Central Bank Balance Sheet Policies Without Rational Expectations*

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Abstract

We study the effects of central bank balance sheet policies—namely, quantitative easing and foreign exchange interventions—in a model where people form expectations through the level-k thinking process, consistent with experimental evidence on the behavior of people in strategic environments. We emphasize two main theoretical results. First, under a broad set of conditions, central bank interventions are effective under level-k thinking, while they are neutral in the rational expectations equilibrium. Second, when preferences exhibit constant relative risk aversion, asset purchases increase aggregate output if they target assets with pro-cyclical returns but reduce it if asset returns are counter-cyclical. Finally, we empirically show that forecast errors about future asset prices are predictable by balance sheet interventions, a property that differentiates our channel from popular alternatives, such as portfolio-balance and signaling channels.

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1 Introduction

The balance sheets of central banks are among the most important stabilization policy tools (Bernanke, 2012; Draghi, 2015; Yellen, 2016). Prominent examples of their use are quantitative easing (QE), which has gained popularity during the 2007-08 global financial crisis, and foreign exchange (FX) interventions, used widely by emerging countries and by some small open advanced economies.

Despite their popularity, central bank balance sheet policies are not yet well understood. From an empirical perspective, identifying a causal effect of these policies is challenging, as they are usually implemented in response to economic events, thereby creating an endogeneity problem. There is nonetheless evidence, which we discuss below, in favor of the effects of QE and FX interventions.

From a theoretical perspective, a wide class of standard macroeconomic models predicts that balance sheet policies are irrelevant, provided that the assets involved are valued for their pecuniary returns only. As noted by Wallace (1981), this *irrelevance result* is like the celebrated Modigliani-Miller proposition applied to central banks. The intuition relies on two key implications of rational expectations. First, investors correctly forecast government behavior. When a central bank acquires, for example, private risky assets and, in exchange, creates reserves or sells short-term public bonds, investors correctly understand that any gains or losses incurred on the central bank's portfolio will be directly transferred to the fiscal authority and indirectly, through taxes, back to investors. As a result, investors reduce their individual demand for risky assets so as to hedge against this new tax risk. Second, investors also correctly forecast other investors' behavior. When an investor correctly believes that all other investors form their expectations rationally, she expects a decrease in the total demand for risky assets, which exactly compensates the higher demand from the central bank. As a consequence, every investor anticipates that future asset prices will be unaffected by the new policy. Taken together, the two effects imply that current prices will be unchanged, making central bank interventions irrelevant.

In this paper, we solve a dynamic stochastic general equilibrium model where we relax the assumption that people foresee all the future consequences of such policies. We show that, with this deviation from rational expectations, central bank interventions cease to be irrelevant and, instead, have an impact on asset prices and aggregate output. What is more, we show that the sign and strength of the effect on aggregate output depends on agents' preferences and on the return profile of the assets targeted by the intervention. We then verify empirically some of our theoretical predictions.

We replace rational expectations with the level-k thinking process of belief formation.

Laboratory experiments have repeatedly demonstrated that standard equilibrium analysis, which is based on rational expectations, is often at odds with actual behavior of subjects. Instead, the evidence suggests that level-k thinking is a better description of how players form their beliefs and, hence, make decisions. More specifically, this process assumes that agents form higher-order beliefs—i.e., beliefs about beliefs ... about the behavior of others—only up to some finite level k, either due to the complexity of the economic environment or because they believe that other agents are lower-order thinkers.

Level-k thinking can be particularly relevant in macroeconomic settings. Rational expectations require people to anticipate all future contingencies of any given policy. This requirement is, however, very demanding as the effects of a policy are the result of intricate general equilibrium (GE) relations connecting the actions of all the players of the "game," such as households and firms, but also policymakers. At the same time, some agents can devote time and resources to predicting the behavior of other agents. Following the announcement of a new policy, these agents can respond in a forward-looking fashion. Level-k thinking can capture both types of behavior: it does not assume that agents can predict all future contingencies, however, at the same time, it allows some agents to be more forward-looking.

We construct beliefs under level-k thinking as follows. First of all, all agents are assumed to be perfectly aware of current balance sheet policies as well as of their own income and asset positions. However, each agent is characterized by a "level of thinking," which determines her expectations about the effects of balance sheet policies on future endogenous variables, such as transfers from the central bank to the treasury, taxes, and asset prices.

More specifically, expectations are constructed according to an iterative procedure. First, "level-1 thinking" posits that, after observing the policy change, agents do not update their expectations about future endogenous variables. As a result, "level-1 thinkers" make consumption and portfolio decisions under their old expectations; in particular, they do not hedge against the future tax risk, as required for the irrelevance result to hold. Next, level-2 thinkers believe that the economy is populated only by level-1 thinkers. Thus, upon observing a policy change, they expect future variables to coincide with the equilibrium outcomes of an economy populated only by such agents. Notice that, unlike level-1 thinkers, these higher-level thinkers do revise their expectations following a policy intervention. However, we show that this revision is not enough to reach rational expectations. Proceeding recursively, we can define the expectations and, hence, the behavior of level-k thinkers, for any finite k. Having characterized the expectations of every agent, we compute the equilibrium of an economy populated by agents with different levels of thinking. We use the notion of equilibrium known as *reflective* equilibrium.[1]

Our first main result shows that, when agents are level-k thinkers, balance sheet policies have an impact on asset prices. Thus, the irrelevance result under rational expectations is overturned. To derive this result in the most transparent way, we first focus on an endowment economy, which we call "the simple model." The result is intuitive. Since agents do not hold rational expectations about future endogenous variables, they underestimate the tax risk emanating from policy interventions and incorrectly forecast the behavior of future assets prices. As a result, they demand lower risk premia, which boosts asset prices and makes balance sheet policies effective. Interestingly, even when all the agents correctly understand the tax risk, which is the case in our simple model when agents are level-2 thinkers or higher, balance sheet policies may still be effective. This happens because higher-level thinkers fail to form all the higher-order beliefs correctly: even though these agents predict the tax risk correctly, they believe that other agents will have incorrect beliefs, or that other agents believe that other agents will have incorrect beliefs, and so on.

Perhaps surprisingly, the strength of balance sheet policies can increase with average level of thinking of agents in the economy, defined as the average level-k across all agents. This apparently counterintuitive result occurs because an increase in the level of thinking has two opposing effects. On the one hand, higher-level thinkers foresee the fiscal consequences of balance sheet policies, bringing the policy closer to full neutrality. On the other hand, higher-level thinkers become endogenously more forward-looking, increasing the strength of persistent balance sheet interventions. When the average level of thinking of agents grows to infinity, however, the reflective equilibrium converges to the rational expectations equilibrium. We also demonstrate that our first main result is unchanged when we extend the model by adding a fraction of rational expectations agents, statistical learning, equilibrium unraveling, domestic and foreign *nominal* assets.

Our second main result characterizes the response of aggregate output to the balance sheet policies. To do this, we extend the simple model to an environment with nominal rigidities where output is determined by aggregate demand. We show that the ability of central bank interventions to stimulate output depends on the agents' attitude towards risk and on the risk-return characteristics of the assets traded by the central bank. We explore this mechanism in detail and provide a complete characterization of the first-order output effects under general preferences and asset characteristics. In particular, the model predicts that, when the central bank purchases a *pro-cyclical* asset, i.e., an asset whose excess returns *co-move positively* with aggregate output (e.g., risky corporate bonds), then

[1] Angeletos and Lian (2017) point out that a notion of reflective equilibrium "smooths out" some of the unappealing properties of level-k equilibria (i.e., a reflective equilibrium with a degenerate distribution over k).

the effect of the intervention on output is positive as long as agent preferences exhibit *decreasing absolute risk aversion*. Instead, when the central bank buys a *counter-cyclical* or *hedge* asset (e.g., long-term nominal public debt), the effect on output is negative.

To understand the intuition suppose that the central bank issues risk-free reserves to purchase a pro-cyclical asset. Some agents fail to understand (or think that other agents fail to understand) that the intervention of the central bank affects future taxes. As a result, the intervention will have two potentially offsetting effects on agents' consumption. First, agents will incorrectly believe that their portfolio (which now includes fewer risky assets and more risk-free reserves) is less risky and, thus, their overall income is less risky. They will thus reduce precautionary saving and increase consumption. Second, since a pro-cyclical asset must command a risk premium, agents will also incorrectly believe that the average return on their portfolio and, thus, their overall income is lower. They will thus smooth out the lower expected future income by lowering current consumption. Which of these two channels prevails depends on how absolute risk aversion varies with wealth. For example, in the empirically plausible case of constant relative risk aversion (CRRA) preferences, a balance sheet policy that targets pro-cyclical instruments will be effective at stimulating output. On the contrary, purchases of hedge instruments have a negative effect on output.

The last part of the paper presents specific predictions of our channel that can differentiate it from other channels of balance sheet policies. Specifically, we focus on the behavior of forecast errors of asset prices after policy interventions. We show that individual and cross-sectional-average forecast errors are related to policy interventions. Predictable forecast errors are absent in the standard models that assume limited market participation or signaling by the central bank, but retain the assumptions of full information and rational expectations. Moreover, in models with incomplete information in which agents form expectations rationally, predictable forecast errors would arise only if agents had imperfect information about policy interventions. If, instead, the policy was well advertised and, as a result, agents had complete information about it, forecast errors would not be predictable. In contrast, in our model agents are fully aware of the policy intervention, yet they make mistakes due to their inability to form rational expectations.

Finally, we empirically document the predictability of cross-sectional-average forecast errors by balance sheet policies. We focus on the mortgage market in the US and proxy balance sheet interventions with purchases of mortgages by government-sponsored enterprises (GSEs), such as Fannie Mae and Freddie Mac. This approach utilizes the fact that GSEs' purchases of mortgages resemble Fed's acquisition of mortgage backed securities, as argued in Fieldhouse, Mertens and Ravn (2018). We follow these authors, who identify "exogenous and unexpected" changes in mortgage purchases by the GSEs using a narrative approach in the spirit of Romer and Romer (2010). As predicted by our model, we first verify that these exogenous changes in mortgage purchases affect the conventional mortgage rate. We then use the Blue Chip Financial Forecasts survey data to show that exogenous purchases by the GSEs also predict conventional mortgage rate forecast errors. Using these empirical estimates together with our stylized model, we calculate that the number of level-1 thinkers (those who do not change their expectations after policy interventions) in the data is 86 percent.

Related literature. Our paper is related to Evans and Ramey (1992), García-Schmidt and Woodford (2019), and Farhi and Werning (2017), who use level-k thinking in macroeconomic models to study the effects of conventional monetary policy and forward guidance, but abstract from aggregate risk.[2] Our mechanism, instead, hinges entirely on aggregate risk: agents incorrectly believe that balance sheet change the risk-return profile of their portfolio. Moreover, the focus of our paper is on balance sheet policies.

Level-k thinking that we use in this paper is related to other deviations from full information rational expectations in macroeconomics. Level-k thinking generates endogenous discounting of expectations about future endogenous variables, which can rationalize the exogenous discounting introduced in, for example, Gabaix (2016). Level-k thinking dampens the response to shocks of higher-order belief in a way similar to models with information frictions, such as noisy information (e.g., Angeletos and Huo, 2018), sticky information (e.g., Mankiw and Reis, 2002), and rational inattention (e.g., Maćkowiak and Wiederholt, 2009). Section 4 highlights the differences between our model with level-k thinking and models with information frictions. Informational frictions are often invoked in the finance literature to explain the phenomenon of slow-moving capital, that is, the sluggish response of investors to the arrival of new investment opportunities (e.g., Duffie 2010). By dampening the response of agents to policy announcements, level-k thinking is another source of slow-moving capital. Last but not least, level-k thinking is a forward-looking process, thus, it is different from the so-called "inductive approaches" to belief formation, such as statistical learning (Evans and Honkapohja, 2012; De Grauwe, 2012). Section 2.6 shows the differences between our model with level-k thinking and models with statistical learning.

[2]The level-k thinking belief-formation process has been widely used in behavioral game theory to rationalize the behavior of subjects playing full-information games in various laboratory and field experiments (Stahl and Wilson, 1995; Nagel, 1995; Bosch-Domenech et al., 2002). Crawford et al. (2013) provide a review of level-k thinking in game theory. For example, Nagel (1995) presents empirical evidence that subjects play according to the level-k belief formation in a simple formalization of Keynes' beauty contest metaphor (Keynes, 1936) called a p-beauty contest game. She documents that deviations from Nash-equilibrium behavior are starkest in the first round of play, when agents face a novel strategic environment. Subjects usually exhibit levels of thinking no higher than 3.

We contribute to the literature that studies the effectiveness of balance sheet policies. On the empirical side, there is some evidence that these policies are effective.[3] In Section 4, we empirically show that balance sheet interventions not only affect asset prices, but also predict asset returns forecast errors of financial market experts. The use of forecast errors connects our exercise to the literature that studies the behavior of forecast errors, e.g., Coibion and Gorodnichenko (2012).

On the theory side of balance sheet interventions, an important starting point is the irrelevance result in Wallace (1981). Backus and Kehoe (1989) show the irrelevance of *sterilized* foreign-exchange interventions in an international setting. To deviate from the irrelevance result, the literature has proposed several frictions, such as incomplete information and market segmentation. The former friction generates the so-called "signaling" channel, whereby changes in the composition of a central bank's balance sheet serve as a signal of the central bank's objectives or information about economic fundamentals (see Mussa, 1981).[4] The latter friction generates the "portfolio balance" channel, whereby changes in the supply of different assets affect asset prices due to the segmentation of assets markets. Kouri (1976) is an early paper and Gabaix and Maggiori (2015) is a more recent contribution that apply this idea to FX interventions. Vayanos and Vila (2009) provide a framework to study asset purchases when markets are segmented and Curdia and Woodford (2011) study quantitative easing with market segmentation in a New Keynesian model. Krishnamurthy and Vissing-Jorgensen (2011) summarize the recent literature on quantitative easing.[5] In this paper, we propose a "bounded rationality" channel of balance sheet policies, derive the implications that can distinguish it from other prominent channels, and provide some empirical support for it.

Our result on the output effects of balance sheet policies is related to Caballero and Farhi (2013), a working paper version of Caballero and Farhi (2017). The authors show that, in an overlapping generation model where some agents are infinitely risk averse and others face financial constraints, central bank purchases of long-term public bonds with counter-cyclical dividends may reduce aggregate output if the economy experiences

[3]In the case of QE policies, Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2011), Hancock and Passmore (2011) present evidence on asset prices, Di Maggio et al. (2020) evaluate mortgage originations and refinancing responses, and Weale and Wieladek (2016) and Fieldhouse et al. (2018) estimate output effects. In the case of FX policies, Dominguez and Frankel (1990, 1993), Kearns and Rigobon (2005), Dominguez et al. (2013), Kohlscheen and Andrade (2014), Chamon et al. (2017) show that FX interventions move exchange rates.

[4]Jeanne and Svensson (2007) and Bhattarai et al. (2015) proposed that central banks cannot commit to a desired monetary policy and use balance sheet policies as a costly signal about future intentions in the case of FX and QE interventions, respectively.

[5]Other recent contributions to FX interventions are Fanelli and Straub (2016), Amador et al. (2017), Cavallino (2017), and to QE policies are Chen et al. (2012), Silva (2016), Goncharov et al. (2017), Reis (2017), Cui and Sterk (2018), and Sterk and Tenreyro (2018).

a "safety trap" (i.e., a shortage of safe assets). By contrast, our result that the output effects of asset purchases might be negative hinges entirely on non-rational expectations. In addition, we show that such a result depends not only on the risk-return profile of the targeted assets but also on households' attitude towards risk.

The rest of the paper is organized as follows. Section 2 presents the simple model and considers purchases of private risky assets by the central bank. Section 3 extends the model by allowing for endogenous output. Section 4 presents and tests the implications of the model. Section 5 concludes. Appendix A presents omitted proofs, Appendix B provides details of our empirical exercise, and the Online Appendix, which is currently attached to the paper, extends the model in various ways.

2 Balance Sheet Policies and Asset Prices

We now present a model that we will refer to as the "simple model." The model is a standard Lucas tree economy, with the exception that agents form their expectations according to the level-k thinking process. We use this model to investigate the effects on the asset prices of purchases of private risky assets by the central bank.

2.1 Assets, Agents, and Beliefs

Time is discrete, infinite, and indexed by $t = 0, 1, 2, \ldots$. There are two assets in the economy. First, there is a one-period riskless asset, available in perfectly elastic supply, that yields a real net return $r > 0$. Second, there is a risky asset, available in fixed supply (which we normalize to 1), that entitles the owner to a stream of dividends $\{D_t\}$ and trades at price q_t in period t. The dividend on the risky asset is $D_t = \overline{D} + \epsilon_t^x$, where $\overline{D}$ is constant and ϵ_t^x is assumed to be random, independent over time and normally distributed, with zero mean and standard deviation σ_x.

Households. There are infinitely-lived households in the economy, who are identical except, perhaps, for their beliefs. We describe beliefs in detail below; for now, we use a "tilde" on top of the variables that the household needs to predict to solve her problem. Households have constant absolute risk aversion (CARA) preferences with risk aversion γ and discount factor $e^{-\rho}$, with $\rho > 0$. At time 0, given beliefs and prices, households choose consumption $\{\widehat{c}_t\}$, investment in the safe asset $\{\widehat{b}_{t+1}\}$, and investment in the risky asset $\{\widehat{x}_{t+1}\}$, to maximize

$$\mathbb{E}_0 \left[-\frac{1}{\gamma} \sum_{t=0}^{\infty} e^{-\rho t - \gamma \widehat{c}_t} \right], \tag{1}$$

subject to the flow budget constraint

$$\widehat{c}_t + \widehat{b}_{t+1} + \widetilde{q}_t \widehat{x}_{t+1} \leq \widetilde{W}_t - \widetilde{T}_t + (1+r)\widehat{b}_t + (\widetilde{D}_t + \widetilde{q}_t)\widehat{x}_t, \tag{2}$$

where T_t is lump-sum taxes, W_t is an endowment process, for example, labor income, which, for simplicity, we assume to be non-stochastic (Section 3 relaxes this assumption). It is important to note that, due to bounded rationality, the optimal decisions that the household plans to implement in the future will differ, in general, from the actual decisions that the agent will take. We denote actual choices with c_t,x_{t+1}, and b_{t+1}, and use a "hat" for planned choices.

We take advantage of the fact that exponential preferences with Gaussian shocks deliver a closed-form solution to the agent's portfolio problem, even in the presence of aggregate risk.[6] In Section 3, we relax these assumptions by resorting to an approximate solution.

Government. The government consists of a treasury and a central bank. The treasury conducts fiscal policy and the central bank implements purchases of private risky assets. Government policies will be denoted by capital letters. The treasury controls real per capita lump-sum taxes $\{T_t\}$ and the *real* amount of one-period safe public bonds $\{B_{t+1}\}$. The central bank manages the real purchases of private risky assets $\{X_{t+1}\}$, the amount of outstanding reserves $\{R_{t+1}\}$, which are perfect substitutes of safe assets, and the transfers to the treasury $\{Tr_t\}$. We let $\Pi_t \equiv \{T_t, B_{t+1}, X_{t+1}, R_{t+1}, Tr_t\}$ denote the collection of government policies.

It is worth noting that here central bank reserves are valued for their pecuniary returns only. This is in contrast to models that assume non-pecuniary benefits of reserves. Importantly, the results in this paper are also relevant to those models. First, many actual instances of balance sheet policies occurred during liquidity traps when non-pecuniary benefits of reserves approach zero. Second, often balance sheet policies change only the composition of central bank assets, as in sterilized interventions, without altering the size of the balance sheet. In our discussion below, we formally model balance sheet policies by assuming that a central bank creates reserves. Because here reserves are perfect substitutes of short-term public debt, the results will be identical if, instead, the central bank financed its purchases by selling treasury bonds.

[6]Note, however, that the particular functional forms will not be responsible for the qualitative results that balance sheet policies are neutral in the rational expectations equilibrium and effective in the reflective equilibrium.

The intertemporal budget constraint of the treasury is

$$(1+r)B_t = \sum_{s=0}^{\infty} \frac{T_{t+s}}{(1+r)^s} + \sum_{s=0}^{\infty} \frac{Tr_{t+s}}{(1+r)^s}, \tag{3}$$

where we assumed that public debt does not grow "too fast," i.e., $\lim_{s\to 0} B_{t+s+1}/(1+r)^s = 0$. The left-hand side of equation (3) represents treasury's outlays, consisting of repayment of outstanding debt, $(1+r)B_t$. The right-hand side is government income. The treasury finances itself by raising taxes and by receiving transfers from the central bank (which can be negative).

We assume that in period 0 the central bank announces, and commits to, the entire path of risky asset purchases $\{X_{t+1}\}$ and finances it by creating reserves such that[7]

$$R_{t+1} = q_t X_{t+1}. \tag{4}$$

We refer to these purchases as "quantitative easing." Equation (4) states that the central bank does not reinvest its profits, but it rebates them to the Treasury. This approximates the way the Federal Reserve and the European Central Bank operate. As a result, the budget constraint of the central bank simplifies to

$$Tr_t = (D_t + q_t)X_t - (1+r)R_t. \tag{5}$$

Beliefs. Household expectations can deviate from rational. We introduce beliefs in two steps by imposing general assumptions here and then describing the iterative process that level-k thinkers use to arrive at their exact forecasts in part "Level-k thinking" of the following Section 2.2.

First, we assume that all agents incorporate the central bank announcements into their beliefs. Second, we require that agent beliefs about *exogenous* variables—that is, W_t and ϵ_t^x—coincide with the *true* process of such variables.[8] Third, beliefs about endogenous variables can deviate from rational. We use the following notation for endogenous beliefs. Agent beliefs are a sequence of functions that map histories of exogenous shocks $\{\epsilon_t^x\}$ into actual realizations. These functions are presented in Section 2.2. We use a "tilde" to stress the fact that the belief about an endogenous variable may not be the same as its equilibrium counterpart. To simplify notation, we drop explicit reference to histories of

[7] Angeletos and Sastry (2018) show when it is optimal to the central bank to announce a policy target in a model without rational expectations.

[8] Gennaioli et al. (2015) and Bordalo et al. (2016) study the consequences of over-weighing the future likelihood of exogenous shocks that occurred in the recent past. It is promising to combine this assumption about exogenous variables beliefs with level-k endogenous variable beliefs in the future research.

shocks. Thus, for example, $\widetilde{q}_{t+s}$ denotes the belief about the asset price at time $t+s$, which may differ from q_{t+s}, i.e. the asset price in equilibrium. Finally, we use the expectation operator $\mathbb{E}_t[\cdot]$ to denote the average of a variable using the probability distribution of exogenous shocks.[9]

Fourth, we restrict possible beliefs further by assuming that households believe that the intertemporal budget constraint of the treasury (3) is satisfied. Intuitively, after observing a change in current revenues or spending by the treasury, households correctly anticipate the changes in future revenues or spending that are required to satisfy the treasury's intertemporal budget constraint. As a result, Ricardian equivalence (Ricardo, 1821; Barro, 1974) will hold when it comes to treasury's policies. This assumption does not change the main findings of this section. However, it becomes relevant in Section 3 where we study the effect of asset purchases on output. In particular, without this assumption, aggregate output will depend on when transfers from the central bank to the Treasury will materialize as taxes on households, making the dynamic implications of the model richer. For simplicity of exposition, we abstract from this additional effect.

