \left(\frac{\epsilon}{\epsilon^*_{jt}} - 1\right) dF, \tag{2.11}$$

and $\Gamma(\cdot)$ is decreasing in ϵ^*_{jt}. As ϵ^*_{jt} approaches to the highest investment shock realization $\overline{\epsilon}$, $\Gamma\left(\epsilon^*_{jt}\right)$ vanishes to zero.

The tightness of the borrowing constraint (2.9) controls how far the equilibrium allocation is away from the efficient one due to the investment misallocation. Similar to Miao and Wang (2018), self-fulfilling stock bubbles can arise in the equilibrium, which increase the firms' value, V_{jt}, and relax the borrowing constraint. Different from their work, in our model, bubbles in our model are sector specific, as characterizes below.

Proposition 2.1. *1. The expected firm value is given by*

$$\mathbb{E}_t \frac{\beta\Lambda_{t+1}}{\Lambda_t} \overline{V}_{j,t+1}\left(k_{j,t+1}, l_{j,t+1}\right) = Q_{jt} k_{j,t+1} - \frac{l_{j,t+1}}{1+r_t} + B_{jt}, \tag{2.12}$$

where the sectoral Tobin's Q and sectoral bubble B_{jt} satisfy

$$Q_{jt} = \mathbb{E}_t \frac{\beta\Lambda_{t+1}}{\Lambda_t}\left[R_{j,t+1}\left(1+\Gamma(\epsilon^*_{j,t+1})\right) + (1-\delta)\,Q_{j,t+1} + \theta^k_j \Gamma(\epsilon^*_{j,t+1}) \sum_{i=1}^{S} \mathcal{M}_{ji} Q_{i,t+1}\right] \tag{2.13}$$

$$B_{jt} = \mathbb{E}_t \frac{\beta\Lambda_{t+1}}{\Lambda_t}\left[B_{j,t+1} + \theta^b_j \Gamma(\epsilon^*_{j,t+1}) \sum_{i=1}^{S} \mathcal{M}_{ji} B_{i,t+1}\right], \quad \text{if } B_{jt} > 0. \tag{2.14}$$

2. The sectoral efficiency cutoff is $\epsilon^*_{jt} = \frac{1}{Q_{jt}}$.

Condition (2.12) states that the expected value of a firm depends on its capital stock minus the debt payment, which can be interpreted as the fundamental evaluation. Moreover, the firm value may loom larger if the sectoral bubble B_{jt} exists.Condition (2.14) reveals the self-fulfilling nature of the sectoral bubble: coordinating to $B_{jt} = 0$ for all j always satisfy this forward-looking Euler equation, and positive bubble may also be consistent with rational expectations. In the next section, we explore under what conditions sectoral bubble can arise. Condition (2.12) also implies the borrowing constraint (2.10) can be written as

$$\frac{l_{j,t+1}(\iota)}{1+r_t} \leq \theta_j^k k_{jt}(\iota) \sum_{i \in \mathbf{S}} \mathcal{M}_{ji} Q_{it} + \theta_j^b \sum_{i \in \mathbf{S}} \mathcal{M}_{ji} B_{it}, \tag{2.15}$$

which makes it clear that sectoral bubble helps to overcome financial frictions.

The cutoff value ϵ_{jt}^* is determined by the sectoral Tobin's Q. To increase one unit of capital for firm ι, the cost is $1/\epsilon_{jt}(\iota)$, and the return is Q_{jt}. It is only profitable to invest if $\epsilon_{jt}(\iota) > 1/Q_{jt} = \epsilon_{jt}^*$. Notice that Q_{jt} is increasing in the misallocation measure $\Gamma_j\left(\epsilon_{jt}^*\right)$, which effectively compensates the individual investment risk by holding capital. In addition, Q_{jt} also depends on other sectors' misallocation measures, as a result of sectoral linkages through the borrowing constraint.

Once the efficiency cutoffs are determined, the sectoral capital and investment can be solved accordingly.

Corollary 1. *The law of motion of sectoral capital K_{jt} and investment I_{jt} are given by*

$$I_{jt} = \left[R_{jt} K_{jt} + \theta_j^k \sum_{i \in \mathbf{S}} \mathcal{M}_{ji} Q_{it} K_{jt} + \theta_j^b \sum_{i \in \mathbf{S}} \mathcal{M}_{ji} B_{it} - L_{jt}\right] \cdot \left[1 - F\left(\epsilon_{jt}^*\right)\right], \tag{2.16}$$

$$K_{j,t+1} = (1-\delta) K_{jt} + \mathbb{E}\left(\epsilon | \epsilon \geq \epsilon_{jt}^*\right) \cdot I_{jt}. \tag{2.17}$$

It is useful to note that the cutoff value affects the capital accumulation both through the fraction of active investing firms $\left[1 - F\left(\epsilon_{jt}^*\right)\right]$, and also the average investment efficiency $\mathbb{E}\left(\epsilon_j | \epsilon_j \geq \epsilon_{jt}^*\right)$.

3. Equilibrium Characterization

In this paper, we focus on the case without aggregate shocks. As a result, along the transition path with rational expectations, whether sectoral bubbles can exist or not boils down to whether they can exist or not in steady states.

To fix the terminology, we classify the equilibria according to their bubble properties. Particularly, we allow the possibility that bubbles may only exist in some but not all sectors.

Definition 1 (Bubbleless and Bubbly Equilibrium)**.** *In bubbleless equilibrium (i.e., fundamental equilibrium), $B_j = 0$ for all sector $j \in \mathbf{S}$. In $\mathfrak{s}$-bubbly equilibrium, $B_j > 0$ for $j \in \mathfrak{s} \subset \mathcal{P}(\mathbf{S})$, where $\mathcal{P}(\mathbf{S})$ denotes the power subset of $\mathbf{S} = \{1, ..., S\}$.*

3.1 Efficiency Cutoff

As hinted in previous analysis, the cutoff values for investment decision govern the capital accumulation process. In steady state, the following lemma shows that the cutoff values are equalized across sectors.

Lemma 1. *In steady state, the Tobin's Q and the cutoff values are equalized across sectors*

$$\epsilon_j = \epsilon^* = \frac{1}{Q(\epsilon^*)},$$

and the interest rate is $1 + r = \frac{1}{\beta[1+\Gamma(\epsilon^*)]}$.

The interest rate is increasing in the single efficiency cutoff as the investment risk reduces. When the misallocation measure approaches to zero, we have $r = \chi$ as in the complete market. It turns out that this efficiency cutoff is also the key determinant for a bubbly equilibrium to exist. To facilitate the characterization of the efficiency cutoff, it is useful to define the following function

$$\Phi(x) = \sum_{j=1}^{S} \frac{\varphi_j}{\chi + \delta - \theta_j^k \Gamma(x)} \left(\frac{\delta x F(x)}{1 - F(x)} - \chi - \theta_j^k \right),$$

where $\chi = 1/\beta - 1$. We make the following assumption on the primitives of the environment.

Assumption. $\Phi(x)$ *is strictly increasing in* x.

As shown in Appendix, this assumption is always satisfied when θ_k^j is relatively small. We also provide the characterization of the equilibrium when this assumption is not satisfied, though quantitatively this may not be the relevant case.

Bubbleless Equilibrium. First consider the bubbleless equilibrium, in which $B_j = 0$ for all j.

Proposition 3.1 (Bubbleless Equilibrium)**.** *There exists a unique bubbleless equilibrium. The cutoff efficiency* ϵ^f *in the bubbleless equilibrium satisfies*

$$\Phi(\epsilon^f) = 0. \tag{3.1}$$

Condition (3.1) implies that ϵ_f^* is determined by the capital pledgeability $\boldsymbol{\theta}^k$. In the special case where $\theta_j^k = \theta^k$, this condition reduces to

$$\frac{\delta \epsilon^f F\left(\epsilon^f\right)}{1 - F(\epsilon^f)} - \chi = \theta^k, \tag{3.2}$$

where it is easy to verify that the cutoff efficiency ϵ^f is increasing in θ^k and denoted as $\epsilon^f\left(\theta^k\right)$. Intuitively, a higher θ^k relaxes the financial constraint, which in turn helps to mitigate the investment misallocaiton. In the general case with heterogeneous θ_j^k, the cutoff efficiency is pinned down by a weighted average of the capital pledgeability across sectors.

Bubbly Equilibrium. Now consider the case where bubbles arise in a subset of sectors, $\boldsymbol{s} \subset \mathcal{P}(\mathbf{S})$. Let $\mathcal{M}_{\boldsymbol{s}}$ denote the financial linkage matrix $\mathcal{M}$ which keeps columns and rows in $\boldsymbol{s}$, and denote diag $\left(\boldsymbol{\theta}_{\boldsymbol{s}}^b\right)$ as the matrix which keeps the equity θ_j^b on the diagonal for $j \in \boldsymbol{s}$. For any sector $i, j \in \boldsymbol{s} \subset \mathcal{P}(\mathbf{S})$, the Euler equation in (2.14) can be rewritten as

$$\frac{r}{\Gamma(\epsilon^*)} B_i = \theta_i^b \sum_{j \in \boldsymbol{s}} \mathcal{M}_{ij} B_j, \tag{3.3}$$

which can written in a more compact way as

$$\left[\frac{r}{\Gamma(\epsilon^*)} \mathbf{I} - \operatorname{diag}(\boldsymbol{\theta}_{\boldsymbol{s}}^b) \mathcal{M}_{\boldsymbol{s}}\right] \mathbf{B}_{\boldsymbol{s}} = 0. \tag{3.4}$$

Then the existence condition of the $\boldsymbol{s}$-bubbly equilibrium is characterized below.

Proposition 3.2 (Bubbly Equilibrium). *If the stochastic matrix $\mathcal{M}$ is irreducible, then for a selection of sectors $\boldsymbol{s} \subset \mathcal{P}(\mathbf{S})$, there exists a unique $\boldsymbol{s}$-bubbly equilibrium if and only if*

$$\rho\left(\operatorname{diag}\left(\boldsymbol{\theta}_{\boldsymbol{s}}^b\right) \mathcal{M}_{\boldsymbol{s}}\right) > \frac{\chi}{\Gamma\left(\epsilon^f\right)}, \tag{3.5}$$

where $\rho(\cdot)$ selects the leading eigenvalue.[8] *The associated cutoff efficiency $\epsilon_{\boldsymbol{s}}^b$ is given by*

$$\epsilon_{\boldsymbol{s}}^b = \Gamma^{-1}\left(\frac{\chi}{\rho\left(\operatorname{diag}(\boldsymbol{\theta}_{\boldsymbol{s}}^b) \mathcal{M}_{\boldsymbol{s}}\right)}\right) > \epsilon_f. \tag{3.6}$$

Condition (3.5) provides the necessary and sufficient condition for the $\boldsymbol{s}$-bubbly equilibrium to exist. The basic logic is that for a bubble to exist, the corresponding efficiency cutoff has to be high enough to exceed the one in the bubbleless equilibrium. This verifies the role of bubbles to help overcome financial frictions and improves allocation efficiency. On one hand, the leading eigenvalue is increasing in the equity pledgeability $\boldsymbol{\theta}_{\boldsymbol{s}}^b$, and bubbles are more likely to arise with larger $\boldsymbol{\theta}_s^b$. On the other hand, the efficiency cutoff in the bubbleless equilibrium is increasing in capital pledgeability $\boldsymbol{\theta}^k$ increases, and bubbles are less likely to arise with larger $\boldsymbol{\theta}_s^k$. These two forces underscore the distinct roles played by equity versus capital pledgeability: as can seen from the borrowing limit (2.15), a higher θ_j^k makes capital more valuable as collateral, which reducing the needs for bubbles; a higher θ_j^b makes bubbles more valuable in facilitating borrowing, which boost the call for bubbles.