Crucially, however, we do not require household beliefs to be consistent with the central bank's budget constraint (5). As a result, some households may fail to understand the exact amount of transfers that the central bank sends to the treasury. By making this assumption, we treat the central bank as an agent of the economy whose future actions, along with the future actions of the other households embedded in endogenous variables, are needed to be forecasted.

2.2 Equilibrium Concepts

When studying deviations from rational expectations, it is useful to start with a more general notion of equilibrium known as temporary equilibrium (Hicks, 1939; Lindahl, 1939; Grandmont, 1977; Woodford, 2013). A temporary equilibrium generalizes the standard notion of rational expectations equilibrium by relaxing the assumption that beliefs about endogenous variables must be consistent with the equilibrium distribution of such variables. Instead, in a temporary equilibrium, agent beliefs are free to deviate from their equilibrium counterparts. More specifically, a temporary equilibrium takes as given household beliefs about future endogenous variables and requires only that *(i)* house-

[9] Another standard way to describe beliefs is to use probability distributions over histories of endogenous variables. This latter formulation and the one employed in the paper are equivalent and amounts to a simple change of variables. To see this consider a static example in which the endogenous variable y is only a function of exogenous shock ϵ. Under our notation, agent's beliefs about y is captured by a function $\widetilde{y} = \widetilde{y}(\epsilon)$ and its expectation is $\mathbb{E}\widetilde{y} = \int \widetilde{y}(\epsilon) dP(\epsilon)$, where P is the cumulative distribution function (cdf) of ϵ. An alternative representation of the same expectations is $\widetilde{\mathbb{E}} y = \int y d\widetilde{P}(y)$, where $\widetilde{P}$ is the cdf of y. The two cdf's are related to each other through the function $d\widetilde{P}(\widetilde{y}(\epsilon)) = dP(\epsilon)$.

holds optimize given these beliefs and that *(ii)* markets clear in every period.

Let Z_t denote the vector of endogenous variables and government policies $Z_t = (q_t, T_t, Tr_t, B_{t+1}, X_{t+1}, R_{t+1})$ and $\widetilde{\mathcal{Z}}_t \equiv \{\widetilde{Z}_{t+1+s}\}_{s>0}$ denote a sequence of beliefs about future Z_t.

Definition (Temporary Equilibrium)**.** Conditional on beliefs $\{\widetilde{\mathcal{Z}}_t\}$, a temporary equilibrium is a collection of household choices $\{c_t, x_{t+1}, b_{t+1}\}$, government policies $\{\Pi_t\}$, and prices $\{q_t\}$ such that

1. Given beliefs and prices, households optimize for all t;
2. The risky-asset market clears for all t
$$x_{t+1} + X_{t+1} = 1, \tag{6}$$
3. The treasury and central bank's budget constraints (3)-(5) are satisfied for all t.

Note that the equilibrium definition does not feature the bonds market clearing condition, because we assumed that the supply of bonds is perfectly elastic and it satisfies any demand from households (at the net return r). Finally, the goods market clears by Walras' law.

We now define a mapping which will be convenient below. Given beliefs $\widetilde{\mathcal{Z}}_t$, we compute the temporary equilibrium and let $\mathcal{Z}_t \equiv \{Z_{t+1+s}\}_{s>0}$ be the sequence of future Z_t in such equilibrium. We can thus define a mapping from beliefs into equilibrium variables:

$$\mathcal{Z}_t = \Psi(\widetilde{\mathcal{Z}}_t, \{X_{t+1}\}). \tag{7}$$

In general, the sequence of equilibrium variables $\mathcal{Z}_t$ may differ from the sequence of household beliefs $\widetilde{\mathcal{Z}}_t$, except when agents hold rational expectations.

Definition (Rational Expectations Equilibrium)**.** A rational expectations equilibrium (REE) is a temporary equilibrium that satisfies

$$\mathcal{Z}_t = \widetilde{\mathcal{Z}}_t, \text{ for all } t.$$

Note that REE beliefs are a fixed point of the mapping (7). For future references we use a "star" to denote REE variables.

Level-k thinking. We now continue the description of household beliefs that we started in Section 2.1. In particular, we introduce beliefs about future *endogenous* variables through a specific process of belief formation, known as level-k thinking.

Suppose that, before the policy intervention, the economy is in its REE, that is, all agents hold rational expectations about future variables. We begin with level-1 agents, the lowest level of thinking.[10] We assume that, after the policy intervention in period $t = 0$, these agents do not change their beliefs about future endogenous variables. Formally, the beliefs of level-1 agents satisfy $\tilde{\mathcal{Z}}_t^1 = \tilde{\mathcal{Z}}_t^{SQ}$, for all periods $t \geq 0$, where the additional superscript denotes "level-1" beliefs and where $\tilde{\mathcal{Z}}_t^{SQ}$ denotes the "status quo" beliefs, which we define as the beliefs of an agent in the REE *before* the policy intervention. Since central bank interventions will be neutral under rational expectations, the sequences $\tilde{\mathcal{Z}}_t^{SQ}$ and $\tilde{\mathcal{Z}}_t^*$ can disagree only over taxes and transfers. Note that the beliefs of level-1 agents do not incorporate the effects of policy interventions, neither at the time when the policy is announced nor at any later date. This will not be the case if agents can update their beliefs over time, for example, through learning. We discuss this possibility in Section 2.6.

Let us emphasize that it is crucial that expectations of level-1 agents do not *fully* incorporate changes in future transfers into their beliefs. Without this assumption, balance sheet interventions would be irrelevant. There are two motivations for this assumption. First, level-1 agents could treat the central bank as another agent whose behavior, like the one of all other agents, needs to be forecasted if it is not credibly announced. In reality, when making announcements about QE, central banks do not usually announce also the path of future transfers. Second, if level-1 agents form beliefs by extrapolating historical data, when facing a new policy or a novel modification of the old policy, they will not be able to fully internalize the future effects of this policy.

Having specified the beliefs for level-1 agents, we can use the mapping (7) to obtain the sequence of equilibrium variables $\mathcal{Z}_t^1$ in an economy populated by level-1 thinkers. We then move to level-2 agents. We assume that any such agent is overconfident and believes that *all* other agents in the economy are level-1 thinkers. As a result, her beliefs about future endogenous variables satisfy $\tilde{\mathcal{Z}}_t^2 = \mathcal{Z}_t^1 = \Psi(\tilde{\mathcal{Z}}_t^{SQ}, \{X_{t+1}\})$. For example, a level-2 agent's belief about the asset price at some future time s satisfies $\tilde{q}_s^2 = q_s^1$, that is, it equals the actual asset price, at time s, in the temporary equilibrium of the economy with only level-1 thinkers.

Proceeding recursively, we define the beliefs of level-k agents for any $k \geq 1$.[11] Formally, for any $k > 1$, we use the mapping (7) to obtain the sequence of endogenous

[10] By convention, the lowest level of thinking is 0 in game theory, while it is 1 in macroeconomics.

[11] Camerer et al. (2004) propose a related model of "cognitive hierarchy" in which level-k thinkers assume that the other players are not only level-$(k-1)$, like in this paper, but also level-$(k-2)$ and so on. This alternative assumption retains most of the tractability of level-k thinking, but outperforms it in some applications. We can incorporate this alternative into our model at the expense of tractability. The main results will be unchanged.

variables $\mathcal{Z}_t^k$ in the temporary equilibrium in which agents are level-k thinkers with beliefs $\widetilde{\mathcal{Z}}_t^k$. We then use the sequence just obtained to define the beliefs $\widetilde{\mathcal{Z}}_t^{k+1}$ of level-$(k+1)$ thinkers. The entire process of belief formation is thus described by the following recursion:

$$\widetilde{\mathcal{Z}}_t^{k+1} = \Psi(\widetilde{\mathcal{Z}}_t^k, \{X_{t+1}\}), \tag{8}$$

for all $k \geq 1$ and $t \geq 0$.

Reflective equilibrium. Having defined the beliefs of level-k thinkers, for any k, we introduce the notion of equilibrium that we will use to investigate central bank interventions. We follow García-Schmidt and Woodford (2019) and consider an economy populated by households who are heterogeneous in their levels of thinking k. In particular, the population is divided into different groups depending on their beliefs. Each group contains households with the same level k. Groups have a mass given by the probability density function $f(\cdot)$, with $f(k) \geq 0$ and $\sum_{k=1}^{\infty} f(k) = 1$.

Definition (Reflective Equilibrium)**.** A reflective equilibrium is a collection of beliefs $\{\widetilde{\mathcal{Z}}_t^k\}_k$, household choices $\{c_t^k, x_{t+1}^k, b_{t+1}^k\}$, government policies $\{\Pi_t\}$, and prices $\{q_t\}$ such that

1. Given beliefs and prices, households optimize for all t;
2. The risky-asset market clears for all t

$$\sum_{k=1}^{\infty} f(k) x_{t+1}^k + X_{t+1} = 1; \tag{9}$$

3. The treasury and central bank budget constraints (3)-(5) are satisfied for all t;
4. Beliefs are generated through the mapping (8), starting from $\widetilde{\mathcal{Z}}_t^1 = \widetilde{\mathcal{Z}}_t^{SQ}$, for all t.

Reflective equilibrium generalizes a notion of temporary equilibrium with level-k thinkers by allowing agents with different levels of thinking to coexist in the economy. When $f(\cdot)$ assigns all the mass to some particular k, then the reflective equilibrium collapses to the temporary equilibrium of an economy where all agents have level-k beliefs. An advantage of using reflective equilibrium is that the economy is not indexed by a discrete level of thinking. Instead, by changing the mean of $f(\cdot)$, we can vary the average level of thinking in the economy and perform comparative statics in a continuous way. Finally, in the definition of reflective equilibrium, agent beliefs are still constructed using the level-k thinking process above. Thus, an agent with level of thinking k believes that all other agents in the economy are level-$(k-1)$ thinkers. This also implies that, although agents with a high k compute beliefs of a very high order, their predictions are not necessarily closer to actual equilibrium outcomes because average k in the population might be low.

2.3 Equilibrium Effects of Risky Assets Purchases

We now solve the household problem and then derive the temporary equilibrium for a general sequence of balance sheet policies.

To simplify the exposition, it is convenient to impose one more requirement on beliefs by assuming that $\widetilde{\mathcal{Z}}_t$ is such that the vector of the endogenous variables $\widetilde{z}_{s+1} \equiv (\widetilde{q}_{s+1}, \widetilde{Tr}_{s+1})$ can be represented as a linear function of the contemporaneous shock:

$$\widetilde{z}_{s+1} = \alpha_s + \beta_s \epsilon^x_{s+1}, \tag{10}$$

where α_s and β_s are vectors of deterministic functions of time. We use subscripts "q" and "Tr" to denote elements of the vectors α_s and β_s. Here, α_s represents the expected value of $\widetilde{z}_{s+1}$, while β_s captures the expected sensitivity of $\widetilde{z}_{s+1}$ to the aggregate shock. Note that we do not restrict endogenous variables other than $\widetilde{z}_{s+1}$, because either they do not enter the household problem directly or they can be expressed through $\widetilde{q}_{s+1}$ and $\widetilde{Tr}_{s+1}$.

The assumption of linearity in (10) is without loss of generality as it will be verified by all our notions of equilibrium. This result follows from the fact that the mapping (8) preserves linearity and that the initial condition is given by REE variables that satisfy (10).

Since preferences are exponential, beliefs are linear, and shocks are normally distributed, the household demand for risky assets has a simple closed-form solution.

Lemma 1. *Suppose household beliefs satisfy (10), then household asset demand is*

$$x_{t+1} = \frac{\mathbb{E}_t[D_{t+1} + \widetilde{q}_{t+1}] - (1+r)q_t}{\gamma \frac{r}{1+r}\sigma_x^2} - \beta_{Tr,t}. \tag{11}$$

This result, which we prove in Appendix A.1, is well known in the finance literature. The first term on the right-hand side of (11) shows that the demand for risky assets is proportional to their excess return (on a unit of risky asset) and inversely proportional to the coefficient of absolute risk aversion γ times the volatility of excess returns. The second term captures a hedging motive coming from the fact that the return on the risky asset may correlate with net labor income, which, in our simple setting where income is non-stochastic, coincides with the correlation between asset returns and taxes. Importantly, this correlation may vary when, as it will be the case here, the central bank conducts balance sheet policies and households realize that assets returns will affect future transfers to the treasury and, hence, taxes. If, for example, households expect future transfers to be positively correlated to the return on risky assets or, equivalently, future taxes to be negatively correlated with the risky return, then the risky assets will be a bad hedge against future tax risk. As a result, demand (11) will be lower.

A convenient property of (11) is that it does not depend on the optimal choice of consumption nor on investment in the riskless asset. As a result, we can impose the market-clearing condition in the risky-asset market, i.e., equation (6), and solve for the endogenous price without reference to the other markets. Formally, using (10), we can express the risky-asset price at time t as

$$q_t = \frac{\overline{D} + \alpha_{q,t} - \frac{r}{1+r}\gamma\sigma_x^2(1 - X_{t+1} + \beta_{Tr,t})}{1+r}. \tag{12}$$

Equation (12) illustrates that the risky-asset price in the temporary equilibrium is, in general, a function of both government holdings of risky assets and of household beliefs about future prices and transfers.

Finally, equilibrium transfers and creation of reserves by the central bank are obtained by substituting the pricing equation (12) into equations (4) and (5), which characterize the behavior of the central bank. Taken together, equations (3)-(5) and (12) define the mapping $\Psi(\cdot)$ in equation (7).

2.4 Neutrality under Rational Expectations

We next solve for the response of the economy to balance sheet policies in the REE, in which expectations are linear in fundamental shocks as in (10). By definition, in a REE subjective beliefs must be equal to equilibrium distributions. In particular, since the contemporaneous realization of the shock does not appear in equation (12), the equilibrium asset price must satisfy $\beta_{q,t} = 0$, implying

$$q_{t+1} = \mathbb{E}_t\widetilde{q}_{t+1} = \alpha_{q,t}.$$

The fact that the asset price is independent of the aggregate shock is not surprising, since neither demand nor supply of risky assets are stochastic. In addition, from the budget constraint of the central bank (5), we conclude that beliefs about transfers must satisfy $\beta_{Tr,t} = X_{t+1}$. With these two observations, we can rewrite equation (12) in a familiar way:

$$q_t = \frac{\overline{D} + q_{t+1} - \frac{r}{1+r}\gamma\sigma_x^2}{1+r}.$$

This standard asset pricing equation shows that the current price equals the discounted sum of expected dividends and future resale price minus the risk premium. We can then solve the above equation forward to obtain

$$q_t = \frac{1}{r}\left(\overline{D} - \gamma \frac{r}{1+r}\sigma_x^2\right) \equiv q^*. \tag{13}$$

The following proposition summarizes the key property of q^*.

Proposition 1. *In the REE, the price of the risky asset does not depend on balance sheet policies.*

Proposition 1 states that, when all the agents in the economy anticipate future central bank transfers correctly, balance sheet interventions are irrelevant. This result is similar to the one obtained by Wallace (1981). The intuition is simple. When expectations are rational, households correctly anticipate that future taxes will inherit the stochastic properties of the assets purchases by the central bank and, thus, adjust their demand for such assets. In the end, equilibrium prices are unaffected.

2.5 Non-neutrality under Level-k Thinking

We now depart from rational expectations and assume that, following an announcement of interventions in period $t = 0$, households form expectations following the level-k process described in Section 2.2. As explained in that section, we assume that, before the announcement, the economy is in the REE. Thus, absent a policy change, households would correctly forecast the behavior of future prices and transfers. Formally, status-quo beliefs $\widetilde{\mathcal{Z}}_t^{SQ}$ are a fixed point of (7) when $X_{t+1} = 0$, for all t.

Let q_t^k denote the asset price in the temporary equilibrium when all agents hold level-k beliefs. The pricing equation (12) implies

$$q_t^k = \begin{cases} \dfrac{\overline{D} + q^* - \frac{r}{1+r}\gamma\sigma_x^2(1 - X_{t+1})}{1+r}, & k = 1, \\ \dfrac{\overline{D} + q_{t+1}^{k-1} - \frac{r}{1+r}\gamma\sigma_x^2}{1+r}, & k > 1. \end{cases} \tag{14}$$

By assumption, this price coincides with level-$(k+1)$ agents' beliefs.

To understand equation (14), remember that, when solving their maximization problem, households need to form expectations about next period's asset price and transfers. The first line of equation (14) reflects the fact that, following a policy of asset purchases, level-1 agents do not revise their expectations. Instead, they think that the following period asset price and transfers will coincide with the status quo before the policy intervention, where $q_{t+1} = q^*$ and $Tr_{t+1} = 0$. In particular, the latter, which follows from equation (3) with $X_{t+1} = 0$, $t \geq 0$, implies that level-1 agents fail to understand that transfers are now risky and, hence, so are taxes. Instead, level-1 behave as if the X_{t+1} units of risky assets purchased by the central bank have disappeared from the economy. In turn, since they expect their future consumption to be less risky, they ask for a lower risk premium and, as a result, the asset price increases. Moving to the second line of equation (14), we see that level-k thinkers, for $k > 1$, expect next period's asset price to coincide with the asset price computed in the previous iteration, q_{t+1}^{k-1}. On the transfer side, these agents incorporate the budget constraint of the central bank into their expectations and, thus, correctly anticipate future tax risk. As a result, they fully hedge against such risk, thus, X_{t+1} disappears from the pricing equation.

The iterative process implied by equation (14) is depicted in Figure 1. The horizontal axis represents time and the vertical axis plots the level of thinking k. A bold dot corresponding to time t and level k represents the temporary equilibrium price q_t^k. The diagram shows visually that, if one wants to compute, for example, the asset price at time 0 in a temporary equilibrium with level-5 agents, one has to "iterate diagonally" and compute the asset price at time 1 in a temporary equilibrium with level-4 agents, q_1^4, the asset price at time-2 in a temporary equilibrium with level-3 agents, q_2^3, and so on.

Importantly, these iterations always stop at the point where the temporary equilibrium with level-1 agents is reached. Beyond that point, agents will no longer revise their expectations following a policy announcement. In fact, Figure 1 suggests that the economy displays a form of *endogenous discounting*. To see this, suppose that the economy is populated by level-k agents, with $k \leq 5$, and that, at time 0, the central bank announces that it will purchase risky assets in period 6. From (14), level-k agents do not react to events happening more than k periods ahead. Therefore, the announcement of the government will have no effect on asset prices at $t = 0$. To see this more formally, we can iterate equation (14) to compute the temporary equilibrium price at time t:

$$q_t^k = q^* + \frac{r}{1+r}\gamma\sigma_x^2 \frac{X_{t+k}}{(1+r)^k}. \tag{15}$$

As suggested by Figure 1, equation (15) depends only on asset purchases k-periods ahead.

Having characterized beliefs for any level of thinking, we turn to the reflective equilibrium, which considers an economy populated by agents who are heterogeneous in their level of thinking. Using Lemma 1, the market-clearing condition in the reflective equilibrium can be written explicitly in period t as

$$X_{t+1} + \sum_{k=1}^{\infty} f(k) \left[\frac{\overline{D} + \alpha_{q,t}^k - q_t(1+r)}{\gamma \frac{r}{1+r}\sigma_x^2} - \beta_{Tr,t}^k \right] = 1. \tag{16}$$

Using the beliefs of level-k households obtained above, we can then solve (16) for the price of the risky asset.

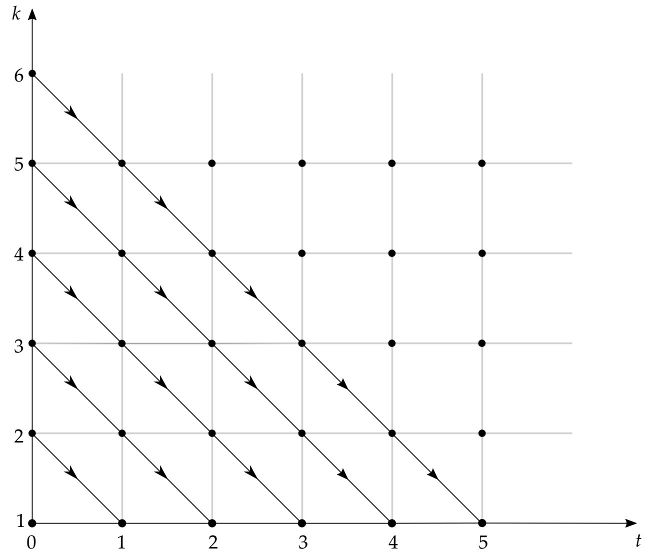

Figure 1: The price q_t^k of the risky asset in the temporary equilibrium where agents hold level-k beliefs. The horizontal axis plots time and the vertical axis plots the level of thinking k. Every bold dot represents q_t^k corresponding to a given pair (t, k). The arrows point to the direction of "diagonal iterations" required to compute the price q_t^k.

Proposition 2. *The asset price in the reflective equilibrium depends on the entire future path of risky asset purchases $\{X_{t+1}\}$ and satisfies*

$$q_t = q^* + \frac{r}{1+r}\gamma\sigma_x^2 \sum_{k=1}^{\infty} f(k) \frac{X_{t+k}}{(1+r)^k}. \tag{17}$$

Proposition 2 shows that, in the reflective equilibrium, balance sheet policies are an effective tool for controlling the price of risky assets. Moreover, since the risk-free real return is fixed at r, any change in the risky asset price can be immediately interpreted as a change in the risk premium required by investors. This is not surprising: the reason for why prices change is that some households fail to understand how asset risk translates into future tax risk, thus, they underestimate the total amount of risk in the economy.

An important consequence of equation (17) is that the average level of thinking of the agents in the economy—defined as the mean of $f(k)$—has two counteracting effects on the strength of balance sheet policies. First, a lower average level of thinking implies that fewer agents internalize the future fiscal consequences of balance sheet policies. This effect tends to make balance sheet policies stronger. Second, following our discussion of Figure 1, a lower average level of thinking implies a higher endogenous discounting, thus, more agents disregard the effects of balance sheet policies in the distant future. This effect tends to make balance sheet policies weaker.