Though being precise, condition (3.5) does not directly reveal which type of the financial linkage matrix $\mathcal{M}$ helps to generate bubbles. The following corollary provides a more intuitive estimate on the efficiency cutoff. Broadly speaking, more connected group of sectors is more likely to allow sectoral bubbles to exist. A sector with almost no connection with other sectors eliminates the possibility to

[8] With $\boldsymbol{\theta}_{\boldsymbol{s}}^b$ and $\mathcal{M}_{\boldsymbol{s}}$ being positive matrices, the eigenvalues are all positive.

have bubbles.

Corollary 2. *The dominant eigenvalue of* $\rho\left(\text{diag}\left(\boldsymbol{\theta}_{\boldsymbol{s}}^{b}\right)\mathcal{M}_{\boldsymbol{s}}\right)$ *is bounded by*

$$\rho\left(\text{diag}\left(\boldsymbol{\theta}_{\boldsymbol{s}}^{b}\right)\mathcal{M}_{\boldsymbol{s}}\right) \in \left[\min_{i\in\boldsymbol{s}}\sum_{j\in\boldsymbol{s}}\theta_i^b\mathcal{M}_{ij}, \max_{i\in\boldsymbol{s}}\sum_{j\in\boldsymbol{s}}\theta_i^b\mathcal{M}_{ij}\right], \tag{3.7}$$

and the $\boldsymbol{s}$*-bubbly equilibrium exists if* $\min_{i\in\boldsymbol{s}}\sum_{j\in\boldsymbol{s}}\theta_i^b\phi_{ij} > \frac{\chi}{\Gamma(\epsilon^f)}$.

Corollary 3 (Stock Pledgeability)**.** *Consider a relaxation of stock pledgeability* $\widetilde{\boldsymbol{\theta}}_{\boldsymbol{s}}^{b}$*, that is,* $\widetilde{\theta}_i^b \geq \theta_i^b$ *for all* $i \in \boldsymbol{s}$ *and* $\widetilde{\theta}_j^b > \theta_j^b$ *for some* $j \in \boldsymbol{s}$*. If the* $\boldsymbol{s}$*-bubbly equilibrium exists under* $\boldsymbol{\theta}_{\boldsymbol{s}}^{b}$*, then it also exists under* $\widetilde{\boldsymbol{\theta}}_{\boldsymbol{s}}^{b}$*, but not vice versa.*

So far we have focused on a particular $\boldsymbol{s}$-bubbly equilibrium. Another question of interest is that among many different groups of sectors, which groups are more likely to sustain sectoral bubbles? The following corollary answers this question.

Corollary 4. ***(Pecking order of efficiency on sectoral bubbles)*** *For any* $\boldsymbol{s}\subset\boldsymbol{s}'\subset\mathcal{P}(\mathbf{S})$*, if the* $\boldsymbol{s}$*-bubbly equilibrium exists, then* $\mathbf{s}'$*-bubbly equilibrium can also be sustained with* $\epsilon_{\boldsymbol{s}}^b < \epsilon_{\boldsymbol{s}'}^b$*.*

That is, if a small group can sustain bubbles, a strict larger group can also sustain bubbles with a higher efficiency cutoff. Particularly, if one is wondering whether bubbles can exist at all, then it is sufficient to compare the leading eigenvalue of the matrix $\text{diag}\left(\boldsymbol{\theta}^b\right)\mathcal{M}$ associated with the entire economy with $\frac{\chi}{\Gamma(\epsilon^f)}$.[9] In Section 4, we visit the reverse question that whether bubbles can be contagious in the sense the burst of the bubble in section j may destroy the bubbles in its connected sectors.

Moreover, the above corollary implies that the upper bound of effficiency is given by $\epsilon_{\mathbf{S}}^b$, which is in turn determined by

$$\rho\left(\text{diag}\left(\boldsymbol{\theta}_{\mathbf{S}}^{b}\right)\mathcal{M}_{\mathbf{S}}\right) = \frac{\chi}{\Gamma\left(\epsilon_{\mathbf{S}}^b\right)}. \tag{3.8}$$

Since $\epsilon_{\mathbf{S}}^b$ denotes the upper bound of efficiency of bubbly equilibrium, we could devote more discussion on $\epsilon_{\mathbf{S}}^b$. In particular, if $\theta_j^b = \theta^b$ for all j, then the above equation can be further simplified as $\theta^b = \frac{\chi}{\Gamma(\epsilon_{\mathbf{S}}^b)}$. Thus $\epsilon_{\mathbf{S}}^b = \Gamma^{-1}\left(\frac{\chi}{\theta^b}\right)$, which strictly increases with θ^b.

3.2 Allocation

In this subsection, we illustrate how the equilibrium allocation depends on the efficiency cutoff. Without financial friction, only firms that receive the highest shock invest. When financial frictions

[9] If $\theta_j^b = 1$ for all $j \in \mathbf{S}$, then $\Gamma(\epsilon^*) = \chi$. This is exactly the bubbly solution in the one-sector model developed by Wang and Wen (AEJ Macro 2012) and a series of works by Miao and Wang. That is, their results are special case of our result. Although we are talking about stock bubbles while most of the aforementioned works are about trading asset bubbles, when $\theta^b = 1$, it means bubbly assets can be sold out with full probability (no resaleability frictions), and thus $\Gamma(\epsilon^*) = \chi$.

are present, it is natural to conclude that a higher efficiency cutoff brings the outcomes closer to the frictionless benchmark monotonically. However, the equilibrium allocation actually depends on the efficiency cutoff in a more subtle way, as characterized below.

Proposition 3.3 (Output)**.** *Given the efficiency cutoff ϵ^*, the aggregate output in both bubbleless and bubbly equilirium is given by*

$$\log Y = \frac{\alpha}{1-\alpha}\left\{\log \epsilon^* + \log(1+\Gamma(\epsilon^*)) - \sum_{j=1}^{S} \varphi_j \log\left(\chi + \delta - \theta_j^k \Gamma(\epsilon^*)\right)\right\} + \texttt{constant}. \tag{3.9}$$

Condition (3.9) shows that the efficiency cutoff affects the aggregate output in several different channels. First, the direct effect of a higher ϵ^* is an improvement of the allocation efficiency, which facilitates capital accumulation and output expansion. Second, recall from Lemma 1, a higher ϵ^* implies a lower a higher risk free rate. What behind this change is a weakened precautionary saving motive due to a reduction of uncertainty, and it discourages capital accumulation. Thirdly, higher efficiency cutoff also reduces the collateral value of capital, which makes capital less attractive. These competing forces paint a mixed picture for how output is varying with the efficiency cutoff.[10]

Proposition 3.2 provide the existence condition for a s-bubbly equilibrium. The following proposition further characterizes the size of bubbles in each sector.

Proposition 3.4 (Sectoral Bubble)**.** *In a $\boldsymbol{s}$-bubbly equilibrium, the vector of bubbles $\{B_j\}$ for $j \in \boldsymbol{s}$ is a leading eigenvector of matrix* $\operatorname{diag}\left(\boldsymbol{\theta}_{\boldsymbol{s}}^b\right)\mathcal{M}_{\boldsymbol{s}}$ *which satisfies*

$$\frac{\sum_{j\in \boldsymbol{s}} \theta_j^b \mathcal{M}_{ij} B_j}{Y} = \alpha\Phi(\epsilon_{\boldsymbol{s}}^b). \tag{3.10}$$

Proposition 3.4 reveals the intimate relationship between the existence condition and the construction of sectoral bubbles, where the former is one of the leading eigenvector of matrix $\operatorname{diag}\left(\boldsymbol{\theta}_{\boldsymbol{s}}^b\right)\mathcal{M}_{\boldsymbol{s}}$, and the latter hinges on its leading eigenvalue. Condition 3.10 then selects the unique eigenvector for the sectoral bubbles, the right-hand side of which is reminiscent of the one in condition (3.1). In a bubbly equilibrium, the efficiency cutoff is necessarily higher than that in a bubbleless equilibrium, the difference in ϵ^f and $\epsilon_{\boldsymbol{s}}^b$ identifies the required size of bubbles to help finance investment demand for more efficient firms.

4. Bubbly Contagion

Previous analysis has implied that with financial linkages, sectors are interconnected. In this section we explore a particular type of connection — bubbly contagion. By contagion, we mean that the

[10] Our numerical analysis has found an inverse-U shape relationship where the latter two channels dominate when ϵ^* is relatively small, and the first channel dominates when ϵ^* is relatively large.

existence of bubbles in a sector depends on the existence of bubbles in another sector or another group of sectors. Alternatively, the burst of the bubble in one sector or a group of sectors pricks the bubbles in other sectors as well. The following corollary provide a mechanical way to examine whether an economy exhibits bubbly contagion.

Corollary 5 (Contagious Bubbles)**.** *Bubbles are contagious if there exits sector collections $\boldsymbol{s} \subset \boldsymbol{s}'$ such that the bubbly equilibrium exists for $\boldsymbol{s}'$ but not for $\boldsymbol{s}$, that is,*

$$\rho\left[\text{diag}\left(\boldsymbol{\theta}^b_{\boldsymbol{s}'}\right)\mathcal{M}_{\boldsymbol{s}'}\right] > \frac{\chi}{\Gamma(\epsilon^f)} > \rho\left[\text{diag}\left(\boldsymbol{\theta}^b_{\boldsymbol{s}}\right)\mathcal{M}_{\boldsymbol{s}}\right]. \tag{4.1}$$

In a bubbly contagious economy, some sectors can be vital: the burst of bubbles in these sectors necessarily prevents the existence bubbles in some other sectors, but not the other way around.

Definition 2. *A collection of sectors $\boldsymbol{s}$ is vital if: (1) $\boldsymbol{s}$-bubbly equilibrium exists; (2) for any $\underline{\boldsymbol{s}} \subset \boldsymbol{s}$, no $\underline{\boldsymbol{s}}$-bubbly equilibrium exists; (3) for some $\overline{\boldsymbol{s}} \supset \boldsymbol{s}$, $\overline{\boldsymbol{s}} \setminus \boldsymbol{s}$-bubbly equilibrium does not exist.*

A crash of a vital sector can drag down the whole stock market. Therefore, additional attention should be paid when altering the financial condition in a vital sector. For example, when the equity constraint is tightened in a non-vital sector, it may only have limited effects on the overall efficiency cutoff, but bubbles still survive. In contrast, when the equity constraint is righted in a vital sector, bubbles in all sectors may burst and a dramatic drop of output may follow.