To illustrate these two effects formally, we compute the risky-asset price following the announcement, at time 0, of a one-time purchase of risky assets in the following period, i.e., $X_2 > 0$ and $X_{t+1} = 0, t \geq 0, t \neq 1$. For the sake of specificity, we also assume that $f(k)$ is the probability distribution function of an exponential distribution, i.e., $f(k) = (1 - \lambda)\lambda^{k-1}$, $\lambda \in [0, 1)$. With this distribution, the average level of thinking in the economy $\bar{k}$ is $1/(1-\lambda)$. From equation (17), the risky asset price at time 0 equals

$$q_0 - q^* = \frac{r}{1+r} \cdot \frac{\gamma\sigma_x^2}{(1+r)^2} \left(\frac{1}{\bar{k}} - \frac{1}{\bar{k}^2} \right) X_2.$$

The nonlinear effect of $\bar{k}$ on the price is clear. If $\bar{k} = 1$, then the policy is completely ineffective because the discounting effect is so strong ($\bar{k} = 1$ if and only if all agents are level-1) that households do not react even to policies implemented in the very near future. As $\bar{k}$ increases above 1, the endogenous discounting effect becomes weaker and the policy gains power, provided that $\bar{k}$ is below 2. The policy strength peaks at $\bar{k} = 2$ and then declines again to zero as $\bar{k}$ approaches infinity, that is, when the equilibrium approaches the REE.

2.6 Discussion of Alternative Assumptions

We now discuss how the results change (or not) under the alternative assumptions that (i) the central bank purchases domestic or foreign nominal bonds instead of real bonds, (ii) agents learn over time, (iii) agents' level of thinking increases over time, and (iv) a fraction of agents holds rational expectations.

Purchases of Domestic and Foreign Nominal Public Bonds. So far, we have focused on purchases of private, real risky assets. Balance sheet policies, however, are not confined to such assets. Central banks often intervene by purchasing public domestic and foreign debt.[12]

Our model can shed light on the effectiveness of all these policies; it is enough to reinterpret the total supply of risky assets as a real stock of domestic or foreign public bonds. With a non-trivial risk of default, domestic and foreign public bonds are risky. Even when default risk is negligible, public bonds (usually fixed-income securities) are risky in *real* terms in the presence of inflation or exchange rate risk. As a result, the insights obtained in the simple model apply to purchases of domestic and foreign public

[12] For example, in November 2010, the Federal Reserve announced the purchase of $600 billion worth of US Treasury securities (this operation was dubbed "QE2"), and in September 2011, disclosed a plan to purchase long-term public bonds by selling short-term bonds (this intervention was dubbed "Operation Twist," after a similar policy action implemented in 1961). See the November 3, 2010 FOMC statement for the details on QE2, the September 21, 2011 FOMC statement for further information about the 2011 Operation Twist, and Alon and Swanson (2011) for the discussion of the 1961 Operation Twist.

bonds. To show this, in Appendix C.1, we add nominal variables and long-term public bonds to the model. In particular, we assume that long-term bonds carry inflation risk. Moreover, in Appendix C.2, we build a two-country extension of the simple model to study the effects of sterilized FX interventions. The two extensions demonstrate formally that the logic of the simple-model carries over to these richer environments.

Learning. In our discussion of beliefs formation through level-k thinking, we assumed that level-1 agents always hold the status-quo beliefs, even though they are proved wrong by unfolding events. The absence of learning by these agents can be thought of as a limit case in which the speed of learning goes to zero and agents *never* change them. We made this assumption both for simplicity and to highlight the role of level-k thinking.

It is easy to extend the model to allow for statistical learning.[13] There are two important consequences that follow from this extension: the first one highlights the key difference between statistical learning (and its close cousin of "adaptive expectations") and level-k thinking, while the second one is about their combination.

When the economy is only populated by level-1 agents who can update their statistical model of the world, the effects of balance sheet policies can change over time and can even disappear if agents learn the fiscal consequences of central bank interventions fully. The exact dynamics depend on the specific model of learning (e.g., on the set of variables considered), however, we can illustrate the point by assuming that agent forecasts are

$$\begin{aligned}\widetilde{q}^1_{s+1|t-1} &= \alpha^1_{q,s|t-1},\\ \widetilde{Tr}^1_{s+1|t-1} &= \alpha^1_{Tr,s|t-1} + \beta^1_{Tr,s|t-1}\epsilon^x_{s+1},\end{aligned}$$

where $s > t$. These forecasts are the analogue of (10), except that now the notation must take into account that expectations depend on the information available to the agent at the time the forecast is made. Note that we have not assumed any particular statistical process for the updating of α's and β's. The demand for the risky asset is given by equation (11) where $\widetilde{q}_{t+1}$ and $\beta_{Tr,t}$ are replaced with $\alpha^1_{q,t|t-1}$ and $\beta^1_{Tr,t|t-1}$, respectively. By market clearing, we thus have

$$q^1_t = \frac{\overline{D}}{1+r} + \frac{1}{1+r}\alpha^1_{q,t|t-1} + \gamma\sigma^2_x\frac{r}{(1+r)^2}(X_{t+1} - 1 - \beta^1_{Tr,t|t-1}).$$

This formula clearly demonstrates that learning adds a backward-looking force to the as-

[13] Barberis et al. (2015) study a CARA-normal framework similar to our simple model but by assuming that some agents form beliefs by extrapolating past realizations of endogenous variables into the future, while others hold rational expectations.

set price, which was absent in the baseline model. The formula also highlights the key difference between statistical learning and level-k thinking. The former is backward looking, while the latter is forward looking in nature. Therefore, if the government announces an unprecedented policy intervention, statistical learners will look at this intervention through the lens of past data, while level-k thinkers will reason through the model of the economy to anticipate the effects of such intervention.

Note that, by making the forecast of transfers dependent on the shock to dividends, we are allowing agents to learn about the correlation between taxes and shocks. In fact, agents may potentially learn the true value $-\beta_{Tr,t} = 1 - X_{t+1}$. If, instead, agents were to use a misspecified model in which the true $\beta_{Tr,t}$ was a constant parameter (rather than a time-dependent function of interventions), they would never uncover the true fiscal consequences of the interventions.

When combined together, learning and level-k thinking may lead to novel insights. As suggested by Figure 1, when forming their beliefs, level-k agents will have to take into account that level-1 agents will learn over time. As a result, level-k thinking "brings to the present" any anticipated behavior of the economy that originates from the learning process of level-1 agents. This effect is more powerful, the higher the level of thinking of the agents. For example, if learning makes future interventions less effective, then the response of the current asset price will be smaller compared to the case in which level-1 agents fail to learn or learn very slowly.

Unraveling. An alternative to statistical learning is a process of *unraveling* that assumes that agents increase their level of thinking over time. One justification for this assumption is that, as time passes, the agents have more time to think through the model and, as a result, they go through more stages of belief formation. In Appendix C.4, we consider such an extension. We show that unraveling leads to converges to the REE, so that balance sheet policies become eventually ineffective. Interestingly, as the power of balance sheet policies fades away, it is no longer enough, for a government wishing to exert a constant effect on asset prices, to keep the size of the intervention constant. Instead, the amount of asset purchases needs to grow over time to counteract the unraveling effect.

Presence of rational expectations agents. In the simple model, we assumed that all agents are level-k thinkers. A natural question is whether the presence of households who hold correct expectations about future endogenous variables can restore the neutrality result of balance sheet policies. The answer to this question is negative. Intuitively, when rational expectations agents are risk averse, they do not take large and risky arbitrage positions. We demonstrate this result formally in Appendix C.5. Interestingly and perhaps surprisingly, the presence of rational-expectations agents can amplify the effects of balance sheet policies on the asset price. The reason is that rational-expectations agents are fully forward looking relative to any level-k thinker with finite k. Thus, when a balance sheet policy persists over time, rational-expectations agents take into account the entire future path of the policy.

3 Balance Sheet Policies and Aggregate Output

In many cases the ultimate goal of balance sheet policies is to affect real economic activity, such as aggregate output. The previous section presented the simple model, where both labor income and dividends were exogenous. That model allowed us to study the impact of these policies only on asset prices and taxes. In this section, we investigate how balance sheet policies affect the real economy.

To answer this question, we extend the simple model so that aggregate output is endogenous due to the presence of nominal price rigidities. This choice is motivated by two reasons. First, central banks routinely use models with nominal rigidities to analyze the effects of balance sheet policies. Second, a strand of the empirical literature has shown that such models are successful at capturing some key properties of the data. We build a model that is both simple enough to be analyzed without having to rely on numerical simulations and yet rich enough to deliver important insights into the effects of balance sheet policies.[14] Specifically, we assume that prices are infinitely rigid. This assumption makes the analysis very tractable by allowing us to abstract from the firms' price-setting decision and, hence, the Phillips curve. Importantly, when prices cannot change, the short-term nominal interest rate coincides with the real interest rate. In addition, we assume that the central bank follows a simple interest rate rule: it keeps the nominal interest rate constant over time.[15] This assumption is motivated by the fact that balance sheet policies—such as quantitative easing—have been implemented precisely when the nominal interest rate was constrained at zero. Together, these two assumptions reduce the standard three-equation New-Keynesian model to a model that can be characterized by just one Euler equation.

We extend the model in two additional ways. We let the utility function to be a any strictly increasing, concave, and twice-differentiable function $u(\cdot)$, instead of CARA. This generality is important: the shape of the utility function will affect the magnitude and

[14] Qiu (2018) quantifies the effects of conventional monetary policies in a calibrated New Keynesian model, extended with level-k thinking, but without aggregate risk.

[15] Instead of the two assumptions of completely sticky prices and a constant nominal interest rate, we could alternatively assume that the central bank sets a constant real interest rate. This would result in identical model predictions, without requiring prices to be fully rigid.

sign of the output effects of balance sheet policies. In addition, we allow dividends to be any differentiable function of aggregate output.[16] In contrast, in Section 2, the central bank could only purchase an asset whose dividends were a linear and increasing function of aggregate output. As we show below, the characteristics of the asset targeted by the central bank are key to understand the effects of balance sheet policies. The addition of all these non-linearities makes it impossible to solve the model in closed form. We thus solve the model by approximating equilibrium relations around the REE where no interventions occur.[17]

3.1 Markets, Beliefs, and Agents

We now lay out the model details and emphasize the differences with the simple model.

We let Y_t denote aggregate output and assume that the endowment ("labor income") equals a fraction of total output: $W_t = (1-\delta)Y_t$, $\delta \in (0,1]$. The remaining fraction is distributed as dividends to asset owners. By assuming that a part of endogenous output is distributed as endowment, we effectively avoid introducing the labor market explicitly (Caballero and Farhi, 2017). Agents can trade two risky assets and one safe asset in every period t. Risky asset i, with $i = 1, 2$, trades at price $q_{i,t}$ and pays dividends $D_{i,t} = D_i(\delta Y_t)$, where $D_i(\cdot)$ is a differentiable function. We finally assume that $D_1(\delta Y) + D_2(\delta Y) = \delta Y$. The safe asset has a fixed real net return of r.

Beliefs are defined as in the simple model. Similarly, we let $\widetilde{\mathcal{Z}}_t^k$, $\widetilde{\mathcal{Z}}_t^{SQ}$, and $\widetilde{\mathcal{Z}}_t^*$ denote level-k beliefs, rational-expectations beliefs before ("status quo") and after the intervention, respectively. In the extended model, $Z_t \equiv (Y_t, q_{1,t}, q_{2,t}, T_t, Tr_t, B_{t+1}, X_{t+1}, R_{t+1})$.

Household problem. At time 0, given beliefs and prices, households choose consumption $\{\widehat{c}_t\}$, investment in risky assets $\{\widehat{x}_{i,t+1}\}$, and net worth $\{\widehat{a}_{t+1}\}$ (defined as the sum of financial wealth and current after-tax income) so as to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} e^{-\rho t + \mathcal{B}_t} u(\widehat{c}_t), \tag{18}$$

subject to the flow budget constraints

$$\widehat{a}_{t+1} = (1+r)(\widehat{a}_t - \widehat{c}_t) + \sum_{i=1,2} \widetilde{ER}_{i,t+1} \widehat{x}_{i,t+1} + (1-\delta)\widetilde{Y}_{t+1} - \widetilde{T}_{t+1}, \tag{19}$$

[16] In a previous version of this paper, we solved the model of this section under CARA preferences and linear assets in closed form. However, those assumptions are too strong and hide many important results. We thank an anonymous referee for pointing this out.

[17] This approximation is accurate in several cases. For example, with its QE1 program, the Federal Reserve purchased only \$1.25 trillion of mortgage-backed securities which is a fraction of the overall value of the MBS market of about \$10 trillion at that time (see the March 16, 2010 FOMC statement).

where $\widetilde{ER}_{i,t+1} \equiv \widetilde{D}_{i,t+1} + \widetilde{q}_{i,t+1} - (1+r)\widetilde{q}_{i,t}$ is the belief about the one-period-ahead excess return on a unit of risky asset i (which we will simply call excess return). Also $\mathcal{B}_t \equiv \sum_{j=0}^{t} \epsilon_{j-1} - \epsilon_{-1}$, where $\{\epsilon_t\}$ are independently and identically distributed shocks over time. The shock ϵ_t affects the subjective discount factor between periods t and $t+1$, which equals $e^{-\rho+\epsilon_t}$: a high realization of ϵ_t increases discounted future marginal utility of consumption and makes households willing to postpone their current consumption.[18] Due to price stickiness, changes in consumption demand will, in turn, affect output in equilibrium.

The problem is recursive in net worth a_t, the current shock ϵ_t, and beliefs $\widetilde{\mathcal{Z}}_t$. As in the simple model, the decisions that the household plans to implement at any point in the future may differ from the decisions that the agent will actually make. We denote the value function at time t as $V_t = V(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t)$ and actual choices as $c_t = c(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t)$, $a_{t+1} = a(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t)$, and $x_{i,t+1} = x_i(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t)$, $i = 1, 2$. They satisfy the Euler equations:

$$u'(c_t) = (1+r)e^{-\rho+\epsilon_t}\mathbb{E}_t[V_a(a_{t+1}, \epsilon_{t+1}; \widetilde{\mathcal{Z}}_{t+1})], \tag{20}$$

$$q_{i,t} = \frac{\mathbb{E}_t[V_a(a_{t+1}, \epsilon_{t+1}; \widetilde{\mathcal{Z}}_{t+1})(\widetilde{D}_{i,t+1} + \widetilde{q}_{i,t+1})]}{(1+r)\mathbb{E}_t[V_a(a_{t+1}, \epsilon_{t+1}; \widetilde{\mathcal{Z}}_{t+1})]}, \tag{21}$$

where V_a denotes the partial derivative of the value function with respect to its first argument.

Government. The government is the same as in the simple model. In particular, the intertemporal budget constraint of the treasury is still given by (3). Moreover, we maintain the assumption that the central bank finances its stream of purchases by creating reserves. Without loss of generality, we assume that the central bank trades the first asset. As a result, equation (4) becomes

$$R_{t+1} = q_{1,t} X_{t+1}, \tag{22}$$

for all s, where $q_{1,t}$ is the equilibrium price of the first asset. Finally, we focus on the effects of small interventions and we will only analyze the case of a geometric path for purchases

$$X_{t+1} = \mu^t \overline{X}, \tag{23}$$

where $0 \leq \mu \leq 1$ and for some initial value $\overline{X} > 0$.[19]

[18] Note that the shock ϵ_t does not bear the index x to differentiate it from the shock to dividends in the simple model.

[19] The results of this section do not change if instead we consider a general intervention of the type $\overline{X} \cdot (\chi_{t+1}, \chi_{t+2}, ...)$ and approximate equilibrium variables with Gateaux derivatives in the direction

3.2 Consumption Demand and Aggregate Output

We now derive the output effects of balance sheet policies, the main result of this section.

Temporary, rational, and reflective equilibria. All the notions of equilibrium introduced in Section 2—temporary, rational-expectations, and reflective—extend naturally to the environment of this section. The main difference is that we need to impose that aggregate demand for consumption equals aggregate output in every period. For brevity, we do not repeat the definitions here. Moreover, we keep the notation of Section 2 for endogenous variables in equilibrium: a "star" for the REE, a superscript "k" for the level-k equilibrium, and no superscript for the reflective equilibrium.

We first discuss the REE benchmark. When solving for the response of the economy to policy changes in a REE, we need to take care of indeterminacy of equilibria. It is well known that, when expectations are rational, a constant real interest rate may lead to multiple equilibria. In standard New-Keynesian models, local uniqueness of the equilibrium is achieved by requiring some form of the *Taylor principle* to hold (Woodford, 2003). Our assumption of a fixed nominal interest rate, together with fixed goods prices, does not satisfy this principle. To get around this complication and, at the same time, retain the tractability brought by our assumptions, we employ a simple equilibrium selection. We first solve the model for a REE in the absence of policy interventions, which will serve as the status quo in the construction of level-k beliefs. Then, after the policy change takes place, we solve for the response of the economy by requiring that, in the long run, the REE after the policy change converges to the chosen REE before the change. This approach, which is illustrated in Farhi and Werning (2017), uniquely pins down the response of the economy to the policy intervention.

We next characterize the marginal effects of balance sheet policies under the assumption that agents hold level-k beliefs. More specifically, we evaluate the Euler equations (20) and (21) at equilibrium values, derive a first-order approximation of them, and use it to characterize the effect of balance sheet interventions on output and asset prices.

Let A_t^* be the equilibrium value of net worth in the REE and let V_t^* be the value function evaluated at such value, i.e., $V_t^* \equiv V(A_t^*, \epsilon_t; \widetilde{\mathcal{Z}}_t^*)$. We also let $V_{a,t}^*$, and $V_{aa,t}^*$ denote, respectively, the equilibrium values of the first and the second partial derivative of V with respect to its first argument, evaluated at the REE. We define the counterparts in the equilibrium with level-k thinkers analogously. Finally, we use the symbol Δ to denote that a variable is in deviation from its REE value.

Our analysis focuses on the first-order effects of asset purchases. We denote with $o(\overline{X})$

$(\chi_{t+1}, \chi_{t+2}, ...)$. This more general formulation, however, comes at the cost of making the notation more cumbersome. As a result, we present our results for the case of $(\chi_{t+1}, \chi_{t+2}, \chi_{t+3}, ...) = (1, \mu, \mu^2, \dots)$.

all the higher-order terms in $\overline{X}$ such that $o(\overline{X})/\overline{X} \to 0$ as $\overline{X} \to 0$. The following lemma summarizes the first-order effects of the assets purchases on output and assets prices.

Lemma 2. *Following a balance sheet policy that satisfies (23), output and asset prices in the temporary equilibrium with level-k thinkers are, respectively,*

$$Y_t^k = Y_t^* + \frac{(1+r)e^{\epsilon_t - \rho}}{u''(Y_t^*)}\mathbb{E}_t[\Delta \widetilde{V}_{a,t+1}^k] + o(\overline{X}), \tag{24}$$

$$q_{i,t}^k = q_{i,t}^* + \frac{\mathbb{E}_t[V_{a,t+1}^* \cdot (D_i'(\delta Y_{t+1}^*)\delta \Delta \widetilde{Y}_{t+1}^k + \Delta \widetilde{q}_{i,t+1}^k)]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]} + \frac{\mathbb{E}_t[\Delta \widetilde{V}_{a,t+1}^k ER_{i,t+1}^*]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]} + o(\overline{X}), \tag{25}$$

where $\Delta \widetilde{Y}_{t+1}^k = \Delta Y_{t+1}^{k-1}$, $\Delta \widetilde{q}_{i,t+1}^k = \Delta q_{i,t+1}^{k-1}$, *for* $k > 1$, *and* $\Delta \widetilde{Y}_{t+1}^k = \Delta \widetilde{q}_{i,t+1}^k = 0$, *for* $k = 1$.

The proof follows by simply computing the differentials of the Euler equations (20) and (21); we present the details in Appendix A.2.2. The lemma states that the change in the expected marginal value of wealth, $\Delta \widetilde{V}_{a,t+1}^k$, is a key object for understanding the first-order impact of the intervention on output and asset prices. In fact, $\Delta \widetilde{V}_{a,t+1}^k$ is a sufficient statistic for the change in future net worth and in the agent's beliefs induced by asset purchases. The next lemma provides a general expression for $\Delta \widetilde{V}_{a,t+1}^k$.

Lemma 3. *Suppose that balance sheet policies satisfy (22) and (23). Then, if* $k > 1$,

$$\Delta \widetilde{V}_{a,t+1}^k = V_{aa,t+1}^*[\Delta Y_{t+1}^{k-1} + \sum_{i=1,2} \Delta q_{i,t+1}^{k-1}] + d\mathcal{Y}_{t+1}'(A_{t+1}^*) + o(\overline{X}), \tag{26}$$

where $d\mathcal{Y}_t(a) \equiv V_a(a, \epsilon_t; \mathcal{Z}_t^*)\mathbb{E}_t \sum_{s=1}^{\infty} m_{t,t+s}[(1-\delta)\Delta Y_{t+s}^{k-1} + \sum_{i=1,2}(\Delta ER_{i,t+s}^{k-1} x_{i,t+s})]$ *is the marginal value function times the present value of all future changes in income and asset prices; if* $k = 1$,

$$\Delta \widetilde{V}_{a,t+1}^1 = -V_{aa,t+1}^* ER_{1,t+1}^* \mu^t \overline{X} + o(\overline{X}). \tag{27}$$

Consider first the case with $k > 1$. The previous lemma states that asset purchases affect marginal utility of wealth through two channels. First, $V_a(A_{t+1}^*, \epsilon_{t+1}; \widetilde{\mathcal{Z}}_{t+1}^*)$ is a function of individual net worth, which, in equilibrium, equals total income plus the total value of the two assets. Since asset purchases impact net worth directly, we have the first term in (26).

Second, following an announcement of asset purchases, agents with $k > 1$ update their beliefs about future income and asset prices. This channel is captured by the second term in (26). The exact dependence of the marginal utility of wealth on future beliefs depends on the specific assumptions on the utility function and the underlying stochastic process of shocks. Below, we consider several special cases in which this channel takes a simple form. There is, however, a general lesson that we can learn from the term $d\mathcal{Y}_t'(A_{t+1}^*)$.

When agent beliefs about future income and asset prices change, the agent responds by changing future portfolio plans, i.e., the amount $\widehat{x}_{i,t+s}$ invested in risky assets. In addition, since markets are incomplete, the value of a risky stream of income cannot, in general, be inferred from market prices. Instead, it depends on the agent's wealth in the different states of the world. As a result, different beliefs about the future also change the agent's stochastic discount factor $m_{t,t+s}$ and, thus, her valuation of future income.

Let's now consider the equilibrium with level-1 thinkers. These agents do not update their beliefs following an announcement of asset purchases. From their point of view, therefore, when the central bank exchanges risky assets for risk-free reserves, it removes both the risk and, at the same time, the average excess return associated with the risk premium on this asset. To illustrate these two forces, we combine equation (24) with equation (27) and rewrite output as

$$Y_t^1 = Y_t^* + \frac{(1+r)e^{\epsilon_t - \rho}}{-u''(Y_t^*)} \{ \underbrace{cov_t(V^*_{aa,t+1}, ER^*_{1,t+1})}_{\equiv \mathcal{R}_t} + \underbrace{\mathbb{E}_t[V^*_{aa,t+1}] \cdot \mathbb{E}_t[ER^*_{1,t+1}]}_{\equiv \mathcal{M}_t} \} \mu^t \overline{X}, \tag{28}$$

where, to simplify the exposition, we have disregarded the term $o(\overline{X})$.

Equation (28) decomposes the excess return on a unit of the targeted asset $ER^*_{1,t+1}$ into a risky mean-zero component $ER^*_{1,t+1} - \mathbb{E}_t[ER^*_{1,t+1}]$, and a riskless component $\mathbb{E}_t[ER^*_{1,t+1}]$. The former yields the term $\mathcal{R}_t$, which captures the impact of the change in portfolio *risk* caused by central bank purchases. The sign and the value of this term depend on how the second derivative of the value function $V^*_{aa,t+1}$ covaries with the excess returns $ER^*_{1,t+1}$. This covariance, in turn, is a function of the *third* derivative of the value function. Thus, the term $\mathcal{R}_t$ captures a *precautionary saving motive*, whereby agents adjust consumption—which, in turn, leads output to change—in response to the risk they face. The term $\mathcal{M}_t$ reflects the impact of the change in the average agent's portfolio return following the central bank intervention. When the value function is concave—which is the case when agents are risk averse—$\mathcal{M}_t$ is negative if the asset commands a risk premium and it is positive if the asset provides a hedge against the overall risk faced by the agent.

The two terms in equation (28) make the sign of $Y_t^1 - Y_t^*$ ambiguous in general. In the following sections, we present examples where the output effect has different signs. Before we proceed, we note that changes in output and asset prices in the reflective equilibrium—where agents with different levels of thinking coexist—are simply equal to a weighted average of their counterparts in the level-k equilibria. The following lemma summarizes this observation; its proof is in Appendix A.2.4.

Lemma 4. *An infinitesimal purchase of risky asset* 1 *by the central bank has the following effects on output and prices in reflective equilibrium:* $Y_t^{RE} = Y_t^* + \sum_{k=1}^{\infty} f(k) \Delta Y_t^k + o(\overline{X})$ *and* $q_{i,t}^{RE} =$

$q^*_{i,t} + \sum_{k=1}^{\infty} f(k)\Delta q^k_{i,t} + o(\overline{X})$, where $f(k)$ is the probability distribution function across levels of thinking.

By definition, in the reflective equilibrium, any agent of any level of thinking k holds the same beliefs as in a level-k equilibrium. The reason why Lemma 4 does not follow immediately from the definition of reflective equilibrium is that, although agents do not change their beliefs about *future* variables, they respond to *current* output and prices, which now must clear markets where agents with different levels of thinking coexist. Lemma 4 states that, at least to a first order, these latter responses cancel each other out.

3.3 The Role of Preferences

We start our investigation of the forces governing the effects of central bank purchases by considering the role of preferences. In particular, we show that household preferences can lead to a negative, zero, or positive effect of central bank asset purchases on output, with the positive outcome achieved with one of the most widely-used preferences in macroeconomics.

We make two assumptions, which we relax in the next section. First, we focus on a temporary intervention that lasts only for one period ($\mu = 0$). Second, we assume that the asset traded by the central bank pays pro-cyclical dividends, that is, we assume that the function $D_1(\cdot)$ is strictly increasing. With i.i.d. shocks, asset prices are constant, thus, pro-cyclical dividends are equivalent to pro-cyclical returns. These assumptions allow us to make the key point of this section in the most transparent way.

Since the intervention lasts only one period, by equation (27), the marginal utility of wealth of level-1 agents will be affected only in period 0. As a result, by Lemma 2, Y^1_t and $q^1_{i,t}$ will differ from their REE counterparts only in period 0. For all other k, by Lemma 3, $Y^k_t = Y^*_t$ and $q^k_{i,t} = q^*_{i,t}$. Thus, according to Lemma 4, the effects of asset purchases in the reflective equilibrium are fully characterized by Y^1_t and $q^1_{i,t}$. In particular, the asset-price effect can be deduced by simply combining equations (25) and (27):

$$q^1_{1,t} = q^*_{1,t} + \frac{1}{1+r} \cdot \frac{\mathbb{E}_t[-V^*_{aa,t+1}(ER^*_{1,t+1})^2]}{\mathbb{E}_t[V^*_{a,t+1}]} \mu^t \overline{X},$$

where, to simplify the exposition, we have disregarded the term $o(\overline{X})$. Therefore, the effect of QE on the price of the asset targeted by the central bank is positive as long as the value function is strictly concave. In fact, this result does not require the assumption that dividends are pro-cyclical. We only need that $ER^*_{1,t+1}$ is non-zero in some states of the world.

To characterize the effect of QE on output, it is convenient to introduce the coefficient of absolute risk aversion (ARA) of the *value function* $g_{t+1} \equiv -V_{aa,t+1}/V_{a,t+1}$. This coefficient is a function, among other variables, of individual net worth and is positive when the value function is concave. We let g^*_{t+1} be its value at the REE. We can then use this coefficient to re-express the terms $\mathcal{R}_t$ and $\mathcal{M}_t$ in equation (28) as follows:

$$Y^1_t = Y^*_t - \frac{(1+r)e^{\epsilon_t - \rho}}{-u''(Y^*_t)} \mathbb{E}_t \left[g^*_{t+1} V^*_{a,t+1} ER^*_{1,t+1} \right] \mu^t \overline{X}, \tag{29}$$

where, again, we have disregarded the term $o(\overline{X})$. This equation provides important insights on the sign of output effect ΔY^1_t. Recall that, by simply rewriting the Euler equation for the risky assets (21) in the REE, we obtain $\mathbb{E}_t[V^*_{a,t+1} ER^*_{1,t+1}] = 0$. Therefore, the expectation in equation (29) and, thus, the first-order effect on output may differ from zero only insofar as the term g^*_{t+1} varies with the realizations of shocks. The next proposition summarizes several important cases in which we can determine the sign of the output effect.

Proposition 3. *When the dividends on targeted assets are strictly increasing in output, the first-order effect on output is positive when the ARA coefficient of the value function is decreasing in wealth ($g_{a,t+1} < 0$); it is negative when the ARA coefficient is increasing in wealth ($g_{a,t+1} > 0$); and it is zero when the ARA coefficient is constant ($g_{a,t+1} = 0$).*

The proof of this proposition is in Appendix A.2.5. The intuition is as follows. When the central bank trades risk-free reserves for risky assets—which, in equilibrium, carry a positive risk premium—it does not change the total value of household net worth at the time the trade occurs. However, the central bank intervention impacts household portfolios by reducing both their average return and their risk in the following period. Proposition 3 states that, to assess the overall impact of these two channels on output, at least to a first order, we need to understand how the agent's attitude towards risk varies with the agent's net worth in the next period.

In particular, when the ARA coefficient is constant, the negative effect on consumption induced by the lower average return is exactly offset by the positive effect induced by the lower risk and the resulting decline in precautionary saving. Instead, when g_{t+1} decreases with individual net worth, by reducing future average returns, central bank purchases make households poorer in the future and, thus, more averse to risk. As a result, the positive effect on consumption arising from the lower risk in the agent's portfolio is amplified by the fact that agents become more responsive to changes in risk, thus, consumption and output increase. The opposite occurs when g_{t+1} increases with net worth.

It is worth highlighting that the coefficient of absolute risk aversion in Proposition 3 is defined using the value function. It turns out, however, that, when the flow utility function exhibits hyperbolic absolute risk aversion (HARA), this coefficient is exactly the same as its analogue defined using the flow utility function. This class of utility functions generalizes many standard preferences, including constant relative risk aversion (CRRA), constant absolute risk aversion, and increasing absolute risk aversion. We prove this result in Appendix A.2.6.

Finally, the general lesson of Proposition 3 is that agents' attitude towards risk is crucial to determine the effects of QE on output. The empirical evidence suggests that household preferences are best described by decreasing absolute risk aversion, such as CRRA preferences (e.g., Guiso and Paiella, 2008). These preferences are also very popular in the macro-finance literature. An important reason for this is that they are consistent with the simple fact that rates of returns have remained reasonably stable in developed countries over the last 150 years despite a tenfold increase in consumption (Jagannathan et al., 1996). As a result, according to our channel, it is reasonable to expect that purchases of pro-cyclical assets will lead to a positive output effect.

3.4 The Role of Assets

We now study how asset characteristics affect the outcome of balance sheet interventions. For concreteness and realism, we focus on the case of preferences with decreasing absolute risk aversion. In the previous section, we studied the effects of a one-period intervention in which the central bank purchased an asset with pro-cyclical payouts. From Proposition 3, this type of intervention increases aggregate output.

Suppose, instead, that the central bank targets an asset with counter-cyclical payouts, that is, we assume that $D_1(\cdot)$ is strictly decreasing. With i.i.d. shocks and, hence, constant asset prices, counter-cyclical dividends are equivalent to counter-cyclical returns. An example of such a security is nominal long-term public bond. In the absence of default risk, such bonds typically pay constant nominal coupons, which imply counter-cyclical returns since inflation falls in recessions and rises in booms. They are thus a natural hedge against business-cycle risk (Campbell et al., 2017, Forthcoming).

When dividends are negatively related to aggregate output, all the conclusions of Proposition 3 hold with the opposite sign. Specifically, when preferences exhibit decreasing absolute risk aversion, an intervention that targets securities that provide a hedge against business cycles lowers aggregate output. The proof and the intuition for this result are mirror images of those for Proposition 3. What is more, the next proposition shows that these conclusions are not affected by the temporary nature of the intervention. Instead, they hold true also for interventions that take place over many periods. For tractability, we focus on CRRA preferences.

Proposition 4. *Suppose balance sheet policies satisfy (22) and (23), the distribution of level-k thinkers is exponential, and preferences are CRRA. Then the the first-order effect of balance sheet interventions on output in the reflective equilibrium is*

$$Y_t^{RE} = Y_t^* + \frac{1-\lambda}{1-\lambda\mu\left[1+(1-\lambda)\left(1-\frac{1}{\gamma}\right)\varphi\right]}\Delta Y_t^1 + o(\overline{X}), \tag{30}$$

where

$$\Delta Y_t^1 = \frac{1}{\gamma}(1+r)e^{-\rho}Y_t^*(\mathcal{R}_t + \mathcal{M}_t)\mu^t\overline{X} + o(\overline{X}),$$

$\varphi \equiv \mathbb{E}_{t+1}[-V^*_{aa,t+2}]r\cdot(Q^*+H^*)e^{-\rho}\mu(1+r)/[(1+r-\mu)\gamma]$ *is a positive constant, and* $\mathcal{R}_t + \mathcal{M}_t = \mathbb{E}_t[V^*_{aa,t+1}ER^*_{1,t+1}]$. *Furthermore,* $Y_t^{RE} - Y_t^* > 0$ *if and only if* $\mathcal{R}_t + \mathcal{M}_t > 0$.

The proof is in Appendix A.2.9. The key message of the proposition is that the output effect in the reflective equilibrium of an economy with heterogeneous agents and a persistent intervention is proportional to its counterpart in a simple economy in which all agents are level-1 thinkers. In other words, all the dynamics arising from the persistence of the intervention and from the forward-looking behavior of the agents with $k > 1$ are captured by the simple factor multiplying ΔY_t^1. What is more, the sign of the output effect is the same as the sign of $\mathcal{R}_t + \mathcal{M}_t$, which, by Proposition 3, is a function of the type of asset targeted by the central bank. We consider a few special cases to uncover important insights.

We start with a temporary intervention that lasts for only one period ($\mu = 0$). In this case, the ratio in equation (30) reduces to $1-\lambda$, which is simply the proportion of level-1 households in the population. This is intuitive. Since the economy returns back to the REE after only one period after the intervention, all agents with $k > 1$ forecast future variables correctly. As a result, the strength of a one-period-long intervention depends only on the relative share of level-1 thinkers.

Consider now the case in which the intervention is permanent ($\mu = 1$). The ratio in equation (30) then becomes $1/(1-\lambda\varphi+\lambda\varphi/\gamma)$. The persistent nature of the intervention has two implications. First, the term $\lambda\varphi$ in the denominator captures the direct effect of increased future income on current consumption demand. Second, the term $-\lambda\varphi/\gamma$ captures the saving effect: a higher expected future output results in higher expected future returns, thus, agents increase investment in the risky assets and reduce current consumption. The strength of both of these effects increases with λ. As λ grows, the average level of thinking in the population increases. Thus, on average, agents become more forward looking and, hence, more sensitive to expected changes in future income. As a result, both the direct and the saving effect become stronger.

It is well known that in the case of log preferences (when $\gamma = 1$) income and substitution effects exactly offset each other. In our case, this property implies that expected changes in future variables do not affect current consumption and the parameter λ becomes irrelevant. Outside of this knife-edge case, the strength of the output effect increases with the average level of thinking if and only if $\gamma > 1$. This restriction is typically satisfied in most macroeconomic models as it implies a non-trivial amount of risk aversion γ and, at the same time, a reasonably low value for the elasticity of intertemporal substitution $1/\gamma$.

We end this section by emphasizing that the effect of balance sheet policies depends on the type of securities that the central bank purchases. This result suggests that the types of interventions, such as QE2 or the Operation Twist, which consisted primarily in purchases of long-term bonds, may potentially have a negative effect on output.

4 Testable Implications

In this last part of the paper, we highlight the differences between our model and the models based on alternative channels of balance sheet policies. We proceed in four steps. First, we present the key difference between our model and those that, while emphasizing other channels, maintain the assumption of full information rational expectations. Second, we contrast our model with those that assume information frictions. Third, we present evidence that support some of the predictions of our model. Finally, we look at this evidence through the lens of our model and conclude that 86 percent of the forecasters in our data are level-1 thinkers.

Full information rational expectations models. When households do not hold rational expectations, they make systematic mistakes. We can use the simple model (or any of its extensions) to derive closed-form expressions for the errors that agents make following central bank interventions. Importantly, these predictions can help us differentiate the mechanism in this paper from other mechanisms that maintain the assumption of *full information rational expectations*, but assume either market segmentation (i.e., the portfolio balance channel) or asymmetric information between the government and private agents (i.e., the signaling channel).[20]

It is instructive to present the asset price forecast errors in our model. Recall that a level-k thinker assumes that the world is populated only by level-$(k-1)$ thinkers; thus, she expects the price of the risky asset at some future period s to be q_s^{k-1}. We denote her forecast error with $u_{t,s}^k \equiv q_s - \mathbb{E}_t \tilde{q}_s^k = q_s - q_s^{k-1}$. Proposition 2 contains an expression for q_s while equation (15) provides the price q_s^{k-1}. We thus have the following expression for

[20]We use the term "full information" to refer to the information available to agents in the economy.

the *average* forecast error:

$$\overline{u}_{t,s} \equiv \sum_{k=1}^{\infty} f(k) u_{t,s}^{k} = \frac{r}{1+r} \gamma \sigma_x^2 \sum_{k=1}^{\infty} f(k) \left[1 - \frac{f(k+1)}{f(k)} \right] \frac{X_{t+k}}{(1+r)^k}. \quad (31)$$

This formula indicates that the size of the average forecast error depends on the size of the intervention. This relation between the average forecast error and the size of balance sheet policies is a peculiar prediction of our mechanism.

Heterogeneous information rational expectations models. There is an alternative class of models in which agents form expectations rationally but possess heterogeneous information. Some of these models can potentially generate non-neutrality of balance sheet policies together with predictable forecast errors (both individually and on average across agents). For example, if some agents do not have accurate information about government interventions, they will make predictable forecast errors from the view point of an econometrician who is perfectly aware of the policy implementation.[21]

Models with heterogeneous information are not observationally equivalent to models with level-k thinking. A crucial difference between them is that, in models with heterogeneous information, individual agents hold rational expectations *conditional* on their information sets. Formally, models with heterogeneous information predict that an agent's forecast error is orthogonal to any variable contained in the agent's information set. For example, if agents are aware of the policy of asset purchases, which is publicly announced, then this policy should not predict future forecast errors. On the contrary, level-k agents use their information "incorrectly" and, hence, make forecast errors that are predictable even with their information. To distinguish the two types of models empirically, therefore, one would need a proxy for individual information about government interventions.[22]

In addition, one can argue that, when it comes to forecasting financial variables by

[21] Prominent examples of information frictions are noisy information (Lucas, 1972; Woodford, 2001; Angeletos and La'O, 2010), sticky information (Mankiw and Reis, 2002; Reis, 2006a,b), and rational inattention (Sims, 2003; Maćkowiak and Wiederholt, 2009).

[22] Using individual forecast updates to proxy for individual information sets, Bordalo et al. (2018) provide evidence on the predictability of individual forecast errors by such variables, which points to non-rational expectations. Moreover, in several papers, laboratory experiments mimic macroeconomic situations. For example, Kneeland (2016) studies coordinated attack games, such as currency attacks, and concludes that a model with level-k thinking fits the responses of subjects to public information better than a model with dispersed information and rational expectations. Giamattei (2015) runs an experiment in which price setters respond to the central bank's attempt to reduce inflation. He concludes that subjects' price choices are better approximated by a model with level-k thinking, rather than with rational expectations.

To differentiate level-k thinking from other behavioral alternatives, we would need not only data on expectations of future asset prices, but also data on higher-order expectations. Unfortunately, most surveys do not include questions about higher order beliefs.

financial institutions, professional forecasters are likely to pay a great deal of attention to government interventions reducing the extent of heterogeneous information about the interventions. In such a case, if incomplete information was the main friction, it would be unlikely that government interventions could predict forecast errors.

Predictability of forecast errors in the data. We present evidence that balance sheet interventions predict forecast errors. We focus on average forecast errors of conventional mortgage rates in the US and show that they respond significantly to "exogenous and unexpected" purchases of mortgages by quasi-government agencies, also known as government-sponsored enterprises (GSEs), such as Fannie Mae and Freddie Mac.

We follow Fieldhouse, Mertens and Ravn (2018) (henceforth referred to as "FMR"), who argue that the purchases of mortgages by GSEs resemble the purchases of private risky assets by the Federal Reserve in the recent financial crisis. The authors use a narrative approach to identify "exogenous and unexpected" shocks to GSE's balance sheets. They then document a significant reaction of mortgage rates after these shocks. We use the data and specification employed by the authors and regress both mortgage rates and their forecast errors on the shocks. To construct forecast errors, we use a survey of expectations by major financial institutions collected in the Blue Chip Financial Forecast (BCFF) database.

Since we adopt the exact econometric approach of FMR, for brevity, we present here only the responses of conventional mortgage rates and their forecast errors to the shocks. Appendix B contains all of the econometric details.

The left panel of Figure 2 shows the impulse response of the mortgage rate after an expansion of the GSEs balance sheet. This plot confirms that the main conclusion in FMR changes very little when we use our restricted data sample due to the shorter sample of the forecast data that we use. The right panel of Figure 2 presents the response of mortgage rate forecast errors at various horizons along with one- and two-standard-error confidence intervals. Consistently with the predictions of our model, forecast errors react negatively and significantly to the GSEs' mortgage purchases, which suggests that forecasters tend to under-react to news about such interventions.

Next, we use these empirical results and our model to compute the average level of thinking of agents in the data.[23] Specifically, we first assume that all of the reaction of forecast errors can be attributed to the bounded rationality channel of this paper. Then we use equations (17) and (31), together with the assumption that the distribution of levels of thinking $f(k)$ is exponential with mean of $\bar{k}$, to show that the average level of thinking

[23]We implicitly assume that the BCFF survey participants have the same average level of thinking as the agents who price mortgages. We leave the alternative assumption of different levels of thinking to future research.

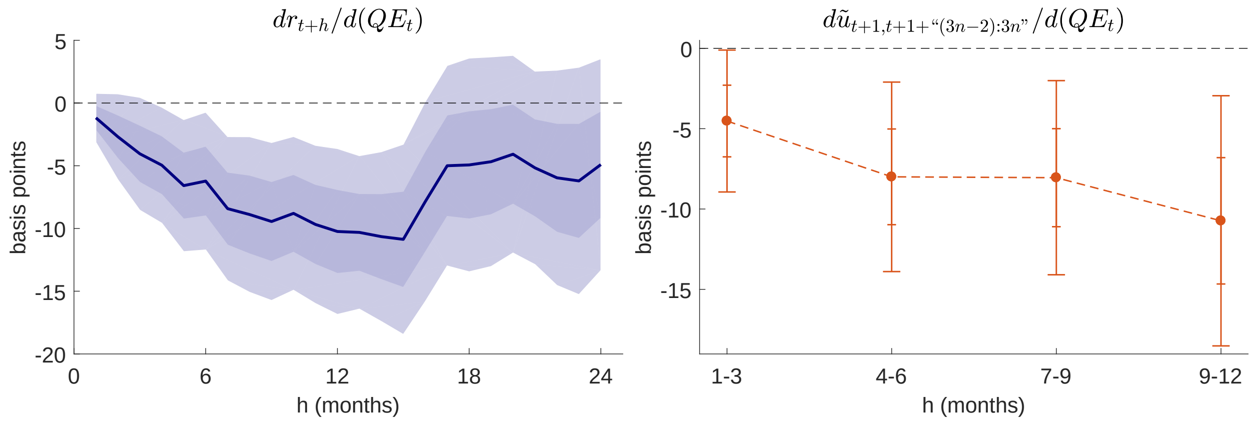

Figure 2: The left panel presents the conventional mortgage-rate impulse response function to an exogenous change in the GSEs' purchases of mortgages. Formally, the notation QE_t refers to the term multiplying $\gamma_h^{(2)}$ on the right-hand side of equation (B.2) in Appendix B. The right panel shows the response of the conventional mortgage-rate forecast errors, at various forecasting horizons, to an exogenous change in the GSEs' purchases of mortgages. The labels on the horizontal axis in the right panel represent the forecast horizon. For example, "1-3 months" refers to one-quarter-ahead forecasts. The appendix states all variable definitions explicitly. In both panels, the confidence intervals show the one and two Newey and West (1987) standard deviation error bands.

$\overline{k}$ equals the ratio of the impulse responses:

$$\overline{k} = \frac{\partial q_s/\partial X_{t+1}}{\partial \overline{u}_{t,s}/\partial X_{t+1}}. \tag{32}$$

The key property of (32) is that it is independent of the specific process of balance sheet policies $\{X_{t+1}\}$, thus, we can use it to estimate $\overline{k}$, independently of the exact details of the asset purchase programs in our data. In addition, the same formula holds true if we replace the price of the risky asset in the numerator with the one-period return on the risky asset. Therefore, if we identify the conventional mortgage rate in our data with the one-period return on the risky assets in our theoretical model, then we can use equation (32) to get an estimate of $\overline{k}$.