The analysis of bubbly contagion hinges on the properties of the financial linkages $\mathcal{M}$ at hand. To illustrate the basic intuition, we provide a detailed characterization of the bubbly contagion in a two-sector example below, and we conduct a dynamic quantitative analysis in Section 5.

Two-Sector Example. In this subsection, we explore the bubbly contagion in a two-sector model. This special case allows some clean analytical results, and the intuition extends to more complicated models. To highlight the role of the financial linkage matrix

$$\boldsymbol{\mathcal{M}} = \begin{bmatrix} \mathcal{M}_{11} & 1 - \mathcal{M}_{11} \\ 1 - \mathcal{M}_{22} & \mathcal{M}_{22} \end{bmatrix},$$

we focus on the case with homogeneous capital and equity pledgeability, i.e., $\theta^b_j = \theta^b$ and $\theta^k_j = \theta^k$.

Proposition 4.1. *Bubbles in the two sectors can coexist iff $\phi^* \equiv \frac{\chi}{\Gamma(\epsilon^f)}\frac{1}{\theta^b} < 1$.*

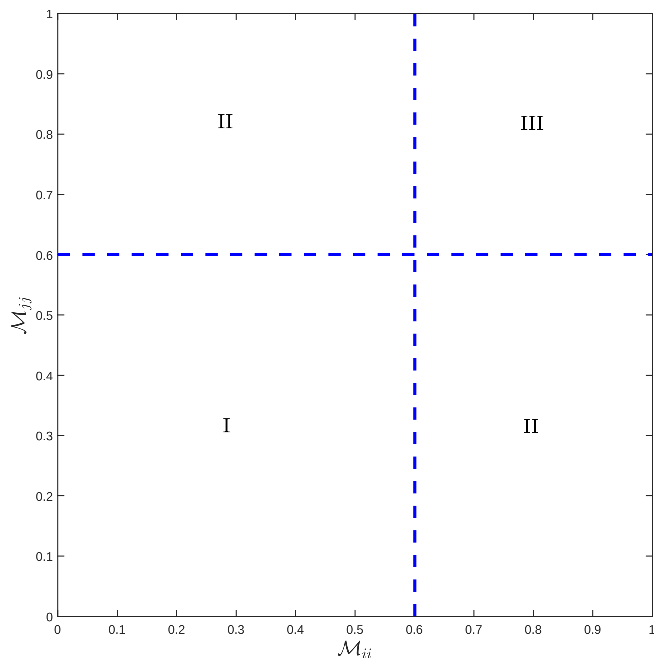
I (no contagion) if $\mathcal{M}_{ii} > \phi^*$ for $i \in \{1, 2\}$, bubbles are not contagious.

II (one-way contagion) if $\mathcal{M}_{ii} < \phi^*$ and $\mathcal{M}_{jj} > \phi^*$, then the burst of bubble in sector j will be contagious to sector i, but not the another way around.

III (*two-way contagion*)*if* $\mathcal{M}_{ii} < \phi^*$ *for* $i \in \{1, 2\}$, *then the bust of bubble in any sector is contagious to the other one.*

Proposition 4.1 implies that the resilience of a sector's bubble to contagion is increasing in its self-dependence in the financial linkage matrix. As shown in Figure 2, when M_{ii} is low, sector i's collateral value relies more on sector j's performance, and a bust of sector j's bubble spillovers to sector i. When M_{ii} is high, the bubble in sector i can survive independent of the other sector's condition. Extrapolating the intuition to a more general environment, bubbly contagion is more likely to happen when the off-diagonal elements are dense.

Figure 2: Interconnectedness and Contagion



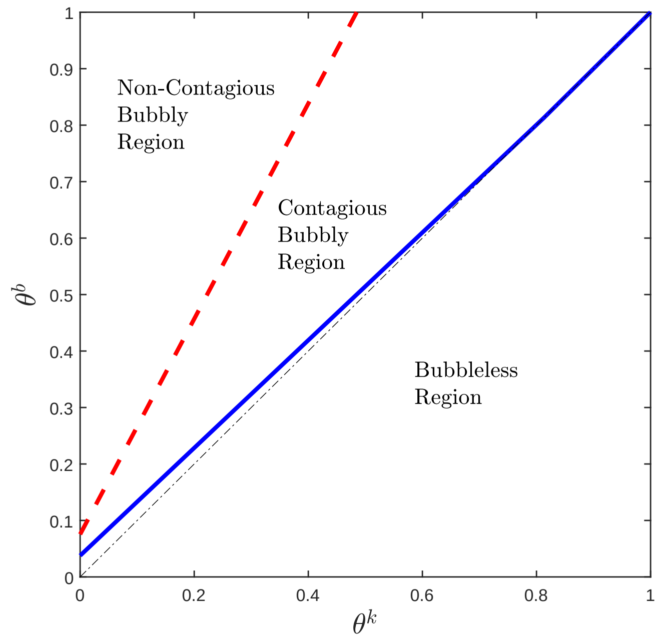
Notes: This figure reports how sector connectedness translates into bubbly contagion.See Proposition 4.1 for details.

Note that in region II, when $M_{ii} < \phi^*$ and $M_{jj} > \phi^*$, then sector j is the vital sector. The bust of bubble in sector j eliminates the possibility of a bubbly equilibrium, but the opposite is not true for sector i. The vital sector has a bigger impact on the whole economy and should be paid with additional attention.

Proposition 4.1 divides the $(\mathcal{M}_{ii}, \mathcal{M}_{jj})$ space into three segments for a fixed combination of θ^b and θ^k. In the space of M_{ii} and M_{jj}, the area of contagious regions (II and III) is $C = 1 - (\phi^*)^2$. Given that a bubbly equilibrium exists, for a fixed combination of $\mathcal{M}_{ii}$ and $\mathcal{M}_{jj}$, the larger C is, the more likely bubbles are contagious. Therefore, C can be interpreted as contagion risk.

Corollary 6. *Assuming* $\phi^* < 1$, *the contagion risk* $C = 1 - (\phi^*)^2$ *increases with a smaller overall borrowing constraint* θ^b, *or a larger capital-specific pledgeability* θ^k.

Figure 3: Pledgeability and Contagion



Notes: This figure reports the parameter region of pledgeability for the existence of bubbles as well as the contagion region of bubble bursts.It is an illustration of Corollary x.x. The distribution is assumed to be Pareto, $F(\epsilon) = 1 - \left(\epsilon/\underline{\epsilon}\right)^{-\eta}$ with $\underline{\epsilon} = 1 - 1/\eta$ for normalization. We set $\eta = 4.2$, $M_{11} = M_{22} = 0.5$, $\beta = 0.99$, and $\delta = 0.04$.

Figure 3 helps to visualize Corollary 6, where the parameter space is portioned into three regions for a fixed financial linkage matrix. The intuition behind Corollary 6 is similar to that of the bubbly equilibrium existence condition. With a lower θ^b, equity is less valuable as collateral, and the existence condition of bubble in a sector becomes more stringent. With a higher θ^k, capital is more powerful in helping overcome the financial friction and the economy relies less on bubbles to provide liquidity, and thus sectoral bubbles will be more fragile to each other.

5. Transition Dynamics with Contagious Bubble

In this section, we use the properties of bubbly equilibrium developed in this paper to shed light on the differential impacts of an unexpected bubble bust in a vital sector versus a non-vital sector on aggregate outcomes. In particular, we discipline the financial linkage matrix $\mathcal{M}$ based on U.S. mergers and acquisitions transaction data and explore the entire transition dynamics after a bubble bust, which helps provide a quantitative assessment on the importance of bubble contagion.

Calibration. Most parameters are standard in the literature. We choose a period to be a year, and we set the discount rate $1/\beta - 1$ to be 4%. We choose the capital share α to be 64%. The depreciation rate δ is set to be 8%, which correspond to the average sector-specific depreciation rates obtained from the BEA for 2001.

Table 1: Targets and Associated Parameters

Parameter and Associated Target	Value
Discount factor, χ	0.04
Capital share, α	0.64
Depreciation rate, δ	0.08
Investment efficiency distribution, η	2.5
Equity pledgeability, θ^b	corporate debt to output ratio
Capital pledgeability, θ^k	10%
Final goods expenditure share, φ_j	WIOD
Financial linkage, $\mathcal{M}_{ij}$	M&A SDC

Turn to the parameters that are more specific to our model environment. When we calibrate the parameters, we consider the bubbly equilibrium and we allow sectoral bubbles to exist whenever it is possible.[11] The M&A Database in SDC contains mergers and acquisitions transaction data between firms from 1999 to 2018. We aggregate firms into three groups: finance, manufacturing, and services, which correspond to the sectors in the model. The mapping from firms SIC code to the three sectors is listed in Table 1. For each sector j, we compute the total transactions value with firms in sector j as targets, and the transaction value associated with acquiring firms in sector i. We interpret the ratio of value acquired by sector i to the total transaction value from sector j as the probability $\mathcal{M}_{ji}$. This leads to the following transition matrix

[11] In our model environment, the bubbly equilibrium is the stable one and the bubbleless equilibrium is unstable in the sense that a local disturbance to the perceived bubble value in the bubbleless equilibrium will make the economy converge to the bubbly one.

$$\mathcal{M} = \begin{bmatrix} 0.959 & 0.021 & 0.020 \\ 0.172 & 0.787 & 0.041 \\ 0.173 & 0.059 & 0.767 \end{bmatrix},$$

where the order of the sectors in $\mathcal{M}$ are finance, service, and manufacturing.

The value added shares φ_j is obtained by aggregating value added in 33 industries obtained from OECD Inter-Country Input-Output (ICIO) Tables into the aforementioned three sectors. We use the information for the latest available year, 2015. Then we have $\varphi = [0.10, 0.23, 0.68]$. For θ_k, we set it to be 10%, consistent with the estimate from Lian and Ma (2018), and we calibrate $\theta^b = 0.42$ such that the debt-to-GDP ratio in the model is 3.36 as in Jermann and Quadrini (2012). Finally, we assume that the across-sectional investment technology shock follows a Pareto distribution, $F(\epsilon) = 1 - (\epsilon/\underline{\epsilon})^{-\eta}$ with $\underline{\epsilon} = 1 - 1/\eta$ for normalization, and we set the the parameter η to be 2.5 so that the proportion of lumpy investment is 5%. In Appendix, we conduct robustness analysis when varying these parameters.

Unexpected Bubble Burst. The economy is initially at its steady state where bubbles exist in all the three sectors. Our exercises explore the consequences of an unexpected bubble burst in one of the sectors, and we assume that the bubbles in remaining sector continue to exist whenever it is possible. That is, we are looking for the most favorable outcomes.