Guided by our theory, we compute $\overline{k}$ by taking the impulse response of the moving average of the realized mortgage rates and dividing it by the responses of the forecast errors at the same horizon from the right panel of Figure 2. We provide all the details in Appendix B. The estimates of $\overline{k}$ for each horizon are presented in Appendix Table B.1 and the average across the four horizons is 1.17. This number implies that 86 percent of the agents in the sample consists of level-1 thinkers, who do not change their forecasts after the policy intervention, while only 14 percent achieves higher levels of thinking. This is quite a low estimate of $\overline{k}$, especially considering that our sample is made up of major financial institutions. At the same time, when interpreting the numbers in this exercise, one may want to keep in mind that we assumed that all the agents are perfectly aware of the policy interventions. If, instead, some agents do not react to new policies simply because they are not aware of them, then these agents would be incorrectly classified as being of the lowest level of thinking.[24]

5 Conclusion

In this paper, we showed that the assumption of rational expectations is essential for the irrelevance of balance sheet policies (e.g., quantitative easing) both for asset prices and for the real activity. Once we assume that agents do not form expectations rationally but, instead, use the level-k thinking process for beliefs formation, balance sheet policies become effective policy tools. What is more, the theory in this paper suggests that the agents' attitude towards risk (i.e., how absolute risk aversion varies with wealth) and the risk-return characteristics of the assets purchased by the central bank (i.e., how excess returns covary with output) are important determinants of the effectiveness of such policies. Next, we contrasted the predictions of our model with the alternative models that emphasize other channels of balance sheet policies. Finally, we tested one of the main implications of our channel, namely, that forecast errors should be predictable by balance sheet policies. Specifically, we show that identified exogenous and unexpected purchases of mortgages by quasi-government agencies predict average forecast errors of asset prices (i.e., mortgage rates) at different horizons.

There are many important directions ahead. For example, we have studied the response of the economy to an exogenous path of asset purchases. A crucial step will be to understand when the central bank finds it optimal to use balance sheet policies as a stabilization tool in an environment when people form expectations as in our paper.

[24]We also abstracted from learning by level-1 agents and from level-k agents' anticipation of such learning. Allowing for these features can have ambiguous effects on the estimate of the fraction of level-1 agents, depending on whether level-1 agents use a well-specified or a mis-specified statistical model.

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Appendix

A Proofs

This part of our appendix details the proofs omitted from the main text in Sections 2 and 3.

A.1 Balance Sheet Policies and Asset Prices

We start from the proofs of the claims in Section 2.

Household Problem and Proof of Lemma 1.

We conjecture that the asset price is a non-stochastic function of time, that is, $\beta_{q,t} = 0$, $t \geq 0$, thus, $\mathbb{E}_t \widetilde{q}_{t+1} = \alpha_{q,t}$. We later verify that this is indeed the case in all the equilibria we consider.

We solve the household problem (1)-(2) recursively. In particular, the value function at time t, $V(a_t; \widetilde{\mathcal{Z}}_t)$, is a solution to the Bellman equation

$$V(a_t; \widetilde{\mathcal{Z}}_t) = \max_{\widehat{c}_t, \widehat{x}_{t+1}, \widehat{a}_{t+1}} \left\{ u(\widehat{c}_t) + e^{-\rho} \mathbb{E}_t [V(\widehat{a}_{t+1}; \widetilde{\mathcal{Z}}_{t+1})] \right\}, \tag{A.1}$$

subject to

$$\widehat{a}_{t+1} = (1+r)(a_t - \widehat{c}_t - q_t \widehat{x}_{t+1}) + (D_{t+1} + \widetilde{q}_{t+1})\widehat{x}_{t+1} + W_{t+1} - \widetilde{T}_{t+1}. \tag{A.2}$$

We conjecture that the value function $V(a_t; \widetilde{\mathcal{Z}}_t)$ depends on individual net worth a_t (which is defined as the sum of financial wealth and current income) and beliefs $\widetilde{\mathcal{Z}}_t$. We also define human capital $\widetilde{H}_t$ as the expected discounted sum of future disposable income:

$$\widetilde{H}_t \equiv \mathbb{E}_t \sum_{j=1}^{\infty} \frac{1}{(1+r)^j} \left(W_{t+j} - \widetilde{T}_{t+j} \right).$$

We guess and verify that

$$V(a; \widetilde{\mathcal{Z}}_t) = -\frac{1}{\gamma} \exp \left(-\gamma A \left(a + \widetilde{H}_t \right) + \widetilde{\vartheta}_t \right),$$

where $\widetilde{\vartheta}_t$ is a deterministic function of time. We can then use standard properties of Normal distributions to rewrite the problem (A.1)-(A.2) as

$$V(a; \widetilde{\mathcal{Z}}_t) = \max_{\widehat{c}_t, \widehat{x}_{t+1}, \widehat{a}_{t+1}} \left\{ -\frac{1}{\gamma} \exp\left(-\gamma \widehat{c}_t\right) - \frac{1}{\gamma} \exp \left(-\rho - \gamma A \mathbb{E}_t \left[\widehat{a}_{t+1} + \widetilde{H}_{t+1} \right] + \frac{1}{2} \gamma^2 A^2 \mathbb{V}_t \left[\widehat{a}_{t+1} + \widetilde{H}_{t+1} \right] + \mathbb{E}_t \widetilde{\vartheta}_{t+1} \right) \right\}.$$

The conditional expectation of the following period's total wealth is

$$\mathbb{E}_t \left[\widehat{a}_{t+1} + \widetilde{H}_{t+1} \right] = (1+r)\left(a_t - \widehat{c}_t - q_t \widehat{x}_{t+1}\right) + (\overline{D} + \alpha_{q,t})\widehat{x}_{t+1} + W_{t+1} - \mathbb{E}_t \widetilde{T}_{t+1} + \mathbb{E}_t \widetilde{H}_{t+1},$$

where $\overline{D} = \mathbb{E}_t \left[D_{t+1} \right]$, since agents are assumed to know the true distribution of exogenous variables.

Similarly, the conditional variance of the following period's total wealth is

$$
\begin{aligned}
\widetilde{\mathbb{V}}_t\left[\widehat{a}_{t+1}+\widetilde{H}_{t+1}\right] =& \widetilde{\mathbb{V}}_t\left[(1+r)(a_t-\widehat{c}_t-q_t\widehat{x}_{t+1})+(D_{t+1}+\alpha_{q,t})\widehat{x}_{t+1}+W_{t+1}-\widetilde{T}_{t+1}+\widetilde{H}_{t+1}\right] \\
=& \widetilde{\mathbb{V}}_t\left[D_{t+1}\widehat{x}_{t+1}-\widetilde{T}_{t+1}+\mathbb{E}_{t+1}\sum_{j=1}^{\infty}\frac{1}{(1+r)^j}\left(-\widetilde{T}_{t+j+1}\right)\right] \\
=& \widetilde{\mathbb{V}}_t\left[D_{t+1}\widehat{x}_{t+1}-(1+r)B_{t+1}+\mathbb{E}_{t+1}\sum_{j=0}^{\infty}\frac{\widetilde{Tr}_{t+j+1}}{(1+r)^j}\right] \\
=& \sigma_x^2(\widehat{x}_{t+1}+\beta_{Tr,t})^2,
\end{aligned}
$$

where the second to last line uses the treasury's budget constraint (3).

The solution to the optimization problem at time t gives realized choices $c_t = c(a_t;\widetilde{\mathcal{Z}}_t)$, $a_{t+1} = a(a_t;\widetilde{\mathcal{Z}}_t)$, and $x_{t+1} = x(a_t;\widetilde{\mathcal{Z}}_t)$ and planned choices, that is, the choices that the agent expects to implement in the future. Due to bounded rationality, the latter may differ from the former. We use a "hat" for planned choices. The solution to the problem satisfies the first-order condition for consumption

$$
\exp(-\gamma c_t) = A(1+r)\exp\left(-\rho-\gamma A\mathbb{E}_t\left[a_{t+1}+\widetilde{H}_{t+1}\right]+\frac{1}{2}\gamma^2A^2\mathbb{V}_t\left[a_{t+1}+\widetilde{H}_{t+1}\right]+\widetilde{\vartheta}_{t+1}\right) \tag{A.3}
$$

and for investment in the risky asset

$$
-(1+r)q_t+\overline{D}+\alpha_{q,t}-\gamma A\sigma_x^2(x_{t+1}+\beta_{Tr,t}) = 0. \tag{A.4}
$$

The latter equation determines the demand for the risky asset

$$
x_{t+1} = \frac{\overline{D}+\alpha_{q,t}-(1+r)q_t}{\gamma A\sigma_x^2}-\beta_{Tr,t}. \tag{A.5}
$$

Taking logs of both sides of equation (A.3) and using the expressions for $\mathbb{E}_t\left[\widehat{a}_{t+1}+\widetilde{H}_{t+1}\right]$ and $\mathbb{V}_t\left[\widehat{a}_{t+1}+\widetilde{H}_{t+1}\right]$, we rewrite (A.3) as

$$
\begin{aligned}
-\gamma c_t =& \log[A(1+r)]-\rho-\gamma A\left[(1+r)(a_t-c_t-q_tx_{t+1})+\left(\overline{D}+\alpha_{q,t}\right)x_{t+1}+W_{t+1}-\mathbb{E}_t\widetilde{T}_{t+1}+\mathbb{E}_t\widetilde{H}_{t+1}\right] \\
&+\frac{1}{2}\gamma^2A^2\sigma_x^2(x_{t+1}+\beta_{Tr,t})^2+\mathbb{E}_t\widetilde{\vartheta}_{t+1}.
\end{aligned}
$$

We next solve this equation for consumption:

$$
\begin{aligned}
c_t =& -\frac{\log[A(1+r)]-\rho}{\gamma[A(1+r)+1]}+\frac{A(1+r)}{A(1+r)+1}\left(a_t+\widetilde{H}_t\right)+\frac{A}{A(1+r)+1}\left(\overline{D}+\alpha_{q,t}-(1+r)q_t\right)x_{t+1} \\
&-\frac{1}{2}\cdot\frac{A^2\gamma}{A(1+r)+1}\sigma_x^2(x_{t+1}+\beta_{Tr,t})^2-\frac{1}{\gamma[A(1+r)+1]}\mathbb{E}_t\widetilde{\vartheta}_{t+1},
\end{aligned}
$$

where we took into account that $(1+r)\widetilde{H}_t = W_{t+1}-\mathbb{E}_t\widetilde{T}_{t+1}+\mathbb{E}_t\widetilde{H}_{t+1}$.

For our guess to be true, equation (A.1) must be satisfied by our conjectured value function, for any value of a_t and t, that is,

$$-\frac{1}{\gamma}\exp\left(-\gamma A\left(a_t+\widetilde{H}_t\right)+\widetilde{\vartheta}_t\right)$$
$$=\max_{\widehat{c}_t,\widehat{x}_{t+1},\widehat{a}_{t+1}}\left\{-\frac{1}{\gamma}\exp\left(-\gamma\widehat{c}_t\right)-\frac{1}{\gamma}\exp\left(-\rho-\gamma A\mathbb{E}_t\left[\widehat{a}_{t+1}+\widetilde{H}_{t+1}\right]+\frac{1}{2}\gamma^2A^2\mathbb{V}_t\left[\widehat{a}_{t+1}+\widetilde{H}_{t+1}\right]+\mathbb{E}_t\widetilde{\vartheta}_{t+1}\right)\right\}.$$

Using (A.3), the latter becomes

$$-\frac{1}{\gamma}\exp\left(-\gamma A\left(a_t+\widetilde{H}_t\right)+\widetilde{\vartheta}_t\right)=-\frac{1}{\gamma}\exp\left\{\log\left[\frac{A(1+r)+1}{A(1+r)}\right]-\gamma c_t\right\}, \tag{A.6}$$

where, with slight abuse of notation, we use c_t to denote optimal consumption. Because equation (A.6) must hold for any value of a_t, the constant A must satisfy $A=\frac{r}{1+r}$. The asset demand (A.5) is thus

$$x_{t+1}=\frac{\mathbb{E}_t[D_{t+1}+\widetilde{q}_{t+1}]-(1+r)q_t}{\gamma\frac{r}{1+r}\sigma_x^2}-\beta_{Tr,t}, \tag{A.7}$$

which is equation (C.38) in the main text. Similarly, consumption becomes

$$\begin{aligned}c_t=&-\frac{\log(r)-\rho}{\gamma(1+r)}+\frac{r}{1+r}\left(a_t+\widetilde{H}_t\right)+\frac{r}{(1+r)^2}\left(\overline{D}+\alpha_{q,t}-(1+r)q_t\right)x_{t+1}\\&-\frac{1}{2(1+r)}\left(\frac{r}{1+r}\right)^2\gamma\sigma_x^2\left(x_{t+1}+\beta_{Tr,t}\right)^2-\frac{1}{\gamma(1+r)}\mathbb{E}_t\widetilde{\vartheta}_{t+1}\end{aligned}$$

and, finally,

$$\begin{aligned}\widetilde{\vartheta}_t=&\log\left(\frac{1+r}{r}\right)+\frac{\log(r)-\rho}{1+r}-\gamma\frac{r}{(1+r)^2}\left(\overline{D}+\alpha_{q,t}-(1+r)q_t\right)x_{t+1}\\&+\frac{1}{2(1+r)}\left(\frac{r}{1+r}\right)^2\gamma^2\sigma_x^2\left(x_{t+1}+\beta_{Tr,t}\right)^2+\mathbb{E}_t\widetilde{\vartheta}_{t+1}.\end{aligned}$$

The latter can be iterated forward to get

$$\begin{aligned}\widetilde{\vartheta}_t=&\log\left(\frac{1+r}{r}\right)+\frac{\log(1+r)-\rho}{r}-\gamma\frac{r}{(1+r)^2}\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left(\overline{D}+\alpha_{q,t+j}-(1+r)\alpha_{q,t+j-1}\right)\widehat{x}_{t+j+1}\\&+\frac{1}{2(1+r)}\left(\frac{r}{1+r}\right)^2\gamma^2\sigma_x^2\sum_{j=0}^{\infty}\frac{1}{(1+r)^j}\left(\widehat{x}_{t+j+1}+\beta_{Tr,t+j}\right)^2,\end{aligned}$$

where, to simplify notation, we used $\alpha_{q,t-1}=q_t$. Since asset demands x_{t+1} and $\{\widehat{x}_{t+j+1}\}_{j>0}$ are a deterministic function of time, so is $\widetilde{\vartheta}_t$, therefore, our conjecture is verified.

Proof of Proposition 1.

In the REE, the market-clearing condition (6) holds for $t\geq 0$. In addition, households take the budget constraints of both branches of the government into account when forming expectations about the future. Thus, we immediately obtain $\beta_{Tr,t}=X_{t+1}$, $t\geq 0$. Plugging the latter and (6) into the asset demand (A.7) and solving for the asset price gives (we add an asterisk to denote REE objects):

$$q_t^* = \frac{\overline{D} + q_{t+1}^* - \gamma \frac{r}{1+r} \sigma_x^2}{1+r},$$

where we used the fact that, since expectations are rational and the price is deterministic, the expectation of next period's asset price coincides with its realized value. The unique non-explosive solution of the above equation is (13) and is independent of asset purchases.

Proof of Proposition 2.

We first prove (14). We proceed recursively, starting with level-1 agents. At time t, level-1 agents solve (A.1) under the belief that the economy will be in the REE without asset purchases. In particular, since the transfers are zero in the absence of asset purchases, level-1 agents believe that $\beta^1_{Tr,s} = 0$, for all $s \geq t$. The value function in (A.1) conveniently encapsulates the agent's beliefs about the future. We can then use the risky-asset demand (A.7), together with market clearing (6) and the asset price in the REE (13), to derive the first line of (14).

Now, let q_t^{k-1} be asset price in the temporary equilibrium of the economy populated by level-$(k-1)$ agents, with $k > 1$. Importantly, agents with $k > 1$ take into account the intertemporal budget constraints of the entire government when forming their expectations. As a result, $\beta_{Tr,s} = X_{s+1}$, $s \geq t$. Moreover, they expect the future asset price to coincide with the asset price in an economy with level-$(k-1)$ agents, that is, $\mathbb{E}_t \left[\widetilde{q}^k_{t+1} \right] = q^{k-1}_{t+1}$. Plugging the latter into (A.7) and imposing market clearing (6) gives the second line of (14).

Equation (16) then follows immediately by using the results above with (A.7) and the market-clearing condition (9).

A.2 Balance Sheet Policies and Aggregate Output

A.2.1 Household Problem Optimality in Section 3.1

We begin with the individual problem at a generic time t. Let $\widehat{\lambda}_{t,t+s}$ be the Lagrange multipliers on the budget constraints (19) and define the Lagrangian

$$\begin{aligned} \mathcal{L}_t = & \mathbb{E}_t \Bigg[\sum_{s=0}^{\infty} \Bigg\{ e^{-\rho s + \sum_{j=1}^{s} \epsilon_{t+j-1}} u(\widehat{c}_{t+s}) + \widehat{\lambda}_{t,t+s+1} \Bigg(-\widehat{a}_{t+s+1} + (1+r)(\widehat{a}_{t+s} - \widehat{c}_{t+s}) + (1-\delta)\widetilde{Y}_{t+s+1} - \widetilde{T}_{t+s+1} \\ & + \sum_{i=1,2} (\widetilde{D}_{i,t+s+1} + \widetilde{q}_{i,t+s+1} - (1+r)\widetilde{q}_{i,t+s}) \widehat{x}_{i,t+s+1} \Bigg) \Bigg\} \Bigg]. \end{aligned} \tag{A.8}$$

By Lagrange duality,

$$V(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t) = \min_{\{\widehat{\lambda}_{t+s+1}\}_{s \geq 0}} \max_{\{\widehat{c}_{t+s}, \widehat{x}_{i,t+s+1}, \widehat{a}_{t+s+1}\}_{s \geq 0}} \mathcal{L}_t.$$

We denote realized choices with $c_t = c(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t)$, $a_{t+1} = a(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t)$, and $x_{i,t+1} = x_i(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t)$, $i = 1, 2$, and use a "hat" for planned choices. The first-order conditions are

$$e^{-\rho s + \sum_{j=1}^{s} \epsilon_{t+j-1}} u'(\widehat{c}_{t+s}) = (1+r)\mathbb{E}_{t+s}[\widehat{\lambda}_{t,t+s+1}], \text{ for } s \geq 0, \tag{A.9}$$

$$\widehat{\lambda}_{t,t+s} = (1+r)\mathbb{E}_{t+s}[\widehat{\lambda}_{t,t+s+1}], \text{ for } s > 0, \tag{A.10}$$

$$\mathbb{E}_{t+s}[\widehat{\lambda}_{t,t+s+1}(\widetilde{D}_{i,t+s+1} + \widetilde{q}_{i,t+s+1} - (1+r)\widetilde{q}_{i,t+s})] = 0, \text{ for } s \geq 0, \text{ and } i = 1, 2. \tag{A.11}$$

In addition, we define $\widehat{\lambda}_{t,t} \equiv (1+r)\mathbb{E}_t[\widehat{\lambda}_{t,t+1}]$. This extra variable is needed because the first order condition (A.10) is only valid for $s > 0$.

Furthermore, by the envelope theorem,

$$V_a(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t) = (1+r)\mathbb{E}_t[\widehat{\lambda}_{t,t+1}].$$

Combining the conditions above, we obtain $V_a(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t) = \widehat{\lambda}_{t,t}$ and, therefore,

$$u'(c_t) = (1+r)e^{\epsilon_t - \rho}\mathbb{E}_t[V_a(a_{t+1}, \epsilon_{t+1}; \widetilde{\mathcal{Z}}_{t+1})], \tag{A.12}$$

which is equation (20) in the main text. Similarly, we can rewrite the asset pricing conditions (A.11) as

$$q_{i,t} = \frac{1}{1+r} \cdot \frac{\mathbb{E}_t[V_a(a_{t+1}, \epsilon_{t+1}; \widetilde{\mathcal{Z}}_{t+1})(\widetilde{D}_{i,t+1} + \widetilde{q}_{i,t+1})]}{\mathbb{E}_t[V_a(a_{t+1}, \epsilon_{t+1}; \widetilde{\mathcal{Z}}_{t+1})]}, \ i = 1, 2, \tag{A.13}$$

which is equation (21) in the main text.

For future references, we introduce the next definition.

Definition 1. The stochastic discount factor of an agent with a sequence of Lagrange multipliers $\{\widehat{\lambda}_{t,t+s}\}$ is defined as

$$m_{t,t+s} = \frac{\widehat{\lambda}_{t,t+s}}{\widehat{\lambda}_{t,t}}, \ s \geq 0.$$

By condition (A.10), the stochastic discount factor satisfies $(1+r)\mathbb{E}_{t+s}[m_{t,t+s+1}] = m_{t,t+s}$ and $m_{t,t} = 1$.

To prove the results on the first-order effects of balance sheet policies, we differentiate equilibrium conditions with respect to the initial size of the intervention $\overline{X}$, around the point $\overline{X} = 0$. Given a function L, we use dL to denote its differential, that is, the function that approximates L and is linear in $\overline{X}$. Finally, when a given function equals its REE counterpart whenever $\overline{X} = 0$, then we also have $\Delta L \equiv L - L^* = dL + o(\overline{X})$, where $o(\overline{X})$ contains all the terms such that $o(\overline{X})/\overline{X} \to 0$ as $\overline{X} \to 0$.

A.2.2 Proof of Lemma 2

We now evaluate equations (20) and (21) at equilibrium values. We begin with the first one. Using market clearing in the goods market, we rewrite (A.12) as

$$u'(Y_t^k) = (1+r)e^{\epsilon_t - \rho}\mathbb{E}_t[\widetilde{V}_{a,t+1}^k].$$

We then compute the differential of the equation above with respect to $\overline{X}$ around the point $\overline{X} = 0$:

$$u''(Y_t^*)dY_t^k = (1+r)e^{\epsilon_t - \rho}\mathbb{E}_t[d\widetilde{V}_{a,t+1}^k].$$

Equation (24) follows from the fact that $\Delta Y_t^k = dY_t^k + o(\overline{X})$ and $\Delta \widetilde{V}_{a,t+1}^k = d\widetilde{V}_{a,t+1}^k + o(\overline{X})$, for all k.

Similarly, to obtain the first-order effect on asset prices, we apply the same arguments to equations (A.13). First, we evaluate them at level-k equilibrium

$$q_{i,t}^k = \frac{1}{1+r} \cdot \frac{\mathbb{E}_t[\widetilde{V}_{a,t+1}^k(\widetilde{D}_{i,t+1}^k + \widetilde{q}_{i,t+1}^k)]}{\mathbb{E}_t[\widetilde{V}_{a,t+1}^k]}, \; i = 1, 2.$$

Then we take logs of both sides and differentiating around $\overline{X} = 0$ gives

$$\frac{dq_{i,t}^k}{q_{i,t}^*} = \frac{\mathbb{E}_t[d\widetilde{V}_{a,t+1}^k(D_{i,t+1}^* + q_{i,t+1}^*)]}{\mathbb{E}_t[V_{a,t+1}^*(D_{i,t+1}^* + q_{i,t+1}^*)]} + \frac{\mathbb{E}_t[V_{a,t+1}^*(\delta D_i'(\delta Y_{t+1}^*)d\widetilde{Y}_{t+1}^k + d\widetilde{q}_{i,t+1}^k)]}{\mathbb{E}_t[V_{a,t+1}^*(D_{i,t+1}^* + q_{i,t+1}^*)]} - \frac{\mathbb{E}_t[d\widetilde{V}_{a,t+1}^k]}{\mathbb{E}[V_{a,t+1}^*]},$$

for $i = 1, 2$. Let $ER_{i,t+1}$ denote the realized excess return (multiplied by the initial price of the asset $q_{i,t}$) of asset i:

$$ER_{i,t+1} \equiv D_{i,t+1} + q_{i,t+1} - (1+r)q_{i,t},$$

for $i = 1, 2$. Then, using the fact that the beliefs of a level-k thinker (for $k > 1$) coincide with the outcome of a level-$(k-1)$ equilibrium, we rewrite the effect on asset prices as

$$dq_{i,t}^k = \frac{1}{1+r} \cdot \frac{\mathbb{E}_t[d\widetilde{V}_{a,t+1}^k ER_{i,t+1}^*]}{\mathbb{E}_t[V_{a,t+1}^*]} + \frac{1}{1+r} \cdot \frac{\mathbb{E}_t[V_{a,t+1}^*(D_i'(\delta Y_{t+1}^*)\delta dY_{t+1}^{k-1} + dq_{i,t+1}^{k-1})]}{\mathbb{E}_t[V_{a,t+1}^*]}, \; i = 1, 2.$$

When $k = 1$, agents expect that asset prices and output will remain unchanged, thus, the second term in the expression above is zero. As a result,

$$dq_{i,t}^1 = \frac{1}{1+r} \cdot \frac{\mathbb{E}_t[d\widetilde{V}_{a,t+1}^1 ER_{i,t+1}^*]}{\mathbb{E}_t[V_{a,t+1}^*]},$$

for $i = 1, 2$.

Equations (25) follows from the fact that $\Delta Y_t^{k-1} = dY_t^{k-1} + o(\overline{X})$, $\Delta q_{i,t}^{k-1} = dq_{i,t}^{k-1} + o(\overline{X})$ and $\Delta \widetilde{V}_{a,t+1}^k = d\widetilde{V}_{a,t+1}^k + o(\overline{X})$, for all k and $i = 1, 2$.

A.2.3 Proof of Lemma 3

Asset purchases affect future marginal value of wealth $V_a(\widetilde{A}_{t+1}^k, \epsilon_{t+1}; \widetilde{\mathcal{Z}}_t^k)$ in two ways. First, as long as $k > 1$, asset purchases will impact beliefs of future variables (a change in argument $\widetilde{\mathcal{Z}}_t^k$). Second, they change the value of wealth at $t+1$ (a change in argument $\widetilde{A}_{t+1}^k$). We compute each channel separately, in Step (ii) and Step (iii), respectively and we start with a preliminary result in Step (i).

Step (i). We begin by showing a preliminary result that the present discounted value of future taxes—defined as $\widetilde{\mathcal{T}}_t^k \equiv \mathbb{E}_t[\sum_{s=1}^{\infty} m_{t,t+s}\widetilde{T}_{t+s}^k]$—for agents with $k > 1$ and the stochastic discount factor $m_{t,t+s}$ equals

$$\widetilde{\mathcal{T}}_{t+s}^k = B_{t+s+1}^{k-1} + R_{t+s+1}^{k-1} - q_{1,t+s}^* \mu^{t+s}\overline{X}, \tag{A.14}$$

for $s > 0$, and

$$\widetilde{\mathcal{T}}_t^k = R_{t+1}^{k-1} - q_{1,t}^* \mu^t \overline{X},$$

for $s = 0$. These expressions state that the present discounted value of future taxes equals the negative of the net worth of the central bank, that is, the total value of assets owned by the central bank minus the value of outstanding reserves.The differential form of the last two equations is

$$d\mathcal{T}^k_{t+s} = dB^{k-1}_{t+s+1} + dR^{k-1}_{t+s+1} - q^*_{1,t+s}\mu^{t+s}d\overline{X}, \tag{A.15}$$

for $s > 0$, and

$$\widetilde{\mathcal{T}}^k_t = dR^{k-1}_{t+1} - q^*_{1,t}\mu^t d\overline{X},$$

for $s = 0$. Without loss of generality, we focused on the case in which taxes are zero in the absence of asset purchases and, in addition, initial government debt B_{t+1} is also zero.

To simplify notation, we drop the superscript k; we focus on time t, the other periods are analogous. Agents with $k > 1$ understand that asset purchases at time t will generate a sequence of future taxes and transfers which must satisfy both the budget constraint of the government and the budget constraint of the central bank. Formally, agents believe that government debt, central bank reserves, taxes, and transfers satisfy

$$(1+r)\widetilde{B}_{t+s+1} = \widetilde{T}_{t+s+1} + \widetilde{Tr}_{t+s+1} + \widetilde{B}_{t+s+2} \tag{A.16}$$

and

$$(1+r)\widetilde{R}_{t+s+1} + \widetilde{q}_{1,t+s+1}\mu^{t+s+1}\overline{X} = -\widetilde{Tr}_{t+s+1} + (\widetilde{D}_{1,t+s+1} + \widetilde{q}_{1,t+s+1})\mu^{t+s}\overline{X} + \widetilde{R}_{t+s+2}, \tag{A.17}$$

for all $s \geq 0$ with $B_{t+1} = 0$. Summing up over s, we obtain the intertemporal budget constraints:

$$0 = \sum_{s=1}^{\infty} \frac{1}{(1+r)^{s-1}} \left(\widetilde{T}_{t+s} + \widetilde{Tr}_{t+s}\right) \tag{A.18}$$

and

$$(1+r)(R_{t+1} - q_{1,t}\mu^t\overline{X}) = \sum_{s=1}^{\infty} \frac{1}{(1+r)^{s-1}} \left\{-\widetilde{Tr}_{t+s} + [\widetilde{D}_{1,t+s} + \widetilde{q}_{1,t+s} - (1+r)\widetilde{q}_{1,t+s-1}]\mu^{t+s}\overline{X}\right\}. \tag{A.19}$$

Notice that there is no expectation operator in equations (A.16)-(A.19). This is because budget constraints must hold for every history of shocks.

Let $\eta_t \equiv \lim_{s\to\infty}(1+r)^s m_{t,t+s}$, where the limit exists almost surely by Doob's convergence theorem (?). Using the properties of the SDF, which we mentioned in Section A.2.1, we can write

$$\begin{aligned} m_{t,t+s} &= (1+r)\mathbb{E}_{t+s}[m_{t,t+s+1}] = \frac{1}{(1+r)^s}\mathbb{E}_{t+s}[(1+r)^{s+1}m_{t,t+s+1}] \\ &= \frac{1}{(1+r)^s}\mathbb{E}_{t+s}[\mathbb{E}_{t+s+1}[(1+r)^{s+2}m_{t,t+s+2}]] = ... = \frac{1}{(1+r)^s}\mathbb{E}_{t+s}[\eta_t]. \end{aligned}$$

In particular, $\mathbb{E}_t[\eta_t] = m_{t,t} = 1$.

Since the budget constraint (A.18) must hold for every history, if we multiply both sides by η_t, take expectations, and use the properties of η_t together with the law of iterated expectations, we obtain

$$0 = \mathbb{E}_t \sum_{s\geq 1} m_{t,t+s}(\widetilde{T}_{t+s} + \widetilde{Tr}_{t+s}).$$

Therefore,

$$\begin{aligned}\widetilde{\mathcal{T}}_t &= \mathbb{E}_t \sum_{s\geq 1} m_{t,t+s}\widetilde{T}_{t+s} \\ &= \mathbb{E}_t \sum_{s\geq 1} m_{t,t+s}(\widetilde{T}_{t+s} + \widetilde{Tr}_{t+s}) - \mathbb{E}_t \sum_{s\geq 1} m_{t,t+s}\widetilde{Tr}_{t+s} \\ &= -\mathbb{E}_t \sum_{s\geq 1} m_{t,t+s}\widetilde{Tr}_{t+s}.\end{aligned}$$

The same logic applied to (A.19) gives

$$(1+r)(R_{t+1} - q_{1,t}\mu^t\overline{X}) = \mathbb{E}_t \sum_{s=1}^{\infty} m_{t,t+s}\{-\widetilde{Tr}_{t+s} + [\widetilde{D}_{1,t+s} + \widetilde{q}_{1,t+s} - (1+r)\widetilde{q}_{1,t+s-1}]\mu^{t+s}\overline{X}\}$$

or, using the Euler equation for risky assets (A.11),

$$(1+r)(R_{t+1} - q_{1,t}\mu^t\overline{X}) = -\mathbb{E}_t \sum_{s=1}^{\infty} m_{t,t+s}\widetilde{Tr}_{t+s}.$$

Therefore, the present discounted value of taxes satisfies

$$\widetilde{\mathcal{T}}_t = R_{t+1} - q_{1,t}\mu^t\overline{X},$$

which is (A.14). Finally, differentiating both sides of the equation above gives (A.15).

Step (ii). As long as $k > 1$, asset purchases will affect agent beliefs of future income and asset prices. Here, we characterize this channel by computing the differential of the marginal value of wealth with respect to asset purchases, keeping fixed the level of wealth. With a slight abuse of notation we acknowledge an explicit depends of beliefs on the interventions as $\widetilde{\mathcal{Z}}_t^k(\overline{X})$. As a result, the object we want to compute is $dV_a(a,\epsilon;\widetilde{\mathcal{Z}}_t^k(\overline{X}))$ where a and ϵ are held fixed and $\overline{X}$ can vary. In addition, we use similar notation for the change in the level of the value function: $dV(a,\epsilon;\widetilde{\mathcal{Z}}_t^k(\overline{X}))$. Finally, we will use the property that cross-partial derivatives are independent of the order of differentiation: $\partial dV(a,\epsilon;\widetilde{\mathcal{Z}}_t^k(\overline{X}))/\partial a = dV_a(a,\epsilon;\widetilde{\mathcal{Z}}_t^k(\overline{X}))$.

When differentiating the value function, we have to take into account that optimal choices and Lagrange multipliers depend on a and $\overline{X}$. We begin with $dV(a,\epsilon,\widetilde{\mathcal{Z}}_t^k(\overline{X}))$. We differentiate the Lagrangian (A.8) and use the envelope theorem to set to zero the terms that do not depend directly on $\overline{X}$:

$$\begin{aligned}dV(a,\epsilon,\widetilde{\mathcal{Z}}_t^k(\overline{X})) =& \lambda_{t,t}\mathbb{E}_t\left[\sum_{s=1}^{\infty} m_{t,t+s}\sum_{i=1,2}\{D_i'(\delta Y_{t+s}^*)\delta dY_{t+s}^{k-1} + dq_{i,t+s}^{k-1} - (1+r)dq_{i,t+s-1}^{k-1}\}x_{i,t+s}\right] \\ &+ \lambda_{t,t}\mathbb{E}_t\left[\sum_{s=1}^{\infty} m_{t,t+s}(1-\delta)dY_{t+s}^{k-1}\right] - \lambda_{t,t}d\widetilde{\mathcal{T}}_t^k. \end{aligned} \quad \text{(A.20)}$$

Note that we next use $\lambda_{t,t} = V_a(a,\epsilon_t;\widetilde{\mathcal{Z}}_t^*)$, which we proved in Section A.2.1, to replace $\lambda_{t,t}$.

We now take the derivative of equation (A.20) with respect to a. We did not explicitly wrote all the arguments of functions in equation (A.20), thus it is useful to recall that the initial wealth a appears in three places in the equation: (i) in the initial Lagrange multiplier $\lambda_{t,t}$; (ii) in all future stochastic discount factors $m_{t,t+s}$; and (iii) in all future investments in the risky assets $x_{i,t+s}$. Denote $d\mathcal{Y}_t(a) \equiv V_a(a,\epsilon_t;\widetilde{\mathcal{Z}}_t^*)\mathbb{E}_t\sum_{s=1}^{\infty} m_{t,t+s}[(1-$

$\delta)dY^{k-1}_{t+s} + \sum_{i=1,2}(dER^{k-1}_{i,t+s}x_{i,t+s})]$. It is easy to see that

$$dV_a(a,\epsilon,\widetilde{\mathcal{Z}}^k_t(\overline{X}))\Big|_{\overline{X}=0} = -V^*_{aa,t}d\widetilde{\mathcal{T}}^k_t + d\mathcal{Y}'_t(A^*_t), \tag{A.21}$$

where

$$\begin{aligned} d\mathcal{Y}'_t(A^*_t) = & V^*_{aa,t}\left(\mathbb{E}_t\sum_{s=1}^{\infty} m_{t,t+s}dY^{k-1}_{t+s} - \sum_{i=1,2} dq^{k-1}_{i,t}\right) \\ & + V^*_{a,t}\mathbb{E}_t\sum_{s=1}^{\infty}\frac{\partial m_{t,t+s}}{\partial a}dY^{k-1}_{t+s} \\ & + V^*_{a,t}\mathbb{E}_t\sum_{s=1}^{\infty} m_{t,t+s}\sum_{i=1,2}[\delta D'_i(\delta Y^*_{t+s})dY^{k-1}_{t+s} + dq^{k-1}_{i,t+s} - (1+r)dq^{k-1}_{i,t+s-1}]\frac{\partial x_{i,t+s}}{\partial a}. \end{aligned}$$

To derive the last formula, we used the following observations: (i) risky assets market clearing condition before the intervention is $x_{i,t+s+1} = 1$ (we approximate around this point); (ii) $D'_1(\delta Y_{t+s}) + D'_2(\delta Y_{t+s}) = 1$; (iii) $\partial m_{t,t+j}/\partial a = (1+r)\mathbb{E}_{t+j}[\partial m_{t,t+j+1}/\partial a]$ for all $j \geq 0$ (this is just a derivative of $m_{t,t+j} = (1+r)\mathbb{E}_{t+j}[m_{t,t+j+1}]$ with respect to a); (iv) $\mathbb{E}_{t+j}[(\partial m_{t,t+j}/\partial a)d\widetilde{q}_{i,t+j}] = \mathbb{E}_{t+j}[(\partial m_{t,t+j+1}/\partial a)(1+r)d\widetilde{q}_{i,t+j}]$ (this is a previous property multiplied by $d\widetilde{q}_{i,t+j}$); and (v) $0 = \mathbb{E}_t[(\partial m_{t,t+1}/\partial a)d\widetilde{q}_{i,t+1}]$ (because $\partial m_{t,t}/\partial a = 0$).

Step (iii). We next compute the effect of asset purchases on future forecasted equilibrium wealth $\widetilde{A}^k_{t+1}$. In equilibrium, agents must hold the entire supply of the risky assets, which must take into account the purchases by the central bank. The agents also hold reserves issued by the central bank and government bonds, which in the first period of the intervention are zero by assumption. Finally, agents receive labor income and pay taxes.

We focus on period t, the other periods are analogous. In a level-k equilibrium, the agent believes that future aggregate variables will coincide with their counterparts in a level-$(k-1)$ equilibrium. At the same time, the agent observes any variable (such as the value of reserves) which is realized at time t. Therefore, expected wealth in period $t+1$ is

$$\widetilde{A}^k_{t+1} = [D_1(\delta Y^{k-1}_{t+1}) + q^{k-1}_{1,t+1}](1-\mu^t\overline{X}) + D_2(\delta Y^{k-1}_{t+1}) + q^{k-1}_{2,t+1} + (1+r)R^k_{t+1} + (1-\delta)Y^{k-1}_{t+1} - T^{k-1}_{t+1}.$$

First, consider the case with $k > 1$. Agents understand the budget constraints (A.16) and (A.17), thus, we can use them to substitute for T^{k-1}_{t+1} and rewrite wealth as

$$\widetilde{A}^k_{t+1} = D_1(\delta Y^{k-1}_{t+1}) + q^{k-1}_{1,t+1} + D_2(\delta Y^{k-1}_{t+1}) + q^{k-1}_{2,t+1} + (1-\delta)Y^{k-1}_{t+1} + B^{k-1}_{t+2} + R^{k-1}_{t+2} - q^{k-1}_{1,t+1}\mu^{t+1}\overline{X}.$$

Finally, using $D_1(\delta Y) + D_2(\delta Y) = \delta Y$ gives

$$\widetilde{A}^k_{t+1} = Y^{k-1}_{t+1} + \sum_{i=1,2} q^{k-1}_{i,t+1} + B^{k-1}_{t+2} + R^{k-1}_{t+2} - q^{k-1}_{1,t+1}\mu^{t+1}\overline{X}.$$

Therefore, the differential of $\widetilde{A}^k_{t+1}$ evaluated at $\overline{X} = 0$ is

$$d\widetilde{A}^k_{t+1} = dY^{k-1}_{t+1} + \sum_{i=1,2} dq^{k-1}_{i,t+1} + dB^{k-1}_{t+2} + dR^{k-1}_{t+2} - q^*_{1,t+1}\mu^{t+1}d\overline{X}. \tag{A.22}$$

When $k = 1$, asset purchases do not affect beliefs of future variables. Future wealth is thus

$$\widetilde{A}^1_{t+1} = [D_1(\delta Y^*_{t+1}) + q^*_{1,t+1}](1 - \mu^t \overline{X}) + D_2(\delta Y^*_{t+1}) + q^*_{2,t+1} + (1+r)R^1_{t+1} + (1-\delta)Y^*_{t+1}.$$

Differentiating around the point with no asset purchases gives the differential of $\widetilde{A}^1_{t+1}$:

$$d\widetilde{A}^1_{t+1} = -[D_1(\delta Y^*_{t+1}) + q^*_{1,t+1}]\mu^t d\overline{X} + (1+r)dR^1_{t+1}.$$

In the first period of the intervention, the supply of government bonds is zero and the central bank issues reserves to purchase risky assets. Thus, $R^1_{t+1} = q^1_{1,t}\mu^t\overline{X}$ and $dR^1_{t+1} = q^*_{1,t}\mu^t d\overline{X}$. As a result,

$$d\widetilde{A}^1_{t+1} = -[D_1(\delta Y^*_{t+1}) + q^*_{1,t+1} - (1+r)q^*_{1,t}]\mu^t d\overline{X}. \tag{A.23}$$

In the following periods, the central bank will, in general, make profits from its trading activity; if these profits are not immediately transferred to the treasury, the value of outstanding reserves will differ from the value of the assets owned by the central bank. This has an effect on level-1 agents, and only on them, since by assumption they do not anticipate that profits accumulated by the central bank will eventually be rebated to them. Formally, $R^1_{t+s+1} \neq q^1_{1,t+s}\mu^{t+s}\overline{X}$, for $s > 0$, and, thus, $dR^1_{t+s+1} \neq q^*_{1,t+s}\mu^{t+s}d\overline{X}$. However, under the assumption of balanced purchases (22), equation (A.23) holds in every period.

The total output effect of asset purchases follow from combining equation (A.21) in step (ii), at time $t+1$, with equation (A.22) in step (iii). For $k > 1$, we have

$$\begin{aligned} d\widetilde{V}^k_{a,t+1} &= V^*_{aa,t+1} d\widetilde{A}^k_{t+1} + dV_a(A^*_{t+1}, \epsilon, \widetilde{\mathcal{Z}}^k_{t+1}(\overline{X}))\Big|_{\overline{X}=0} \\ &= V^*_{aa,t+1}(dY^{k-1}_{t+1} + \sum_{i=1,2} dq^{k-1}_{i,t+1}) + d\mathcal{Y}'_{t+1}(A^*_{t+1}). \end{aligned} \tag{A.24}$$

Equation (26) follows from the fact that $\Delta Y^{k-1}_t = dY^{k-1}_t + o(\overline{X})$ and $\Delta q^{k-1}_{i,t} = dq^{k-1}_{i,t} + o(\overline{X})$, for all k and $i = 1, 2$.

For level $k = 1$, future beliefs coincide with those in the REE. In particular, they do not depend on $d\overline{X}$. Therefore, asset purchases will affect the *marginal value of wealth* only through future wealth. Using equation (A.23) in step (iii), we have

$$\begin{aligned} d\widetilde{V}^1_{a,t+1} &= -V^*_{aa,t+1}[D^*_{1,t+1} + q^*_{1,t+1} - (1+r)q^*_{1,t}]\mu^t d\overline{X} \\ &= -V^*_{aa,t+1} ER^*_{1,t+1}\mu^t d\overline{X}. \end{aligned} \tag{A.25}$$

Equation (27) follows from the fact that $\overline{X} = d\overline{X}$.

A.2.4 Proof of Lemma 4

We now show that the changes in output and prices in reflective equilibrium is a weighted average of changes in these variables in level-k equilibria.

Step 1. The goods market clearing condition is

$$Y^{RE}_t = \sum_{k=1}^{\infty} f(k) c^{k,RE}_t,$$

where consumption demand of level-k agents $c_t^{k,RE}$ can depend on outcomes in reflective equilibrium, which is highlighted with additional superscript RE which use throughout this proof. We next take differential when asset purchases change by $d\overline{X}$

$$dY_t^{RE} = \sum_{k=1}^{\infty} f(k) dc_t^{k,RE}.$$

Using the safe assets Euler equation for each level-k agent, we can write

$$dY_t^{RE} = \frac{(1+r)e^{\epsilon_t - \rho}}{u''(Y_t^*)} \mathbb{E}_t \sum_{k=1}^{\infty} [f(k) d\widetilde{V}_{a,t+1}^{k,RE}]. \tag{A.26}$$

At this point, it is not clear that the right-hand side of the last equation is just a sum of output changes in different level-k equilibria because $d\widetilde{V}_{a,t+1}^{k,RE}$ may depend on reflective equilibrium outcomes. We next show that this is not the case.

To do this, recall that a change in individual marginal value function is $d\widetilde{V}_{a,t+1}^{k,RE} = V_{aa,t+1}^* d\widetilde{A}_{t+1}^{k,RE} + dV_a(A_{t+1}^*, \epsilon, \widetilde{\mathcal{Z}}_{t+1}^{k,RE}(\overline{X}))\Big|_{\overline{X}=0}$. Notice that expectations of future equilibrium variables do not depend on the current realizations of equilibrium variables in reflective equilibrium because level-k agents form beliefs without understanding that they live in reflective equilibrium. As a result, $\widetilde{\mathcal{Z}}_{t+1}^{k,RE}(\overline{X}) = \widetilde{\mathcal{Z}}_{t+1}^{k}(\overline{X})$. This implies that we only need to understand how $\widetilde{A}_{t+1}^{k,RE}$ depends on reflective equilibrium outcome. Moreover, equation (A.26) suggests that we need to only understand how $\sum_{k=1}^{\infty} f(k) \widetilde{A}_{t+1}^{k,RE}$ depends on reflective equilibrium outcome. This is what we do next.