Figure 4 displays the evolution of sectoral bubbles. When the bubble in the finance sector bursts, the bubbles in the other two sectors burst as well. In contrast, when the bubble in the service sector or in the manufacturing sector bursts, the switch of the nature of equilibrium from bubbly one to a bubbleless one is limited to its own sector without spilling over to other sectors. Therefore, we conclude in our model economy, there is a one-way bubble contagion, and the finance sector is the vital sector.

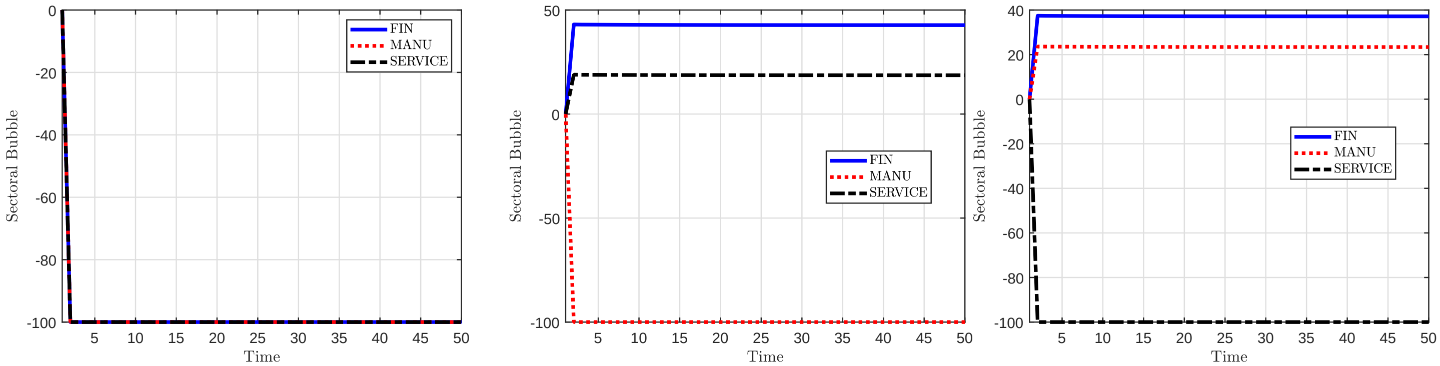
What explains this result is the particular structure of the financial linkage matrix $\mathcal{M}$. As characterized in Section 3 and 4, the diagonal elements is responsible for whether a sector's bubble can survive by itself, and the off-diagonal elements determines which sector is vital for other sector's bubble existence. In our economy, the within-sector reestablishment probability in the finance sector, $\mathcal{M}_{11}$, is large enough, and the other two sectors' depend on the finance sector in the default scenario, captured by sizable off-diagonal elements $\mathcal{M}_{j1}$. Note that in terms of value added, the finance sector does not dominates the other two sectors, which illustrates the distinct role played by the financial linkage matrix.

In terms of the transitional dynamics, the sizes of sectoral bubbles jump immediately after the unexpected burst. This is because the bubble is a purely forward-looking variable, the law of motion of which follows condition (2.14). Interestingly, when the bubble in service or manufacturing sector bursts, the bubble size in the finance sector actually increases.

Different from bubbles, the outputs in different sectors decline gradually towards their new steady

Figure 4: Transition of Sector Bubbles

(a) finance sector bust (b) manufacturing sector bust (c) service sector bust



Notes: TBA

states. This is because both the TFP and labor are constant in the model, and only the capital goes down over time. As the bubbles bursts, the borrowing constraints are tightened, and those firms that are efficient in investing cannot borrow as much as they could. The average investment efficiency declines as time goes on, as shown in the right panel of Figure 5, eventually leads to a lower steady-state capital level. The burst of sectoral bubbles therefore acts as an adverse financial shock. Different from standard shocks to firms' collateral constraints, the recession induced by bubble burst is endogenous as it does not require an exogenous change of the primitives. Also, the severity of the outcomes depends on whether the triggering sector is a vital sector or not. If originated from the finance sector, the eventual loss of GDP is 5.2%, while if originated from the manufacturing or the service sector, the eventual loss is much milder with only 0.2%.

Figure 5: Output Loss and Efficiency Cutoff

(a) output (b) efficiency cutoff

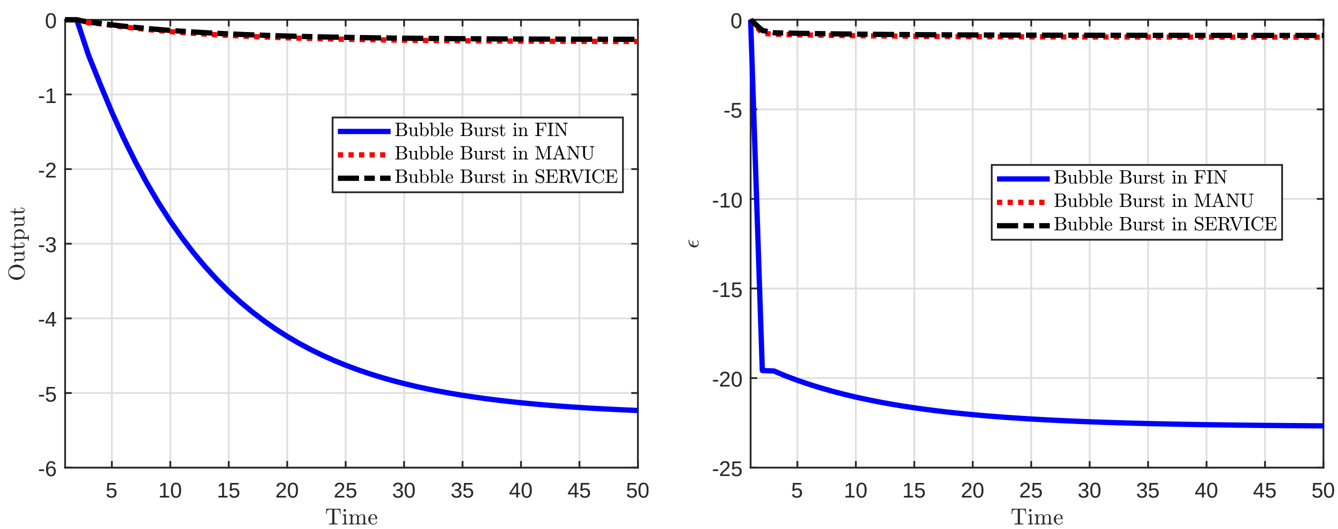
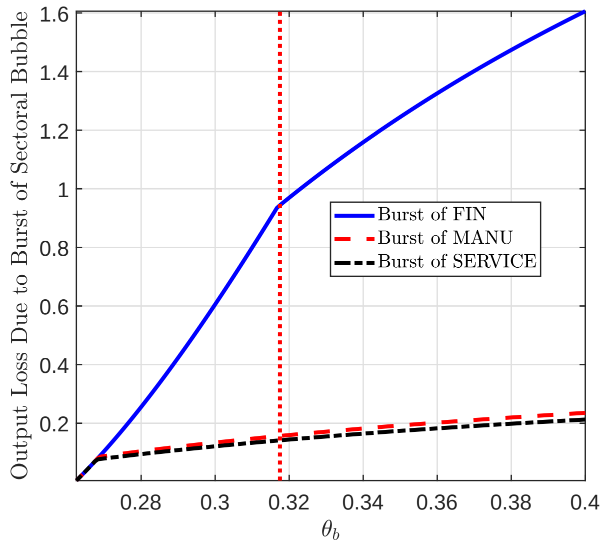


Figure 6: Output Loss Due to Burst of Different Sectoral Bubbles



This exercise illustrates that when the confidence on the performance of a vital sector is weak, it can lead to a sequence of chained effects on other sectors and drag the entire economy into a recession. When considering macro-prudential policies, special attention should be paid to such sectors.

Role of θ^b in Contagion. One of the key parameters in determining whether there exists bubbly contagion is the equity pledgeability θ^b. Figure 6 displays how the output loss due to a sector's bubble burst depends on θ^b. Overall, the loss in output is increasing in θ^b. This is due to that when the economy burst into the bubbleless equilibrium, the output is independent of θ^b, but the output level is increasing in θ^b in the bubbly equilibrium.

The vertical dashed lines partition the figure into three regions. When θ^b is relatively small, the economy lives in a two-way bubbly contagion region. Any sector's bubble burst will eliminate the bubbles in other sectors. Therefore, we see the effects on total output are similar across sectors. When θ^b in an intermediate range, the economy is in the one-way bubbly contagion region with the finance sector as the vital sector. In this region, the burst of the bubble in the finance sector has a much larger impact on the economy than other sectors. Finally, when when θ^b is relatively large, all the sectoral bubble can survive independent of other sectors and there does not exist bubbly contagion anymore. As a result, we see flatter lines in the relationship between output loss and θ^b.

6. Conclusion

This paper proposes a framework to study contagious stock bubbles in a multi-sector production economy with heterogeneous investment technologies. Due to financial frictions, stock bubbles arise endogenously that help inject additional liquidity. Due to financial linkages, the existence of bubbles in different sectors may be interdependent. Theoretically, we characterize the entire set of bubbly equilibria, and provide the condition under which the burst of bubbles in one sector spikes bubbles in other sectors. Quantitatively, we show that due to an unexpected burst of bubbles, it can generate severe a recession only when the burst happens in a vital sector and transmits to the rest of the economy.