Step 2. The average across agents with different levels of sophistication forecast of their individual wealth at time $t+1$ is

$$\begin{aligned}
\widetilde{A}_{t+1}^{RE} = \sum_{k=1}^{\infty} f(k) \widetilde{A}_{t+1}^{k,RE} \quad &= \quad \sum_{k=1}^{\infty} f(k)[D_1(\delta \widetilde{Y}_{t+1}^k) + \widetilde{q}_{1,t+1}^k] x_{1,t+1}^{k,RE} + \sum_{k=1}^{\infty} f(k)[D_2(\delta \widetilde{Y}_{t+1}^k) + \widetilde{q}_{2,t+1}^k] x_{2,t+1}^{k,RE} \\
&\quad + (1+r) \sum_{k=1}^{\infty} f(k) b_{t+1}^{k,RE} + (1-\delta) \sum_{k=1}^{\infty} f(k) \widetilde{Y}_{t+1}^k - \sum_{k=1}^{\infty} f(k) \widetilde{T}_{t+1}^k,
\end{aligned}$$

where $b_{t+1}^{k,RE} \equiv a_t^{k,RE} - c_t^{k,RE} - \sum_{i=1,2} q_{i,t}^k x_{i,t+1}^{k,RE}$ is investment in safe assets.

Next we use the government and central bank budget constraints (A.16) and (A.17) to substitute for $\widetilde{T}_{t+1}^k$ in the case of $k>1$ and rewrite $\widetilde{A}_{t+1}^{RE}$ as

$$\begin{aligned}
\widetilde{A}_{t+1}^{RE} \quad = \quad & \sum_{k=2}^{\infty} f(k)[D_1(\delta Y_{t+1}^{k-1}) + q_{1,t+1}^{k-1}](x_{1,t+1}^{k,RE} + \mu^t \overline{X}) + \sum_{k=2}^{\infty} f(k)[D_2(\delta Y_{t+1}^{k-1}) + q_{2,t+1}^{k-1}] x_{2,t+1}^{k,RE} \\
& + f(1)[D_1(\delta Y_{t+1}^*) + q_{1,t+1}^*] x_{1,t+1}^{1,RE} + f(k)[D_2(\delta Y_{t+1}^*) + q_{2,t+1}^*] x_{2,t+1}^{1,RE} + (1-\delta) f(1) Y_{t+1}^* \\
& + (1+r) \sum_{k=1}^{\infty} f(k) b_{t+1}^{k,RE} \\
& + (1-\delta) \sum_{k=2}^{\infty} f(k) Y_{t+1}^{k-1} + \sum_{k=2}^{\infty} f(k)[B_{t+2}^{k-1} + R_{t+2}^{k-1} - q_{1,t+1}^{k-1} \mu^t \overline{X} - (1+r) R_{t+1}^{RE}],
\end{aligned} \tag{A.27}$$

where R_{t+1}^{RE} is the the amount of reserves issued at time t and observed by agents.

Step 3. In equilibrium, asset markets must clear for all $\overline{X}$:

$$\sum_{k=1}^{\infty} f(k) x_{1,t+1}^{k,RE} = 1 - \mu^t \overline{X},$$
$$\sum_{k=1}^{\infty} f(k) x_{2,t+1}^{k,RE} = 1,$$
$$\sum_{k=1}^{\infty} f(k) b_{t+1}^{k,RE} = (1+r) R_{t+1}^{RE} = (1+r) q_{1,t}^{RE} \mu^t \overline{X}.$$

Using these market clearing conditions, we can differentiate $\widetilde{A}_{t+1}^{RE}$ from equation (A.27) to get

$$\begin{aligned} d\widetilde{A}_{t+1}^{RE} &= \sum_{k=2}^{\infty} f(k) [dY_{t+1}^{k-1} + \sum_{i=1,2} dq_{i,t+1}^{k-1}] \\ &\quad - f(1)[D_1(\delta Y_{t+1}^*) + q_{1,t+1}^* - (1+r) q_{1,t}^*] \mu^t d\overline{X} + \sum_{k=2}^{\infty} f(k) [dB_{t+2}^{k-1} + dR_{t+2}^{k-1} - q_{1,t+1}^* \mu^{t+1} d\overline{X}] \end{aligned}$$

Notice that this last expression is nothing more than just a weighted average of changes in assets in different level-k equilibria $d\widetilde{A}_{t+1}^k$, summarized in equations (A.22) and (A.23), so that $d\widetilde{A}_{t+1}^{RE} = \sum_{k=1}^{\infty} f(k) d\widetilde{A}_{t+1}^k$. Finally, using $\Delta Y_t^{RE} = dY_t^{RE} + o(\overline{X})$ and $\Delta Y_t^k = dY_t^k + o(\overline{X})$, we obtain

$$\Delta Y_t^{RE} = \sum_{k=1}^{\infty} f(k) \Delta Y_t^k + o(\overline{X}).$$

Step 4. The proof that $\Delta q_{i,t}^{RE} = \sum_{k=1}^{\infty} f(k) \Delta q_{i,t}^k + o(\overline{X})$, $i = 1, 2$, follows from analogous steps applied to the Euler equation (A.13).

A.2.5 Proof of Proposition 3

To prove this proposition, we first show that the asset prices in REE are independent of current realization of output and, hence, of the shock ϵ_t. We then prove the statement of the proposition.

Step 1. We conjecture that equilibrium output is only a function of the shock ϵ_t and time: $Y_t^* = \nu_t(\epsilon_t)$. Using the safe-asset Euler equation

$$u'(Y_t^*) = e^{\epsilon_t - \rho}(1+r) \mathbb{E}_t[u'(Y_{t+1}^*)],$$

we can write

$$u'\left[\nu_t(\epsilon_t)\right] = e^{\epsilon_t - \rho}(1+r) \mathbb{E}_t\left[u'\left[\nu_{t+1}(\epsilon_{t+1})\right]\right]$$

or

$$\nu_t(\epsilon_t) = \left(u'\right)^{-1} \left\{ e^{\epsilon_t - \rho}(1+r) \mathbb{E}_t\left[u'\left[\nu_{t+1}(\epsilon_{t+1})\right]\right] \right\}.$$

Since shocks are i.i.d., the last equation confirms our conjecture.

The risky-assets Euler equations are

$$q_{i,t}^* = \frac{1}{1+r} \cdot \frac{\mathbb{E}_t[u'(Y_{t+1}^*)(D_i(\delta Y_{t+1}^*) + q_{i,t+1}^*)]}{\mathbb{E}_t[u'(Y_{t+1}^*)]},$$

for $i = 1, 2$. Since output depends only on the current shock and shocks are i.i.d., asset prices do not depend on ϵ_t and, hence, on output Y_t^*.

Step 2. Assume that $g_{t+1}^* = g_{t+1}(A_{t+1}^*)$, with $g_{t+1}(\cdot)$ being a *decreasing* function of equilibrium net worth, i.e. $A_{t+1}^* = Y_{t+1}^* + q_{1,t+1}^* + q_{2,t+1}^*$. The proof for an increasing function is analogous. We add a time subscript on $g_{t+1}(\cdot)$ to indicate that this function is allowed to depend on time.

Let $[Y^L, Y^H]$ denote the range of possible realizations of output Y_{t+1}^*. We also allow $Y^L = -\infty$ and $Y^H = \infty$. As a result,

$$\begin{aligned}\mathbb{E}_t\left[g_{t+1}^* V_{a,t+1}^* ER_{1,t+1}^*\right] =& \mathbb{E}_t\left[g_{t+1}\left(Y_{t+1}^* + q_{1,t+1}^* + q_{2,t+1}^*\right) V_{a,t+1}^* \cdot \left(D_1\left(\delta Y_{1,t+1}^*\right) + q_{1,t+1}^* - (1+r)q_{1,t}^*\right)\right] \\ =& \int_{Y^L}^{Y^H} g_{t+1}\left(y + q_{1,t+1}^* + q_{2,t+1}^*\right) V_{a,t+1}^* \cdot \left(D_1(\delta y) + q_{1,t+1}^* - (1+r)q_{1,t}^*\right) \phi_{Y_{1,t+1}^*|I_t}(y)dy,\end{aligned}$$

where $V_{a,t+1}^*$ may also depend on y and $\phi_{Y_{t+1}^*|I_t}(y)$ is the probability density function of Y_{t+1}^*, conditional on the information available at time t, i.e. I_t. Let $\overline{B} \equiv \max\{D_1^{-1}[(1+r)q_{1,t}^* - q_{1,t+1}^*]/\delta, Y^L\}$. Notice that this value is well defined since $D_1(\cdot)$ is strictly increasing. We can then split the integral in the last line into two parts:

$$\begin{aligned}&\mathbb{E}_t\left[g_{t+1}^* V_{a,t+1}^* ER_{1,t+1}^*\right] \\ =& \int_{Y^L}^{\overline{B}} g_{t+1}\left(y + q_{1,t+1}^* + q_{2,t+1}^*\right) V_{a,t+1}^* \cdot \left(D_1(\delta y) + q_{1,t+1}^* - (1+r)q_{1,t}^*\right) \phi_{Y_{t+1}^*|I_t}(y)dy \\ &+ \int_{\overline{B}}^{Y^H} g_{t+1}\left(y + q_{1,t+1}^* + q_{2,t+1}^*\right) V_{a,t+1}^* \cdot \left(D_1(\delta y) + q_{1,t+1}^* - (1+r)q_{1,t}^*\right) \phi_{Y_{t+1}^*|I_t}(y)dy.\end{aligned}$$

Since $g_{t+1}(\cdot)$ is a decreasing function and the excess return $D_1(\delta y) + q_{1,t+1}^* - (1+r)q_{1,t}^*$ is non-positive on the interval $y \in [Y^L, \overline{B}]$, we can bound the first integral in the last expression as follows:

$$\begin{aligned}&\int_{Y^L}^{\overline{B}} g_{t+1}\left(y + q_{1,t+1}^* + q_{2,t+1}^*\right) V_{a,t+1}^* \cdot \left(D_1(\delta y) + q_{1,t+1}^* - (1+r)q_{1,t}^*\right) \phi_{Y_{t+1}^*|I_t}(y)dy \\ &< g_{t+1}\left(\overline{B} + q_{1,t+1}^* + q_{2,t+1}^*\right) \int_{Y^L}^{\overline{B}} V_{a,t+1}^* \cdot \left(D_1(\delta y) + q_{1,t+1}^* - (1+r)q_{1,t}^*\right) \phi_{Y_{t+1}^*|I_t}(y)dy.\end{aligned}$$

Similarly, since the excess return is non-negative on the interval $y \in [\overline{B}, Y^H]$, the second integral satisfies

$$\begin{aligned}&\int_{\overline{B}}^{Y^H} g_{t+1}\left(y + q_{1,t+1}^* + q_{2,t+1}^*\right) V_{a,t+1}^* \cdot \left(D_1(\delta y) + q_{1,t+1}^* - (1+r)q_{1,t}^*\right) \phi_{Y_{t+1}^*|I_t}(y)dy \\ &< g_{t+1}\left(\overline{B} + q_{1,t+1}^* + q_{2,t+1}^*\right) \int_{\overline{B}}^{Y^H} V_{a,t+1}^* \cdot \left(D_1(\delta y) + q_{1,t+1} - (1+r)q_{1,t}\right) \phi_{Y_{t+1}^*|I_t}(y)dy.\end{aligned}$$

As a result,

$$\mathbb{E}_t\left[g_{t+1}^* V_{a,t+1}^* ER_{1,t+1}^*\right] < g_{t+1}\left(\overline{B} + q_{1,t+1}^* + q_{2,t+1}^*\right) \int_{Y^L}^{Y^H} V_{a,t+1}^* \cdot \left(D_1(\delta y) + q_{1,t+1}^* - (1+r)q_{1,t}^*\right) \phi_{Y_{t+1}^*|I_t}(y)dy = 0.$$

Finally, the last inequality combined with the expression of the first-order effect on output implies

$$dY_t^1 = -\frac{(1+r)e^{\epsilon_t - \rho}}{-u''(Y_t^*)}\mathbb{E}_t\left[g_{t+1}^* V_{a,t+1}^* ER_{1,t+1}^*\right] > 0.$$

Therefore, the output effect is positive if the value function exhibits decreasing absolute risk aversion in net worth.

A.2.6 HARA Preferences

We assume that preferences exhibit hyperbolic absolute risk aversion (HARA), which nests several well-known preferences, such as constant relative risk aversion (CRRA) when $\beta = 0, \gamma > 0$; constant absolute risk aversion (CARA) when γ tends to infinity; quadratic when $\gamma = -1$. Specifically,

$$u(c) = \frac{\gamma}{1-\gamma}\left(\frac{c}{\gamma} + \beta\right)^{1-\gamma}, \tag{A.28}$$

for some scalars β and γ. The main advantage of the HARA preferences is that the value function inherits the shape of the flow utility. The following Lemma presents the value function and the proof is in Section A.2.7 of this Appendix.

Lemma A.1. *Agent's value function in the case of HARA preferences is*

$$V(a, \epsilon_t; \widetilde{\mathcal{Z}}_t) = u\left(\frac{r}{1+r}\left[a + H(\epsilon_t, \widetilde{\mathcal{Z}}_t)\right]\right) v(\epsilon_t, \widetilde{\mathcal{Z}}_t), \tag{A.29}$$

where human wealth $H(\epsilon_t, \widetilde{\mathcal{Z}}_t)$ and the function $v(\epsilon_t, \widetilde{\mathcal{Z}}_t)$ do not depend on individual net worth.

By taking derivatives of (A.29) with respect to assets a and using market clearing conditions in the REE, it is straightforward to establish the following result.

Lemma A.2. *With HARA preferences, the risk aversion coefficient and the marginal value of wealth in the REE are*

$$g^*_{t+1} = \left[\frac{1}{\gamma}(Y^*_{t+1} + Q^* + H^*) + \frac{1+r}{r}\beta\right]^{-1}, \tag{A.30}$$

$$V^*_{a,t+1} = \left(\frac{Y^*_{t+1}}{\gamma} + \beta\right)^{-\gamma}. \tag{A.31}$$

where Q^* and H^* are, respectively, the (constant) total value of stocks and human capital in the REE.

The proof is in Appendix A.2.8. Finally, note that, with i.i.d. shocks, the price of this asset is constant, thus, the excess return $ER^*_{1,t+1}$ is simply $\delta Y^*_{t+1} - rQ^*$.

A.2.7 Proof of Lemma A.1

First, observe that net labor income is spanned by traded assets. Formally, let $\widetilde{H}_{t+s}$ be the expected value of human capital at time $t+s$—i.e. the present discounted value of net labor income—we require that there exist shares $0 \leq \phi_{1,t+s+1}, \phi_{2,t+s+1} \leq 1$, with $\phi_{1,t+s+1} + \phi_{2,t+s+1} = 1$, such that

$$(1-\delta)\widetilde{Y}_{t+s+1} - \widetilde{T}_{t+s+1} + \widetilde{H}_{t+s+1} - (1+r)\widetilde{H}_{t+s} = \sum_{i=1,2} \phi_{i,t+s+1}(\widetilde{D}_{i,t+s+1} + \widetilde{q}_{i,t+s+1} - (1+r)\widetilde{q}_{i,t+s}),$$

for all $s \geq 0$ (for $s = 0$ this value is observed by the agent and, thus, does not have a "tilde").

Then, by using the change of variables $\widehat{\widehat{x}}_{i,t+s+1} \equiv \widehat{x}_{i,t+s+1} + \alpha_{i,t+s+1}$, we can rewrite the budget constraints (19) as

$$\widehat{a}_{t+s+1} = (1+r)(\widehat{a}_{t+s} - \widehat{c}_{t+s} + \widetilde{H}_{t+s}) + \sum_{i=1,2} (\widetilde{D}_{i,t+s+1} + \widetilde{q}_{i,t+s} - (1+r)\widetilde{q}_{i,t+s})\widehat{\widetilde{x}}_{i,t+s+1} - \widetilde{H}_{t+s+1},$$

or, letting $\widehat{\widehat{a}}_{t+s} \equiv \widehat{a}_{t+s} + \widetilde{H}_{t+s}$,

$$\widehat{\widehat{a}}_{t+s+1} = (1+r)(\widehat{\widehat{a}}_{t+s} - \widehat{c}_{t+s}) + \sum_{i=1,2} (\widetilde{D}_{i,t+s+1} + \widetilde{q}_{i,t+s} - (1+r)\widetilde{q}_{i,t+s})\widehat{\widetilde{x}}_{i,t+s+1}. \tag{A.32}$$

Now, suppose c°_{t+s}, $\overline{x}^{\circ}_{i,t+s+1}$, $\overline{a}^{\circ}_{t+s+1}$, $\lambda^{\circ}_{t,t+s+1}$, $m^{\circ}_{t,t+s}$ is the solution to the agent's problem when $\overline{a}_t = 1$. Then, we show that the solution for any $\bar{a}_t$ is given by

$$\begin{aligned} c_{t+s} &= F(\overline{a}_t)(c^{\circ}_{t+s} + \beta\gamma) - \beta\gamma, \\ \overline{x}_{i,t+s+1} &= F(\overline{a}_t)\overline{x}^{\circ}_{i,t+s+1}, \\ \overline{a}_{t+s+1} &= F(\overline{a}_t)\left(\overline{a}^{\circ}_{t+s+1} + \frac{1+r}{r}\beta\gamma\right) - \frac{1+r}{r}\beta\gamma, \\ \lambda_{t,t+s+1} &= F(\overline{a}_t)^{-\gamma}\lambda^{\circ}_{t,t+s+1}, \\ m_{t,t+s} &= m^{\circ}_{t,t+s}, \\ V(a_{t+s}, \epsilon_{t+s}; \widetilde{\mathcal{Z}}_{t+s}) &= F(\overline{a}_{t+s})^{1-\gamma}\widehat{V}(\epsilon_{t+s}; \widetilde{\mathcal{Z}}_{t+s}). \end{aligned} \tag{A.33}$$

for $s \geq 0$ and where

$$F(\overline{a}_t) = \frac{1}{\frac{1}{\gamma}\cdot\frac{r}{1+r} + \beta}\left(\frac{1}{\gamma}\cdot\frac{r}{1+r}\overline{a}_t + \beta\right). \tag{A.34}$$

Note that function $F(\cdot)$ satisfies

$$F(\overline{a}_{t+s})^{1-\gamma} = F(\overline{a}_t)^{1-\gamma}F(\overline{a}^{\circ}_{t+s})^{1-\gamma}.$$

To see this, we evaluate the first-order conditions at the proposed solution (A.33) and show that they are equivalent to the first order conditions for solution under $\overline{a}_t = 1$.The first-order condition with respect to consumption in equation (A.9) is

$$e^{\sum_{j=0}^{s}\epsilon_{t+j-1}-\rho-(\epsilon_{t-1}-\rho)}u'(F_t(\overline{a}_t)(c^{\circ}_{t+s} + \beta\gamma) - \beta\gamma) = (1+r)\mathbb{E}_{t+s}[F_t(\overline{a}_t)^{-\gamma}\lambda^{\circ}_{t,t+s+1}],$$

which is clearly equivalent to

$$e^{\sum_{j=0}^{s}\epsilon_{t+j-1}-\rho-(\epsilon_{t-1}-\rho)}u'(c^{\circ}_{t+s}) = (1+r)\mathbb{E}_{t+s}[\lambda^{\circ}_{t,t+s+1}],$$

Similarly, the first order condition with respect to future assets allocation and risky assets purchases, equations (A.10) and (A.11) are

$$m_{t,t+s} = (1+r)\mathbb{E}_{t+s}[m_{t,t+s+1}] \;\Leftrightarrow\; m^{\circ}_{t,t+s} = (1+r)\mathbb{E}_{t+s}[m^{\circ}_{t,t+s+1}]$$

and

$$\mathbb{E}_{t+s}[m_{t,t+s+1}(\widetilde{D}_{i,t+s+1} + \widetilde{q}_{i,t+s+1} - (1+r)\widetilde{q}_{i,t+s})] = 0 \;\Leftrightarrow\; \mathbb{E}_{t+s}[m^{\circ}_{t,t+s+1}(\widetilde{D}_{i,t+s+1} + \widetilde{q}_{i,t+s+1} - (1+r)\widetilde{q}_{i,t+s})] = 0$$

for $s \geq 0$ and $i = 1, 2$.

Since $u'(c) = (c/\gamma + \beta)^{-\gamma}$, the term $F_t(\bar{a}_t)$ drops out from the budget constraints (A.32), when we evaluate them at the proposed solution. By definition c°_{t+s}, $\bar{x}^{\circ}_{i,t+s+1}$, $\bar{a}^{\circ}_{t+s+1}$, $\lambda^{\circ}_{t,t+s+1}$, $m^{\circ}_{t,t+s}$ satisfy the first-order conditions and the budget constraints, thus, the proposed solution (A.33) must also satisfy them. Since the first-order conditions and the budget constraints are both necessary and sufficient for the optimum, (A.33) is the solution for any $\bar{a}_t$.

Finally, to verify that the value function takes the proposed form, we write the Bellman equation for the proposed solution. Notice that the objective function in the household problem can be rewritten as

$$\mathbb{E}_t \sum_{s=0}^{\infty} e^{-\rho s + \sum_{j=1}^{s} \epsilon_{t+j-1}} u(\widehat{c}_{t+s}) = u(\widehat{c}_t) + e^{-(\rho - \epsilon_t)} \mathbb{E}_t \sum_{s=0}^{\infty} e^{-\rho s + \sum_{j=1}^{s} \epsilon_{t+j}} u(\widehat{c}_{t+1+s}).$$

By standard arguments, the recursive version of the household problem is

$$V(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t) = \max_{\widehat{c}_t, \{\widehat{\bar{x}}_{i,t+1}\}, \widehat{a}_{t+1}} \left\{ u(\widehat{c}_t) + e^{-\rho + \epsilon_t} \mathbb{E}_t V(\widehat{a}_{t+1}, \epsilon_{t+1}; \widetilde{\mathcal{Z}}_{t+1}) \right\}, \tag{A.35}$$

subject to

$$\widehat{a}_{t+1} = (1+r)(\widehat{a}_t - \widehat{c}_t + \widetilde{H}_t) + \sum_{i=1,2} (\widetilde{D}_{i,t+1} + \widetilde{q}_{i,t} - (1+r)\widetilde{q}_{i,t})\widehat{\bar{x}}_{i,t+1} - \widetilde{H}_{t+1}.$$

We now guess $V(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t) = F(a_t + H_t)^{1-\gamma} \cdot \widehat{V}(\epsilon_t; \widetilde{\mathcal{Z}}_t)$ and use the change of variables: $\widehat{c}_t = F(a_t + H_t)(\widehat{c}^{\circ}_t + \beta\gamma) - \beta\gamma$, $\widehat{\bar{x}}_{i,t+1} = F(a_t + H_t)\widehat{\bar{x}}^{\circ}_{i,t+1}$, $\widehat{a}_{t+1} = F(a_t + H_t)\left(\widehat{a}^{\circ}_{t+1} + \widetilde{H}_{t+1} + \frac{1+r}{r}\beta\gamma\right) - \frac{1+r}{r}\beta\gamma - \widetilde{H}_{t+1}$. After simple steps of algebra, we can rewrite the Bellman equation (A.35) as

$$\widehat{V}(\epsilon_t; \widetilde{\mathcal{Z}}_t) = \max_{\widehat{c}^{\circ}_t, \{\widehat{\bar{x}}^{\circ}_{i,t+1}\}, \widehat{a}^{\circ}_{t+1}} \left\{ u(\widehat{c}_t) + e^{-\rho + \epsilon_t} \mathbb{E}_t [F(\widehat{a}^{\circ}_{t+1} + \widetilde{H}_{t+1})^{1-\gamma} \cdot \widehat{V}(\epsilon_{t+1}; \widetilde{\mathcal{Z}}_{t+1})] \right\},$$

subject to

$$\widehat{a}^{\circ}_{t+1} = (1+r)(1 - \widehat{c}^{\circ}_t) + \sum_{i=1,2} (\widetilde{D}_{i,t+1} + \widetilde{q}_{i,t} - (1+r)\widetilde{q}_{i,t})\widehat{\bar{x}}^{\circ}_{i,t+1} - \widetilde{H}_{t+1},$$

which is the Bellman equation for $\bar{a}_t = 1$.