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Appendix

Proofs

Proof of Proposition 2.1 and Corollary 1:. Since production functions are constant return to scale at firm level, we guess that the value function of firm ι in sector j adopts a linear form such that

$$V_{jt}\left(k_{jt}(\iota), l_{jt}(\iota), \epsilon_{jt}(\iota)\right) = v_{jt}^{k}\left(\epsilon_{jt}(\iota)\right) k_{jt}(\iota) - v_{jt}^{l}\left(\epsilon_{jt}(\iota)\right) l_{jt}(\iota) + b_{jt}\left(\epsilon_{jt}(\iota)\right). \tag{.1}$$

The additional term is $b_{jt}\left(\epsilon_{jt}(\iota)\right)$. We interpret $b_{jt}\left(\epsilon_{jt}(\iota)\right) > 0$ as stock bubble. Since $b_{jt}\left(\epsilon_{jt}(\iota)\right)$ is indexed with sector j, we call it *sectoral bubble*. Using (.1), we can rewrite equation (**??**), i.e., the recursive problem of firm-ι in sector j, as below

$$\begin{aligned} V_{jt}\left(k_{jt}(\iota), l_{jt}(\iota), \epsilon_{jt}(\iota)\right) &= \max_{i_{jt}(\iota), l_{j,t+1}(\iota)} \{R_{jt}k_{jt}(\iota) - i_{jt}(\iota) - l_{jt}(\iota) + l_{j,t+1}(\iota)/R_{ft} \\ &\quad + Q_{jt}\left[(1-\delta)k_{jt}(\iota) + \epsilon_{jt}(\iota) i_{jt}(\iota)\right] + B_{jt} - Q_{jt}^{L} l_{j,t+1}(\iota)\}, \end{aligned} \tag{.2}$$

subject to

$$0 \le i_{jt}(\iota) \le R_t k_{jt}(\iota) - l_{jt}(\iota) + l_{j,t+1}(\iota)/R_{ft}. \tag{.3}$$

Define

$$Q_{jt} = \mathbb{E}_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} v_{j,t+1}^{k}\left(\epsilon_{j,t+1}(\iota)\right) dF\left(\epsilon_{j,t+1}(\iota)\right), \tag{.4}$$

$$B_{jt} = \mathbb{E}_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} b_{j,t+1}\left(\epsilon_{j,t+1}(\iota)\right) dF\left(\epsilon_{j,t+1}(\iota)\right), \tag{.5}$$

$$Q_{jt}^{L} = \mathbb{E}_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} v_{j,t+1}^{l}\left(\epsilon_{j,t+1}(\iota)\right) dF\left(\epsilon_{j,t+1}(\iota)\right). \tag{.6}$$

Then the borrowing constraint (2.10) can be rewritten as

$$l_{j,t+1}(\iota)/R_{ft} \le \xi_j \left(\sigma_j \overline{Q}_{jt} k_{jt}(\iota) + \overline{B}_{jt}\right) = \theta_j^{k} \overline{Q}_{jt} k_{jt}(\iota) + \theta_j^{b} \overline{B}_{jt}, \tag{.7}$$

where

$$\theta_j^{k} \equiv \xi_j \sigma_j,\ \theta_j^{b} \equiv \xi_j,\ \overline{Q}_{jt} \equiv \sum_{i \in \mathbf{S}} \phi_{ij} Q_{it},\ \overline{B}_{jt} = \sum_{i \in \mathbf{S}} \phi_{ij} B_{it}. \tag{.8}$$

Case A: $\epsilon_{jt}(\iota) < \epsilon_{jt}^{*} \equiv 1/Q_{jt}$. For this case, $i_{jt}(\iota) = 0$. Then optimizing over $l_{j,t+1}(\iota)$ yields $Q_{jt}^{L} = 1/R_{ft}$

Case B: $\epsilon_{jt}(\iota) > \epsilon_{jt}^{*}$. For this case, optimal investment hits the upper bound of the above constraint on investment. Moreover, the credit constraint (.7) is binding such that

$$l_{j,t+1}(\iota)/R^{j}_{ft} = \theta^{k}_{j}\overline{Q}_{jt}k_{jt}(\iota) + \theta^{b}_{j}\overline{B}_{jt}, \tag{.9}$$

Therefore

$$i_{jt}(\iota) = \begin{cases} R_{jt}k_{jt}(\iota) + \theta^{k}_{j}\overline{Q}_{jt}k_{jt}(\iota) + \theta^{b}_{j}\overline{B}_{jt} - l_{jt}(\iota), & \text{if } \epsilon_{jt}(\iota) \geq \epsilon^{*}_{jt} \\ 0, & \text{if } \epsilon_{jt}(\iota) < \epsilon^{*}_{jt} \end{cases}. \tag{.10}$$

Substituting the decision rules on investment and $Q^{L}_{jt} = 1/R^{j}_{ft}$ into (.2) yields

$$\begin{aligned} V_{jt}\left(k_{jt}(\iota), l_{jt}(\iota), \epsilon_{jt}(\iota)\right) &= R_{jt}k_{jt}(\iota) + Q_{jt}(1-\delta)k_{jt}(\iota) + B_{jt} - l_{jt}(\iota) &(.11)\\ &+ \max\left(\epsilon_{jt}(\iota)/\epsilon^{*}_{jt} - 1, 0\right)\cdot\left(R_{jt}k_{jt}(\iota) - l_{jt}(\iota) + l_{j,t+1}(\iota)/R_{ft}\right). &(.12)\end{aligned}$$

and $l_{j,t+1}(\iota)/R_{ft}$ is given by equation (.9). Then for any $\epsilon_{jt}(\iota)$, matching coefficients on both sides of (.11) with the conjecture in (**??**) yields

$$\begin{aligned} v^{k}_{jt}\left(\epsilon_{jt}(\iota)\right) &= R_{jt} + (1-\delta)Q_{jt} + \max\left(\epsilon_{jt}(\iota)/\epsilon^{*}_{jt} - 1, 0\right)\cdot\left(R_{jt} + \theta^{k}_{j}\overline{Q}_{jt}\right), &(.13)\\ b_{jt}\left(\epsilon_{jt}(\iota)\right) &= B_{jt} + \theta^{b}_{j}\max\left(\epsilon_{jt}(\iota)/\epsilon^{*}_{jt} - 1, 0\right)\overline{B}_{jt}, &(.14)\\ v^{l}_{jt}\left(\epsilon_{jt}(\iota)\right) &= 1 + \max\left(\epsilon_{jt}(\iota)/\epsilon^{*}_{jt} - 1, 0\right). &(.15)\end{aligned}$$

In turn, using the definition of Q_{jt}, B_{jt}, B_{t}, and Q^{L}_{jt} in (.4) - (.6) immediately implies that

$$\begin{aligned} Q_{jt} &= \mathbb{E}_{t}\frac{\beta\Lambda_{t+1}}{\Lambda_{t}}\left[R_{j,t+1}\left(1+\Gamma_{j,t+1}\right) + (1-\delta)Q_{j,t+1} + \theta^{k}_{j}\Gamma_{j,t+1}\overline{Q}_{j,t+1}\right], &(.16)\\ B_{jt} &= \mathbb{E}_{t}\frac{\beta\Lambda_{t+1}}{\Lambda_{t}}\left(B_{j,t+1} + \theta^{b}_{j}\Gamma_{j,t+1}\overline{B}_{j,t+1}\right), &(.17)\\ \frac{1}{R_{ft}} &= \mathbb{E}_{t}\frac{\beta\Lambda_{t+1}}{\Lambda_{t}}\left(1+\Gamma_{j,t+1}\right), &(.18)\end{aligned}$$

where

$$\Gamma_{jt} \equiv \mathbb{E}\max\left(\epsilon_{j}/\epsilon^{*}_{jt} - 1, 0\right) = \int_{\epsilon^{*}_{jt}}^{\overline{\epsilon}_{j}}\left(\epsilon_{j}/\epsilon^{*}_{jt} - 1\right)dF_{j}. \tag{.19}$$

Moreover, integrating the individual investment decision and the individual law of motion in sector j immediately yields

$$I_{jt} \quad = \quad \left[R_{jt}K_{jt} + \theta_j^k \overline{Q}_{jt}K_{jt} + \theta_j^b \overline{B}_{jt} - L_{jt}\right] \cdot \left[1 - F\left(\epsilon_{jt}^*\right)\right], \tag{.20}$$

$$K_{j,t+1} \quad = \quad (1-\delta) K_{jt} + \mathbb{E}_j \left(\epsilon_j | \epsilon_j \geq \epsilon_{jt}^*\right) I_{jt}. \tag{.21}$$

Proof of Corollary ??:. Combining the definition of $\overline{B}_{jt}$ in equation (**??**) and the Euler equation of B_{jt} in (2.14) immediately yields that

$$\overline{B}_{jt} = \mathbb{E}_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} \left[\overline{B}_{j,t+1} + \sum_{i \in \mathbf{S}} \phi_{ij} \theta_i^b \Gamma_{i,t+1} \overline{B}_{i,t+1}\right]. \tag{.22}$$

Proof of Proposition ??:. For any firm-$\iota \in [0,1]$ in sector $i \in \mathbf{S}$, the FOC on $\left(n_{it}(\iota), s_{ijt}(\iota)\right)$ on (2.4) is given by

$$W_t \quad = \quad \alpha_i^l \frac{P_{it} o_{it}(\iota)}{n_{it}(\iota)} = \alpha_i^l \frac{P_{it} O_{it}}{N_{it}}, \tag{.23}$$

$$P_{jt} \quad = \quad \alpha_i^s \omega_{ij} \frac{P_{it} o_{it}(\iota)}{s_{ijt}(\iota)} = \alpha_i^s \omega_{ij} \frac{P_{it} O_{it}}{S_{ijt}}, \tag{.24}$$

Besides, $O_{it} \equiv \int_0^1 o_{it}(\iota) d\iota$, and similar definition for L_{it}, S_{ijt} and K_{it}. In turn, equation (2.4) implies that

$$R_{it} = \alpha_i^k \frac{P_{it} o_{it}(\iota)}{k_{it}(\iota)} = \alpha_i^k \frac{P_{it} O_{it}}{K_{it}}. \tag{.25}$$

The resource constraint of sector j is given by

$$O_{jt} = X_{jt} + \sum_{i \in \mathbf{S}} S_{ijt}, \tag{.26}$$

which can be rewritten as

$$\frac{P_{jt} O_{jt}}{Y_t} = \frac{P_{jt} X_{jt}}{Y_t} + \sum_{i \in \mathbf{S}} \frac{P_{jt} S_{ijt}}{Y_t}. \tag{.27}$$

To recap, we have shown in equation (**??**) that

$$\frac{P_{jt} X_{jt}}{Y_t} = \varphi_j. \tag{.28}$$

Denote $\gamma_{jt} = P_{jt} O_{jt} / Y_t$. Then the resources constraint (.27) becomes

$$\gamma_{jt} = \varphi_j + \sum_{i \in \mathbf{S}} \alpha_i^s \omega_{ij} \widetilde{\gamma}_{it}, \tag{.29}$$

Then we have

$$\boldsymbol{\gamma}_t = \boldsymbol{\gamma} \equiv \left(\mathbf{I} - \mathbf{Diag}\left(\boldsymbol{\alpha}^s\right)\boldsymbol{\Omega}'\right)^{-1}\boldsymbol{\varphi}, \tag{.30}$$

and thus γ_{jt} is constant and is denoted as γ_j, where $\mathbf{I}$ denotes the $S \times 1$ unity matrix, $\boldsymbol{\gamma}$ denotes $S \times 1$ modified Dommar weight. After solving $\boldsymbol{\gamma}$, we know that, for all t, the expenditure ration of sector j is given by

$$\frac{P_{jt}O_{jt}}{Y_t} = \gamma_j. \tag{.31}$$

Besides, the demand of $\left(N_{it}, S_{ijt}, K_{it}\right)$ in (.23), (.24), and (.25) can be further written as

$$N_{it} = \alpha_i^l \gamma_i \frac{Y_t}{W_t}, \tag{.32}$$

$$S_{ijt} = \alpha_i^s \omega_{ij} \frac{\gamma_i}{\gamma_j} O_{jt}, \tag{.33}$$

$$K_{it} = \alpha_i^k \gamma_i \frac{Y_t}{R_{it}}. \tag{.34}$$

Then the clearing condition in the labor markets is given by

$$N_t = \sum_{i \in \mathbf{S}} N_{it} = \boldsymbol{\gamma}' \boldsymbol{\alpha}^l \frac{Y_t}{W_t}, \tag{.35}$$

which implies

$$W_t = \boldsymbol{\gamma}' \boldsymbol{\alpha}^l \frac{Y_t}{N_t}, \tag{.36}$$

In turn, equation (.32) can be rewritten as

$$N_{it} = \lambda_i N_t, \tag{.37}$$

where $\lambda_i = \frac{\gamma_i \alpha_i^l}{\boldsymbol{\gamma}' \boldsymbol{\alpha}^l}$.

Now we characterize the aggregation on Y_t. The sectoral technology in equation (2.3) implies that

$$\ln O_{it} = \ln A_{it} + \alpha_i^k \ln K_{it} + \alpha_i^l \ln N_{it} + \sum_{j \in \mathbf{S}} \alpha_i^s \omega_{ij} S_{ijt}. \tag{.38}$$

Substituting equation (.33) into (.38) with some algebraic manipulation yields

$$\ln O_{it} = \alpha_i^s \ln o_i + \ln A_{it} + \alpha_i^k \ln K_{it} + \alpha_i^l \ln N_{it} + \sum_{j \in \mathbf{S}} \alpha_i^s \omega_{ij} \ln O_{jt}, \tag{.39}$$

where

$$\ln o_i \equiv \sum_{j \in \mathbf{S}} \ln \left(\alpha_i^s \omega_{ij} \gamma_i / \gamma_j\right)^{\omega_{ij}}. \tag{.40}$$

Writing equation (.39) in a more compact way yields

$$\ln \mathbf{O}_t = \mathbf{Diag}(\boldsymbol{\alpha}^s)\ln \mathbf{o} + \ln \mathbf{A}_t + \mathbf{Diag}\left(\boldsymbol{\alpha}^k\right)\ln \mathbf{K}_t + \mathbf{Diag}\left(\boldsymbol{\alpha}^l\right)\ln \mathbf{N}_t + \mathbf{Diag}(\boldsymbol{\alpha}^s)\boldsymbol{\Omega}\ln \mathbf{O}_t. \quad (.41)$$

Therefore

$$\ln \mathbf{O}_t = (\mathbf{E} - \mathbf{Diag}(\boldsymbol{\alpha}^s)\boldsymbol{\Omega})^{-1}\left(\mathbf{Diag}(\boldsymbol{\alpha}^s)\ln \mathbf{o} + \ln \mathbf{A}_t + \mathbf{Diag}\left(\boldsymbol{\alpha}^k\right)\ln \mathbf{K}_t + \mathbf{Diag}\left(\boldsymbol{\alpha}^l\right)\ln \mathbf{N}_t\right). \quad (.42)$$

Meanwhile, combining equation (**??**) and (.31) yields

$$X_{jt} = \frac{\varphi_j}{\widetilde{\gamma}_j} O_{jt}. \quad (.43)$$

In turn,

$$\ln \mathbf{X}_t = \ln \mathbf{x} + \ln \mathbf{O}_t, \quad (.44)$$

where a typical element of $\mathbf{x}$ is $x_j = \varphi_j/\widetilde{\gamma}_j$. Consequently,

$$\ln Y_t = \boldsymbol{\varphi}' \ln \mathbf{X}_t = \boldsymbol{\varphi}'(\ln \mathbf{x} + \ln \mathbf{O}_t) = \ln \overline{A}_t + \boldsymbol{\gamma}'\boldsymbol{\alpha}^k \ln K_t + \boldsymbol{\gamma}'\boldsymbol{\alpha}^l \ln N_t, \quad (.45)$$

where

$$\boldsymbol{\gamma} \equiv (\mathbf{E} - \mathbf{Diag}(\boldsymbol{\alpha}^s)\boldsymbol{\Omega}')^{-1}\boldsymbol{\varphi}, \quad (.46)$$

and

$$\ln \overline{A}_t = \boldsymbol{\varphi}'\ln \mathbf{x} + \boldsymbol{\gamma}'\mathbf{Diag}(\boldsymbol{\alpha}^s)\ln \mathbf{o} + \boldsymbol{\gamma}'\ln \mathbf{A}_t + \boldsymbol{\gamma}'\mathbf{Diag}\left(\boldsymbol{\alpha}^k\right)\ln \boldsymbol{\kappa}_{jt} + \boldsymbol{\gamma}'\mathbf{Diag}\left(\boldsymbol{\alpha}^l\right)\ln \boldsymbol{\lambda}_{jt}. \quad (.47)$$

Proof.

$$\begin{aligned}
\boldsymbol{\gamma}'\boldsymbol{\alpha}^k + \boldsymbol{\gamma}'\boldsymbol{\alpha}^l &= \boldsymbol{\gamma}'\left(\boldsymbol{\alpha}^k + \boldsymbol{\alpha}^l\right) && (.48)\\
&= \boldsymbol{\varphi}'(\mathbf{E} - \mathbf{Diag}(\boldsymbol{\alpha}^s)\boldsymbol{\Omega})^{-1}(\mathbf{1} - \boldsymbol{\alpha}^s) && (.49)\\
&= \sum_{n=0}^{+\infty}\boldsymbol{\varphi}'(\mathbf{Diag}(\boldsymbol{\alpha}^s)\boldsymbol{\Omega})^n\mathbf{1} - \sum_{n=0}^{+\infty}\boldsymbol{\varphi}'(\mathbf{Diag}(\boldsymbol{\alpha}^s)\boldsymbol{\Omega})^n\boldsymbol{\alpha}^s && (.50)\\
&= \boldsymbol{\varphi}'\mathbf{1} + \sum_{n=1}^{+\infty}\boldsymbol{\varphi}'(\mathbf{Diag}(\boldsymbol{\alpha}^s)\boldsymbol{\Omega})^{n-1}(\mathbf{Diag}(\boldsymbol{\alpha}^s)\boldsymbol{\Omega}\mathbf{1}) - \sum_{n=0}^{+\infty}\boldsymbol{\varphi}'(\mathbf{Diag}(\boldsymbol{\alpha}^s)\boldsymbol{\Omega})^n\boldsymbol{\alpha}^s. && (.51)
\end{aligned}$$

Since $\sum_{j=1}^{S}\omega_{ij} = 1$ for all i, we easily know that

$$\mathbf{Diag}(\boldsymbol{\alpha}^s)\boldsymbol{\Omega}\mathbf{1} = \mathbf{Diag}(\boldsymbol{\alpha}^s)\mathbf{1} = \boldsymbol{\alpha}^s. \quad (.52)$$

Moreover, we have normalized that $\sum_{j=1}^{S}\varphi_j = \boldsymbol{\varphi}'\mathbf{1} = 1$. Consequently, we have $\boldsymbol{\gamma}'\boldsymbol{\alpha}^k + \boldsymbol{\gamma}'\boldsymbol{\alpha}^l = 1$. ∎

Denote $\alpha = \gamma' \boldsymbol{\alpha}^k$. Then the above lemma immediately implies that $\gamma' \boldsymbol{\alpha}^l = 1 - \alpha$. In turn, the aggregate output in equation (.45) can be rewritten as

$$Y_t = \overline{A}_t \left(\prod_{j \in \mathbf{S}} K_{jt}^{\gamma_j \alpha_j^k} \right) N_t^{1-\alpha} = \overline{A}_t \overline{K}_t^{\alpha} N_t^{1-\alpha}, \tag{.53}$$

where

$$\overline{A}_t \equiv \exp\left(\boldsymbol{\varphi}' \ln \mathbf{x} + \boldsymbol{\gamma}' \ln \mathbf{o} + \boldsymbol{\gamma}' \ln \mathbf{A}_t + \boldsymbol{\gamma}' \mathbf{Diag}\left(\boldsymbol{\alpha}^l\right) \ln \boldsymbol{\lambda} \right) = \prod_{j \in \mathbf{S}} x_j^{\varphi_j} o_j^{\gamma_j \alpha_j^s} \lambda_j^{\gamma_j \alpha_j^l} \cdot \prod_{j \in \mathbf{S}} A_{jt}^{\gamma_j}, \tag{.54}$$

$$\overline{K}_t \equiv \left[\prod_{j \in \mathbf{S}} K_{jt}^{\gamma_j \alpha_j^k} \right]^{\frac{1}{\alpha}}, \tag{.55}$$

with

$$x_j = \frac{\varphi_j}{\widetilde{\gamma}_j}, \tag{.56}$$

$$o_j \equiv \prod_{i \in \mathbf{S}} \left(\widetilde{\alpha}_j^s \omega_{ji} \widetilde{\gamma}_j / \widetilde{\gamma}_i \right)^{\omega_{ji}}. \tag{.57}$$

where $\overline{Z} \equiv \prod_{j \in \mathbf{S}} \left(\varphi_j / \widetilde{\gamma}_j \right)^{\varphi_j} \lambda_j^{\gamma_j \alpha_j^l} o_j^{\gamma_j \alpha_j^s}$. Moreover, we know that

$$\kappa_{jt} \equiv \frac{K_{jt}}{\sum_{i \in \mathbf{S}} K_{it}} = \frac{K_{jt}}{K_t}, \tag{.58}$$

$$\lambda_{jt} \equiv \frac{N_{jt}}{\sum_{i \in \mathbf{S}} N_{it}} = \frac{N_{jt}}{N_t} = \frac{\widetilde{\alpha}_j^l \widetilde{\gamma}_j}{\sum_{i \in \mathbf{S}} \widetilde{\alpha}_i^l \widetilde{\gamma}_i} = \lambda_j, \tag{.59}$$

where $\boldsymbol{\kappa}_t$ is an $S \times 1$ vector of state variable, and we have used equation (.32) to simplify λ_{jt}.

Besides, the investment in sector j is obtained as

$$I_{jt} \equiv \int_0^1 i_{jt}(\iota)\, d\iota \tag{.60}$$

$$= \left[\left(R_{jt} + \theta_j^k \overline{Q}_{jt} \right) K_{jt} + \theta_j^b \overline{B}_{jt} - L_{jt} \right] \int_{\epsilon_{jt}^*}^{\overline{\epsilon}_j} dF(\epsilon) \tag{.61}$$

$$= \left[\left(R_{jt} + \theta_j^k \overline{Q}_{jt} \right) K_{jt} + \theta_j^b \overline{B}_{jt} - L_{jt} \right] \left[1 - F\left(\epsilon_{jt}^* \right) \right], \tag{.62}$$

where $K_{jt} \equiv \int_0^1 k_{jt}(\iota)\, d\iota$, and $L_{jt} \equiv \int_0^1 l_{jt}(\iota)\, d\iota = 0$.

In turn, the law of motion of capital in sector $j \in \mathbf{S}$ is given by

$$K_{j,t+1} = (1-\delta) K_{jt} + \int_0^1 \int_{\underline{\epsilon}_j}^{\overline{\epsilon}_j} \epsilon_{jt}(\iota)\, i_{jt}(\iota)\, dF_j\left(\epsilon_{jt}(\iota)\right) d\iota \tag{.63}$$

$$= (1-\delta) K_{jt} + \mathbb{E}\left(\epsilon_j | \epsilon_j \geq \epsilon_{jt}^*\right) \cdot I_{jt}. \tag{.64}$$

Proof of Proposition ??:. NEW PROOF UNDER COMMON BOND:

In steady, combining equation (**??**) and (2.11) yields that $\epsilon_j^* = \epsilon^*$ for all j, where the common cutoff ϵ^* is characterized by

$$\Gamma = \Gamma(\epsilon^*) = \int_{\epsilon^*}^{\overline{\epsilon}} \left(\frac{\epsilon}{\epsilon^*} - 1\right) dF(\epsilon) = \frac{1}{\beta R_f} - 1.$$

In turn, $Q_j = Q = \overline{Q}_j = 1/\epsilon^*$. Moreover, the Euler equation of Q_{jt} in (2.13) reveals that the sector MPK is given by

$$R_j = \frac{r + \delta - \theta_j^k \Gamma}{1 + \Gamma} Q, \tag{.65}$$

Then combining equation (2.16) and (2.17) yields

$$\theta_j^b \overline{B}_j - L_j = \frac{\delta K_j}{\int_{\epsilon^*}^{\overline{\epsilon}} \epsilon dF} - R_j K_j - \theta_j^k Q K_j \tag{.66}$$

$$= \left[\frac{\frac{\delta}{\int_{\epsilon^*}^{\overline{\epsilon}} \epsilon dF} - \theta_j^k Q}{R_j} - 1\right] R_j K_j \tag{.67}$$

$$= \left[\frac{\left(\frac{\delta}{\int_{\epsilon^*}^{\overline{\epsilon}} \epsilon dF} \frac{1}{Q} - \theta_j^k\right)(1+\Gamma) - \left(r + \delta - \theta_j^k \Gamma\right)}{r + \delta - \theta_j^k \Gamma}\right] R_j K_j \tag{.68}$$

$$= \left(\overline{\theta}^k(\epsilon^*) - \theta_j^k\right) \frac{\varphi_j \alpha}{r + \delta - \theta_j^k \Gamma(\epsilon^*)} Y, \tag{.69}$$

where

$$\overline{\theta}^k(\epsilon^*) \equiv \frac{\delta(1+\Gamma)}{\int_{\epsilon^*}^{\overline{\epsilon}} \epsilon dF} - (r + \delta) = \frac{\delta F(\epsilon^*)}{\int_{\epsilon^*}^{\overline{\epsilon}} \frac{\epsilon}{\epsilon^*} dF} - r. \tag{.70}$$

Consequently,

$$\overline{B}_j - \frac{L_j}{\theta_j^b} = \frac{\overline{\theta}^k(\epsilon^*) - \theta_j^k}{\theta_j^b} \frac{\varphi_j \alpha}{r + \delta - \theta_j^k \Gamma(\epsilon^*)} Y. \tag{.71}$$

*************The following are old stuff: to be update later.

Moreover, in steady state, the Euler equation of Q_{jt} in (2.13) reveals that the sector MPK is given by

$$R_j = \frac{(r+\delta) Q_j - \theta_j^k \Gamma_j \overline{Q}_j}{1+\Gamma_j}, \tag{.72}$$

where $\theta_j^k = \xi_j \sigma_j$. Then substituting the above equation into the sectoral law of motion in equation (**??**) yields that

$$\xi_j \overline{B}_j = \left[\frac{\frac{\delta}{\int_{\epsilon_j^*}^{\bar{\epsilon}_j} \epsilon_j dF_j} - \xi_j \sigma_j \overline{Q}_j}{R_j} - 1 \right] R_j K_j \tag{.73}$$

$$= \left[\frac{\left(\frac{\delta}{\int_{\epsilon_j^*}^{\bar{\epsilon}_j} \epsilon_j dF_j} \frac{1}{\overline{Q}_j} - \xi_j \sigma_j \right) (1+\Gamma_j) - \left((r+\delta_j) Q_j / \overline{Q}_j - \xi_j \sigma_j \Gamma_j \right)}{(r+\delta_j) Q_j / \overline{Q}_j - \xi_j \sigma_j \Gamma_j} \right] R_j K_j \tag{.74}$$

$$= \left(\overline{\xi}_j - \xi_j \right) \frac{\sigma_j \widetilde{\gamma}_j \widetilde{\alpha}_j^k}{(r+\delta_j) Q_j / \overline{Q}_j - \xi_j \Gamma_j} Y, \tag{.75}$$

where

$$\overline{\xi}_j \equiv \left[\frac{\delta (1+\Gamma_j)}{\int_{\epsilon_j^*}^{\bar{\epsilon}_j} \epsilon_j dF_j} - (r+\delta_j) \right] \frac{1}{\sigma_j} \frac{Q_j}{\overline{Q}_j} = \left[\frac{\delta F_j \left(\epsilon_j^* \right)}{\int_{\epsilon_j^*}^{\bar{\epsilon}_j} \epsilon_j / \epsilon_j^* dF_j} - r \right] \frac{1}{\sigma_j} \frac{Q_j}{\overline{Q}_j}, \tag{.76}$$

and we have used the fact that $R_j K_j = \widetilde{\gamma}_j \widetilde{\alpha}_j^k Y$, which we have proved previously. Therefore we can formalize $\overline{B}_j$ as

$$\overline{B}_j = \left(\frac{\overline{\xi}_j}{\xi_j} - 1 \right) \frac{\sigma_j \widetilde{\gamma}_j \widetilde{\alpha}_j^k}{(r+\delta_j) Q_j / \overline{Q}_j - \xi_j \Gamma_j} Y, \tag{.77}$$

or equivalently.

Finally, since $\theta_j^b = \xi_j$ and $\theta_j^k = \xi_j \sigma_j$, we can rewrite equation (.77) as

Proof of Corollary ??:. When $\mathbf{\Phi} = \mathbf{E}$, we immediately have $B_j = \overline{B}_j$ and $Q_j = \overline{Q}_j$. Then, on the one hand, the sectoral bubble j exists iff

$$\overline{\xi}_j \equiv \left[\frac{\delta F_j \left(\epsilon_{j,b}^* \right)}{\int_{\epsilon_{j,b}^*}^{\bar{\epsilon}_j} \epsilon_j / \epsilon_{j,b}^* dF_j} - r \right] \frac{1}{\sigma_j} > \xi_j, \tag{.78}$$

which can be rewritten as

$$\frac{F_j \left(\epsilon_{j,b}^* \right)}{\int_{\epsilon_{j,b}^*}^{\bar{\epsilon}_j} \epsilon_j / \epsilon_{j,b}^* dF_j} > \frac{\theta_j^k + r}{\delta}, \tag{.79}$$

where we have used the definition that $\theta_j^k = \xi_j \sigma_j$.

On the other hand, if there were no sectoral bubble, i.e., $B_j = \overline{B}_j = 0$, we have $\overline{\xi}_j = \xi_j$, and thus

$$\frac{F_j\left(\epsilon_{j,f}^*\right)}{\int_{\epsilon_{j,f}^*}^{\bar{\epsilon}_j} \epsilon_j/\epsilon_{j,f}^* dF_j} = \frac{\theta_j^k + r}{\delta}. \tag{.80}$$

We can easily verify that $\frac{F_j\left(\epsilon_{j,b}^*\right)}{\int_{\epsilon_{j,b}^*}^{\bar{\epsilon}_j} \epsilon_j/\epsilon_{j,b}^* dF_j}$ strictly increases with ϵ_j^*. Then comparing the above two condition immediately yields that, given coexistence, we have $\epsilon_{j,b}^* > \epsilon_{j,f}^*$. That is, the average investment efficiency under bubbly equilibrium is higher than that under bubbleless equilibrium. Meanwhile, since $\frac{F_j\left(\epsilon_{j,f}^*\right)}{\int_{\epsilon_{j,f}^*}^{\bar{\epsilon}_j} \epsilon_j/\epsilon_{j,f}^* dF_j} \in (0, \infty)$, there always exists a unique solution $\epsilon_{j,f}^*$ to equation (.80). Additionally, using the Implicit Function Theorem on (.80) reveals that $\epsilon_{j,f}^*$ strictly increases with θ_j^k. Then we denote the solution as $\epsilon_{j,f}^* = \epsilon_{j,f}^*\left(\theta_j^k\right)$.

Moreover, under bubbly equilibrium, the Euler equation (x.x) can be simplified as $\xi_j = \theta_j^b = \frac{r}{\Gamma_j\left(\epsilon_{j,b}^*\right)}$, and thus

$$\epsilon_{j,b}^* = \Gamma_j^{-1}\left(r/\theta_j^b\right). \tag{.81}$$

As shown above, $B_j > 0$ iff $\epsilon_{j,b}^* > \epsilon_{j,f}^*$. Therefore $B_j > 0$ iff $\left(\theta_j^b, \theta_j^k\right) \in \mathcal{B}_j^{bk}$, where

$$\mathcal{B}_j^{bk} = \left\{\left(\theta_j^b, \theta_j^k\right) | \theta_j^b > \max\left\{\theta_j^k, \Theta_j^b\left(\theta_j^k\right)\right\}, \text{and } \theta_j^b, \theta_j^k \in (0,1)\right\}, \tag{.82}$$

with $\Theta_j^b\left(\theta_j^k\right) \equiv \frac{r}{\Gamma_j\left(\epsilon_{j,f}^*\left(\theta_j^k\right)\right)}$, and we have used the fact that $\Gamma_j\left(\epsilon_j^*\right)$ is a decreasing function of ϵ_j^*. In turn, we can easily verify that $\Theta_j^b\left(\theta_j^k\right)$ increases with θ_j^k.

Given the existence of bubbles, we know that $\overline{\theta}_j^k\left(\epsilon_{j,b}^*\right) \equiv \frac{\delta F_j\left(\epsilon_{j,b}^*\right)}{\int_{\epsilon_{j,b}^*}^{\bar{\epsilon}_j} \epsilon_j/\epsilon_{j,b}^* dF_j} - r$ and thus

$$\overline{\theta}_j^k\left(\epsilon_{j,b}^*\right) - \theta_j^k = \frac{\delta F_j\left(\epsilon_{j,b}^*\right)}{\int_{\epsilon_{j,b}^*}^{\bar{\epsilon}_j} \epsilon_j/\epsilon_{j,b}^* dF_j} - \frac{\delta F_j\left(\epsilon_{j,f}^*\right)}{\int_{\epsilon_{j,f}^*}^{\bar{\epsilon}_j} \epsilon_j/\epsilon_{j,f}^* dF_j}. \tag{.83}$$

Consequently, we can further formalize $b_j = B_j/Y$ as

$$b_j = \left[\frac{F_j\left(\epsilon_{j,b}^*\right)}{\int_{\epsilon_{j,b}^*}^{\bar{\epsilon}_j} \epsilon_j/\epsilon_{j,b}^* dF_j} - \frac{F_j\left(\epsilon_{j,f}^*\right)}{\int_{\epsilon_{j,f}^*}^{\bar{\epsilon}_j} \epsilon_j/\epsilon_{j,f}^* dF_j}\right] \cdot \frac{\widetilde{\gamma}_j \widetilde{\alpha}_j^k}{\theta_j^b}. \tag{.84}$$

Furthermore, if ϵ_j conforms to a standard Pareto distribution, we can obtain that

$$\mathcal{B}_j^{bk} = \left\{ \left(\theta_j^b, \theta_j^k\right) | \theta_j^b > \max\left\{\theta_j^k, \frac{\eta_j r}{\delta_j}\theta_j^k + r\left(\eta_j - 1 + \frac{\eta_j r}{\delta_j}\right)\right\} \text{ and } \theta_j^b, \theta_j^k \in (0,1) \right\}. \tag{.85}$$

Finally, since $\theta_j^b = \xi_j$ and $\theta_j^k = \xi_j \sigma_j$, then we know that the sectoral bubble j exists iff $\left(\xi_j, \sigma_j\right) \in \mathcal{B}_j^{\xi\sigma}$, where

$$\mathcal{B}_j^{\xi\sigma} = \left\{ \left(\xi_j, \sigma_j\right) | \xi_j \in \left(0, \frac{\left(r + \delta_j\right)\eta_j - \delta_j}{\delta_j - \eta_j r \sigma_j} r\right) \text{ and } \sigma_j \in \left(0, \min\left\{1, \frac{\delta_j}{\eta_j r}\right\}\right) \right\}. \tag{.86}$$

Proof of Proposition ??:. We start by characterizing Y. To recap, we have proved previously that

$$Y = \overline{A}\overline{K}^{\alpha} N^{1-\alpha}. \tag{.87}$$

where

$$\overline{A} \equiv \prod_{j \in \mathbf{S}} \left(\varphi_j / \widetilde{\gamma}_j\right)^{\varphi_j} \lambda_j^{\gamma_j \alpha_j^l} o_j^{\gamma_j} \cdot \prod_{j \in \mathbf{S}} A_j^{\gamma_j}, \tag{.88}$$

$$\overline{K} \equiv \left[\prod_{j=1}^{S} K_j^{\gamma_j \alpha_j^k}\right]^{\frac{1}{\alpha}}. \tag{.89}$$

and $\alpha = \boldsymbol{\gamma}' \boldsymbol{\alpha}^k$, $\lambda_j \equiv \frac{\widetilde{\gamma}_j \widetilde{\alpha}_j^l}{\sum_{i \in \mathbf{S}} \widetilde{\gamma}_i \widetilde{\alpha}_i^l}$.

Equation (.34) suggests that, in steady state, we have

$$K_j = \frac{\widetilde{\gamma}_j \widetilde{\alpha}_j^k}{R_j} Y, \tag{.90}$$

and

$$\kappa_j = \frac{\widetilde{\gamma}_j \widetilde{\alpha}_j^k / R_j}{\sum_{i \in \mathbf{S}}^k \widetilde{\gamma}_i \widetilde{\alpha}_i^k / R_i}, \tag{.91}$$

where R_j is pinned down by equation (**??**).

Equation (**??**) implies

$$I_j = \frac{\delta_j}{\mathbb{E}_j\left(\epsilon_j | \epsilon_j \geq \epsilon_j^*\right)} K_j = \frac{\delta_j^k \widetilde{\gamma}_j \widetilde{\alpha}_j^k}{\mathbb{E}_j\left(\epsilon_j | \epsilon_j \geq \epsilon_j^*\right) R_j} Y. \tag{.92}$$

Besides, given N, substituting equation (.90) into (.87) yields

$$Y = \overline{A}\left[\prod_{j \in \mathbf{S}} \left(\frac{\widetilde{\gamma}_j \widetilde{\alpha}_j^k}{R_j}\right)^{\gamma_j \alpha_j^k}\right] Y^{\alpha} N^{1-\alpha}, \tag{.93}$$

Besides, since we have normalized $N = 1$, we immediately have

$$Y = \left\{ \overline{A} \left[\prod_{j \in \mathbf{S}} \left(\frac{\widetilde{\gamma}_j \widetilde{\alpha}_j^k}{R_j} \right)^{\gamma_j \alpha_j^k} \right] \right\}^{\frac{1}{1-\alpha}}, \tag{.94}$$

Then the resource constraint for final goods implies that

$$C = Y - \sum_{j \in \mathbf{S}} I_j = \left(1 - \sum_{j \in \mathbf{S}} \frac{\delta_j \widetilde{\gamma}_j \widetilde{\alpha}_j^k}{\mathbb{E}_j \left(\epsilon_j | \epsilon_j \geq \epsilon_j^* \right) R_j} \right) Y. \tag{.95}$$

Finally, equation (.77) reveals that B_j is a function of the vector of the investment cutoff $\boldsymbol{\epsilon}^* \equiv \left\{ \epsilon_j^* \right\}_{j \in \mathbf{S}}$, and therefore equation (**??**) suggests a simultaneous equation system such that

$$r \widetilde{B}_j (\boldsymbol{\epsilon}^*) = \Gamma_j \left(\epsilon_j^* \right) \sum_{i \in \mathbf{S}} \phi_{ij} \widetilde{B}_i (\boldsymbol{\epsilon}^*) \text{ for all } j \in \mathbf{S}, \tag{.96}$$

which can be rewritten in more compact way as

$$\overline{\mathbf{b}} (\boldsymbol{\epsilon}^*) = \frac{\mathbf{\Phi}' \mathbf{Diag} \left(\boldsymbol{\theta}^b \right) \mathbf{Diag} \left(\mathbf{\Gamma} (\boldsymbol{\epsilon}^*) \right)}{r} \overline{\mathbf{b}} (\boldsymbol{\epsilon}^*). \tag{.97}$$

where $\overline{b}_j (\boldsymbol{\epsilon}^*) = \overline{B}_j / Y$.

Proof of Corollary ??:. If $\delta_j = \delta$ for all $j \in \mathbf{S}$, the substituting equation (**??**) into (**??**) immediately yields that

$$\frac{I}{Y} = \frac{\alpha \delta}{r + \delta} \cdot \sum_{j \in \mathbf{S}} \left[\frac{\gamma_j \alpha_j^k}{\alpha} \frac{1}{\mathbb{E}_j \left(\frac{\epsilon_j}{\epsilon_j^*} | \epsilon_j \geq \epsilon_j^* \right)} \frac{1 + \Gamma_j \left(\epsilon_j^* \right)}{1 - \frac{\overline{Q}_j / Q_j}{r + \delta} \xi_j \sigma_j \Gamma_j \left(\epsilon_j^* \right)} \right], \tag{.98}$$

where $\alpha = \boldsymbol{\gamma}' \boldsymbol{\alpha}^k$, which denotes the capital share in frictionless economy, and the aggregate investment is given by $I \equiv \sum_{j \in \mathbf{S}} I_j$.

Proof of Corollary ??:. As proved in Proposition **??**,

$$r B_j = \Gamma_j \sum_{i \in \mathbf{S}} \xi_i \phi_{ij} B_i \text{ for } j \in \mathbf{S}. \tag{.99}$$

Since $B_i \geq 0$ for all $i \in \mathbf{S}$, the above equation implies that, when $B_j = 0$, provided $\phi_{ij} > 0$, it must be that $B_i = 0$.

Proof of Bubbly Contagion in x.x (The link is to be added):. The contagious region is given by

$$\theta_b^j \geq \frac{1}{1-\phi_{j\to i}} \max\left\{\frac{\eta_j r}{\delta_j}\theta_j^k + \frac{\eta_j r}{\delta_j}\left(\delta_j + r - \frac{\delta_j}{\eta_j}\right)\right\}. \tag{.100}$$

where $\phi_{j\to i} = \phi_{ij} = 1 - \phi_{jj}$. Symmetric then $\frac{Q_j}{\overline{Q}_j} = 1$, and thus

$$\overline{\xi}_j = \left[\frac{\delta F\left(\epsilon_j^*\right)}{\int_{\epsilon_j^*}^{\bar{\epsilon}} \epsilon_j/\epsilon_j^* dF} - r\right]\frac{1}{\sigma}, \tag{.101}$$

$$r\overline{b}_j = \sum_{i\in\mathbf{S}} \phi_{ij}\xi_i\Gamma_i\overline{b}_i$$

Bubble i burst such that $b_i = 0$. Therefore

$$\overline{\xi}_i = \xi. \tag{.102}$$

However, $b_j > 0$, and thus

$$\frac{r}{\phi_{jj}\xi} = \Gamma_j\left(\epsilon_j^*\right). \tag{.103}$$

In turn,

$$\overline{b}_j = \left(\overline{\xi}_j/\xi - 1\right)\frac{\sigma\gamma\alpha^k}{(r+\delta)\,Q_j/\overline{Q}_j - \xi\Gamma\left(\epsilon_j^*\right)}. \tag{.104}$$

At the boundary, $\overline{b}_j \longrightarrow 0$, and thus

$$\overline{\xi}_j = \xi. \tag{.105}$$

Due to symmetry, we know that $\epsilon_j^* = \epsilon_i^* = \epsilon^*$ at the boundary. In turn, ϵ^* is determined by

$$\frac{F\left(\epsilon^*\right)}{\int_{\epsilon^*}^{\bar{\epsilon}} \epsilon/\epsilon^* dF} = \frac{\xi\sigma + r}{\delta}. \tag{.106}$$

If bubble burst in sector i, then it is contagious to the bubble in sector j iff $\phi_{ij} > \phi^*$.

Then the cutoff for ϕ^* satisfies

$$\frac{r}{\left(1-\phi^*\right)\xi} = \Gamma\left(\epsilon^*\right), \tag{.107}$$

and therefore

$$\phi^* = 1 - \frac{r}{\Gamma\left(\epsilon^*\right)\xi}. \tag{.108}$$

Furthermore, for Pareto, we know that

$$\frac{F(\epsilon^*)}{1-F(\epsilon^*)} = \frac{\xi\sigma + r}{\delta\epsilon_{\min}}, \tag{.109}$$

and thus

$$1-F(\epsilon^*) = \frac{\delta\epsilon_{\min}}{\xi\sigma + r + \delta\epsilon_{\min}}.$$

In turn,

$$\begin{aligned}
\phi^* &= 1 - \frac{(\eta-1)\, r}{(1-F(\epsilon^*))\,\xi} \\
&= 1 - (\eta-1)\, r \frac{\xi\sigma + r + \delta\epsilon_{\min}}{\xi\delta\epsilon_{\min}} \\
&= 1 - \frac{r}{\delta} \cdot \frac{(\theta_k + r + \delta)\,\eta - \delta}{\theta_b},
\end{aligned}$$