For future reference, notice that, by using the envelope theorem on equation (A.35), we obtain

$$V_a(a_t, \epsilon_t; \widetilde{\mathcal{Z}}_t) = u'(c_t). \tag{A.36}$$

A.2.8 Proof of Lemma A.2

It is convenient to begin with a characterization of aggregate output in the REE. If we combine the first-order conditions (A.9)-(A.11) and use market clearing in the goods market, we obtain the following Euler equation:

$$u'(Y^*_t) = e^{\epsilon_t - \rho}(1+r)\mathbb{E}_t[u'(Y^*_{t+1})].$$

This equation must hold in all periods and for all histories of shocks. We can thus use it to obtain an expression for output. We conjecture that output is a function only of time and the current shock: $Y^*_t = \nu_t(\epsilon_t)$. Using the assumption on preferences, we can rewrite the Euler equation as

$$\left(\frac{1}{\gamma}\nu_t(\epsilon_t) + \beta\right)^{-\gamma} = e^{\epsilon_t - \rho}(1+r)\mathbb{E}_t\left[\left(\frac{1}{\gamma}\nu_{t+1}(\epsilon_{t+1}) + \beta\right)^{-\gamma}\right].$$

Since shocks are iid, the expectation is constant and, thus,

$$\nu_t(\epsilon_t) = \gamma \left(\mathbb{E}_t \left[\left(\frac{1}{\gamma} \nu_{t+1}(\epsilon_{t+1}) + \beta \right)^{-\gamma} \right] \right)^{-\frac{1}{\gamma}} (1+r)^{-\frac{1}{\gamma}} e^{\frac{1}{\gamma}\rho} e^{-\frac{1}{\gamma}\epsilon_t} - \beta\gamma,$$
$$\equiv \gamma \overline{Y}_t e^{-\frac{1}{\gamma}\epsilon_t} - \beta\gamma.$$

Thus, the function $\nu_t(\epsilon_t) = \gamma \overline{Y}_t e^{-\frac{1}{\gamma}\epsilon_t} - \beta\gamma$ satisfies the Euler equation and verifies our conjecture. In addition, if we further assume that

$$\frac{1}{1+r} = \mathbb{E}_t \left[e^{\epsilon_{t+1} - \rho} \right], \tag{A.37}$$

the function $\overline{Y}_t$ becomes independent of time, i.e. $\overline{Y}_t \equiv \overline{Y}$. Finally, notice that the scalar $\overline{Y}$ is not defined and output is determined up to a constant. We need to ensure that marginal utility is positive, that is, $\frac{1}{\gamma}\nu_t(\epsilon_t) + \beta > 0$ or, equivalently, $\overline{Y} > 0$. Therefore, without loss of generality and to simplify the expressions below, we henceforth set $\overline{Y} = 1/|\gamma|$. We conclude that

$$Y_t^* = \frac{\gamma}{|\gamma|} e^{-\frac{1}{\gamma}\epsilon_t} - \beta\gamma. \tag{A.38}$$

We now use (A.38) to show that asset prices are independent of time. Combining the first-order condition for the risky assets (A.11) with the one for consumption (A.9) and using market clearing in the goods market gives

$$q_{i,t}^* = \frac{1}{1+r} \cdot \frac{\mathbb{E}_t[u'(Y_{t+1}^*)(D_i(\delta Y_{t+1}^*) + q_{i,t+1}^*)]}{\mathbb{E}_t[u'(Y_{t+1}^*)]}, \; i = 1, 2.$$

Since output depends only on the current shock and shocks are iid, the asset price is independent of time. We thus use q_i^* to denote the price of asset i. We also let Q^* be the total value of stocks. It satisfies

$$Q^* = \frac{1}{1+r} \cdot \frac{\mathbb{E}_t[u'(Y_{t+1}^*)(\delta Y_{t+1}^* + Q^*)]}{\mathbb{E}_t[u'(Y_{t+1}^*)]} = \frac{1}{r} \cdot \frac{\mathbb{E}_t[u'(Y_{t+1}^*)\delta Y_{t+1}^*]}{\mathbb{E}_t[u'(Y_{t+1}^*)]}. \tag{A.39}$$

Finally, we show that, in the REE, the value of human capital is constant and proportional to Q^*. By definition, human capital equals the present discounted sum of future net labor income:

$$H_t^* = \sum_{s=1}^{\infty} \mathbb{E}_t[m_{t,t+s}((1-\delta)Y_{t+s}^* - T_{t+s}^*)]$$
$$= (1-\delta) \sum_{s=1}^{\infty} \mathbb{E}_t[m_{t,t+s} Y_{t+s}^*] - \mathcal{T}_t^*,$$

where $\mathcal{T}_t^*$ is the present discounted value of future taxes in the REE, which we defined in the proof of Lemma 3. The proof of Lemma 3 shows that, in a level-k equilibrium, $\widetilde{\mathcal{T}}_t^k = R_{t+1}^{k-1} - q_{1,t}^* \mu^t \overline{X}$. It is however immediate to adapt the arguments to the REE case and show that $\mathcal{T}_t^* = R_{t+1}^* - q_{1,t}^* \mu^t \overline{X}$ or, using (22), $\mathcal{T}_t^* = 0$. Therefore,

$$H_t^* = \sum_{s=1}^{\infty} \mathbb{E}_t[m_{t,t+s}(1-\delta)Y_{t+s}^*]$$

or, using the definition of $m_{t,t+s}$ together with the first-order conditions (A.9), (A.10) and market clearing in the goods market,

$$H_t^* = \frac{1}{e^{-\rho+\epsilon_t}\mathbb{E}_t[u'(Y_{t+1}^*)]} \sum_{s=1}^{\infty} \mathbb{E}_t\left[e^{-\rho s+\sum_{j=1}^{s}\epsilon_{t+j-1}} u'(Y_{t+s}^*)(1-\delta)Y_{t+s}^*\right].$$

Since shocks are iid and output depends only on the current shock, $\mathbb{E}_t[u'(Y_{t+s}^*)Y_{t+s}^*] = \mathbb{E}_{t+s-1}[u'(Y_{t+s}^*)Y_{t+s}^*] = \mathbb{E}_t[u'(Y_{t+1}^*)Y_{t+1}^*]$. Therefore, by the law of iterated expectations,

$$\begin{aligned} H_t^* &= \frac{\mathbb{E}_t[u'(Y_{t+1}^*)(1-\delta)Y_{t+1}^*]}{e^{-\rho+\epsilon_t}\mathbb{E}_t[u'(Y_{t+1}^*)]} \sum_{s=1}^{\infty} \mathbb{E}_t\left[e^{-\rho\cdot s+\sum_{j=1}^{s}\epsilon_{t+j-1}}\right] \\ &= \frac{\mathbb{E}_t[u'(Y_{t+1}^*)(1-\delta)Y_{t+1}^*]}{\mathbb{E}_t[u'(Y_{t+1}^*)]} \sum_{s=0}^{\infty} \frac{1}{(1+r)^{s+1}} \\ &= \frac{1}{r}\cdot\frac{\mathbb{E}_t[u'(Y_{t+1}^*)(1-\delta)Y_{t+1}^*]}{\mathbb{E}_t[u'(Y_{t+1}^*)]}, \end{aligned}$$

where the second line uses (A.37). Since shocks are iid, H_t^* is independent of time. We use H^* to denote this constant value.

Finally, notice that $H^* = Q^*(1-\delta)/\delta$ and, thus,

$$Q^* + H^* = \frac{1}{r}\cdot\frac{\mathbb{E}_t[u'(Y_{t+1}^*)Y_{t+1}^*]}{\mathbb{E}_t[u'(Y_{t+1}^*)]}.$$

To compute the coefficient of absolute risk aversion g_{t+1}^*, we need to find the derivatives of the value function and evaluate them at the REE. In Section A.2.7, we showed that $V(a_t,\epsilon_t;\widetilde{\mathcal{Z}}_t) = F(\overline{a}_t)^{1-\gamma}\cdot\widehat{V}(\epsilon_t;\widetilde{\mathcal{Z}}_t)$. Thus, using (A.34), $V_a(a_t,\epsilon_t;\widetilde{\mathcal{Z}}_t) = (1-\gamma)F(\overline{a}_t)^{-\gamma}\cdot\widehat{V}(\epsilon_t;\widetilde{\mathcal{Z}}_t)\frac{1}{\gamma}\cdot\frac{r}{1+r}/(\frac{1}{\gamma}\cdot\frac{r}{1+r}+\beta)$. Differentiating again yields

$$V_{aa}(a_t,\epsilon_t;\widetilde{\mathcal{Z}}_t) = -\gamma(1-\gamma)\frac{\left(\frac{1}{\gamma}\cdot\frac{r}{1+r}\right)^2}{\left(1+\frac{1+r}{r}\beta\gamma\right)^2}F(\overline{a}_t)^{-\gamma-1}\widehat{V}(\epsilon_t;\widetilde{\mathcal{Z}}_t).$$

To compute g_{t+1}^*, we evaluate the latter in the REE, which requires $\overline{a}_{t+1} = A_{t+1}^* + H^*$ with $A_{t+1}^* = Y_{t+1}^* + Q^*$. Therefore,

$$g_{t+1}^* \equiv -\frac{V_{aa,t+1}^*}{V_{a,t+1}^*} = \gamma\left(Y_{t+1}^* + Q^* + H^* + \frac{1+r}{r}\beta\gamma\right)^{-1}. \tag{A.40}$$

Finally, notice that, if we evaluate equation (A.36) at the REE, we immediately have

$$V_{a,t+1}^* = u'(Y_{t+1}^*) = \left(\frac{Y_{t+1}^*}{\gamma}+\beta\right)^{-\gamma},$$

which is equation (A.31). For future reference, we combine the latter with (A.30) to obtain an easy expression for the second derivative of the value function:

$$V_{aa,t+1}^* = -V_{a,t+1}^*\cdot g_{t+1}^* = -\gamma\left(\frac{Y_{t+1}^*}{\gamma}+\beta\right)^{-\gamma}\frac{1}{Y_{t+1}^*+Q^*+H^*+\frac{1+r}{r}\beta\gamma}. \tag{A.41}$$

A.2.9 Proof of Proposition 4

We prove this proposition in four steps. Step 1 builds on our exposition in the case of the HARA utility and, then, it solves for output in level-1 equilibrium. Step 2 presents the output effect for level-2 equilibrium, which becomes the basis of the induction method used in Step 3. Step 3 derives level-$(k+1)$ equilibrium output change assuming that we know level-k equilibrium output effect. Step 4 concludes the proof by aggregating the behavior of all level-k agents in reflective equilibrium.

Step 1: preliminaries and level-1. We now take advantage of the results in Lemma A.2 in Appendix A.2.6. We set $\beta = 0$ and ignore the constant coefficient γ^γ in front of the preferences in equation (A.28). In this case, equations (A.30) and (A.31) result in the following expressions for first and second derivatives of the value functions evaluated at REE

$$V^*_{a,t+1} = (Y^*_{t+1})^{-\gamma},$$
$$V^*_{aa,t+1} = -\gamma \frac{(Y^*_{t+1})^{-\gamma}}{Y^*_{t+1} + Q^* + H^*}.$$

Moreover, as shown in equation (A.38) in the proof of Lemma A.2,

$$Y^*_t = \overline{Y} e^{-\frac{1}{\gamma}\epsilon_t},$$

which implies

$$m_{t+1,t+s+1} = e^{\sum_{j=1}^{s} \epsilon_{t+1+j} - \rho \cdot s}.$$

We will also need to use the following variables from equation (28),

$$\mathcal{R}_t = \mathcal{R} \equiv \mathbb{E}_t[V^*_{aa,t+1}(ER^*_{1,t+1} - \mathbb{E}_t ER^*_{1,t+1})],$$
$$\mathcal{M}_t = \mathcal{M} \equiv \mathbb{E}_t[V^*_{aa,t+1}] \cdot \mathbb{E}_t[ER^*_{1,t+1}],$$

where $\mathcal{R}_t$ and $\mathcal{M}_t$ are independent of time because asset prices are constant, output depends only on the current shock, and shocks are i.i.d. The reason why prices are constant and output is only a function of time were laid out in the proof of Lemma A.2 in Appendix A.2.8. Therefore, equation (28), in differential form, becomes

$$\frac{dY^1_t}{Y^*_t} = \frac{1}{\gamma} \overline{Y}^\gamma (1+r) e^{-\rho} (\mathcal{R} + \mathcal{M}) \mu^t d\overline{X}, \tag{A.42}$$

Note that

$$\begin{aligned}\mathcal{R} + \mathcal{M} &= \mathbb{E}_t[V^*_{aa,t+1} ER^*_{1,t+1}] \\ &= -\mathbb{E}_t \left[\gamma \frac{(Y^*_{t+1})^{-\gamma}}{Y^*_{t+1} + Q^* + H^*} \left[D_1(\delta Y^*_{t+1}) - rQ^* \right] \right].\end{aligned}$$

We now exploit the properties of CRRA preferences to find an expression for $d\widetilde{V}^k_{a,t+1}$, $k \geq 2$. More specifically, we will prove that

$$d\widetilde{V}^k_{a,t+1} = \left[-(1+\varphi_{k-1}) e^{\epsilon_{t+1}} (1+r) e^{-\rho} - V^*_{aa,t+1} \frac{1}{E} (\varphi_{k-1} - \varphi_k) \right] (\mathcal{R} + \mathcal{M}) \mu^{t+k-1} d\overline{X}, \tag{A.43}$$

where

$$E \equiv \mathbb{E}_t[-V^*_{aa,t+1}] = \mathbb{E}_t\left[\frac{\gamma\overline{Y}^{-\gamma}e^{\epsilon_{t+1}}}{\overline{Y}e^{-\frac{\epsilon_{t+1}}{\gamma}} + Q^* + H^*}\right],$$

for coefficients $\{\varphi_k\}_{k\geq 1}$ such that

$$\varphi_{k+1} - \varphi_k = \left(\frac{r}{1+r} - \varphi\left(1-\frac{1}{\gamma}\right)\frac{1+r-\mu}{\mu}\right)\sum_{j=1}^{k-1}\frac{\varphi_{k-j}}{(1+r)^j} + \frac{\varphi\left(1-\frac{1}{\gamma}\right)}{(1+r)^{k-1}}\cdot\frac{1+r-\mu}{r\mu} - \varphi_k\left(\frac{1}{1+r} - \varphi\right) - \varphi\left(1-\frac{1}{\gamma}\right)\frac{(1-\mu)(1+r)}{r\mu}. \tag{A.44}$$

with $\varphi_1 = 0$. We start with $k = 2$ (step 2) and then extend the proof to any $k > 2$ by induction (step 3). We finally compute the reflective equilibrium (step 4).

Note that, by Lemma 2,

$$dY^k_t = \frac{1}{\gamma}(1+r)e^{-\rho}Y^*_t\mathbb{E}_t[-d\widetilde{V}^k_{a,t+1}],$$

thus, equation (A.43), together with (A.37) implies

$$dY^k_t = \frac{1}{\gamma}(1+r)e^{-\rho}Y^*_t(1+\varphi_k)(\mathcal{R}+\mathcal{M})\mu^{t+k-1}d\overline{X}. \tag{A.45}$$

Step2: level-2. We begin with the level-2 equilibrium in period t. By Lemma 3, we can write

$$d\widetilde{V}^2_{a,t+1} = V^*_{aa,t+1}\cdot\left[dY^1_{t+1} + \sum_{i=1,2}dq^1_{i,t+1}\right] + d\mathcal{Y}'_{t+1}(A^*_{t+1})$$

where

$$\begin{aligned} d\mathcal{Y}'_t(A^*_t) =& V^*_{aa,t}\left(\mathbb{E}_t\sum_{s=1}^{\infty}m_{t,t+s}dY^1_{t+s} - \sum_{i=1,2}dq^1_{i,t}\right) \\ &+ V^*_{a,t}\mathbb{E}_t\sum_{s=1}^{\infty}\frac{\partial m_{t,t+s}}{\partial a}dY^1_{t+s} \\ &+ V^*_{a,t}\mathbb{E}_t\sum_{s=1}^{\infty}m_{t,t+s}\sum_{i=1,2}[\delta D'_i(\delta Y^*_{t+s})dY^1_{t+s} + dq^1_{i,t+s} - (1+r)dq^1_{i,t+s-1}]\frac{\partial x_{i,t+s}}{\partial a}. \end{aligned}$$

We now simplify the last expression by making two observations: $\partial m_{t,t+s}/\partial a = 0$ and $\partial x_{i,t+s}/\partial a = 1/[\delta(A^*_t + H^*)]$, which are the standard consequence of CRRA preferences. First, the result that $\partial m_{t,t+s}/\partial a = 0$ can be obtained from the fifth line of equation (A.33) in the proof of Lemma A.1 that implies that $m_{t,t+s}$ is independent of a_t. Second, using $x_{i,t+s+1} = \overline{x}_{i,t+s+1} - (1-\delta)/\delta$ together with the second line of equation (A.33) and equation (A.34), we have $x_{i,t+s+1} = \overline{a}_t\overline{x}^\circ_{i,t+s+1} - (1-\delta)/\delta$, where $\overline{x}^\circ_{i,t+s+1}$ is independent of a_t. Thus, $\partial x_{i,t+s+1}/\partial a_t = \overline{x}^\circ_{i,t+s+1}$. Moreover, for the asset markets to clear, it must be that, when individual wealth equals its equilibrium value $a_t = A^*_t$ and $\overline{a}_t = A^*_t + H^*$, asset demand must equal to supply: $x_{i,t+s+1} = 1$ for $i = 1, 2$. Thus $\overline{x}^\circ_{i,t+s+1} = 1/[\delta(A^*_t + H^*)]$.

Using the observation in the last paragraph, we conclude

$$d\widetilde{V}^2_{a,t+1} = V^*_{aa,t+1}\cdot\left[dY^1_{t+1} + \frac{1}{\gamma}\cdot\frac{1}{\delta}\sum_{i=1,2}dq^1_{i,t+1} + \left(1-\frac{1}{\gamma}\right)\mathbb{E}_{t+1}\sum_{s=2}^{\infty}m_{t+1,t+s}dY^1_{t+s}\right]. \tag{A.46}$$

We now compute each term separately in equation (A.46). First, by (A.42) the change in output equals

$$dY^1_{t+1} = \frac{1}{\gamma}(1+r)e^{-\rho}Y^*_{t+1}(\mathcal{R}+\mathcal{M})\mu^{t+1}d\overline{X}.$$

Second, combining equation (25) from Lemma 2 and equation (27) from Lemma 3, the change in the sum of asset prices is

$$\begin{aligned}
\frac{1}{\delta}\sum_{i=1,2} dq^1_{i,t+1} =& \frac{1}{\delta}\cdot\frac{\mathbb{E}_{t+1}[-V^*_{aa,t+2}ER^*_{1,t+2}(ER^*_{1,t+2}+ER^*_{2,t+2})]}{(1+r)\mathbb{E}_{t+1}[V^*_{a,t+2}]}\mu^{t+1}d\overline{X}\\
=& \frac{1}{\delta}\cdot\frac{\mathbb{E}_{t+1}[-V^*_{aa,t+2}ER^*_{1,t+2}(\delta Y^*_{t+2}-r\cdot Q^*)]}{(1+r)\mathbb{E}_{t+1}[V_{a,t+2}]}\mu^{t+1}d\overline{X}\\
=& \frac{\mathbb{E}_{t+1}[-V^*_{aa,t+2}ER^*_{1,t+2}(\delta Y^*_{t+2}-r\cdot\frac{Q^*}{\delta})]}{(1+r)\mathbb{E}_{t+1}[V_{a,t+2}]}\mu^{t+1}d\overline{X}\\
=& \frac{\mathbb{E}_{t+1}[-V^*_{aa,t+2}ER^*_{1,t+2}(Y^*_{t+2}-r(Q^*+H^*))]}{(1+r)\mathbb{E}_{t+1}[V^*_{a,t+2}]}\mu^{t+1}d\overline{X}\\
=& \frac{\mathbb{E}_{t+1}[\frac{\gamma\overline{Y}^{-\gamma}e^{\epsilon_{t+2}}}{Y^*_{t+2}+Q^*+H^*}ER^*_{1,t+2}(Y^*_{t+2}+Q^*+H^*)]}{(1+r)\mathbb{E}_{t+1}[V^*_{a,t+2}]}\mu^{t+1}d\overline{X} + \frac{\mathbb{E}_{t+1}[V^*_{aa,t+2}ER^*_{1,t+2}(1+r)(Q^*+H^*)]}{(1+r)\mathbb{E}_{t+1}[V^*_{a,t+2}]}\mu^{t+1}d\overline{X}\\
=&(Q^*+H^*)\frac{\mathbb{E}_{t+1}[V^*_{aa,t+2}ER^*_{1,t+2}]}{\mathbb{E}_{t+1}[V^*_{a,t+2}]}\mu^{t+1}d\overline{X},
\end{aligned}$$

where we used the facts that $Q^*\delta = H^*/(1-\delta)$ and $\mathbb{E}_{t+1}[\gamma\overline{Y}^{-\gamma}e^{\epsilon_{t+2}}ER^*_{1,t+2}] = 0$, which is the risky assets Euler equation (A.11). Using the definition of $\mathcal{R}$ and $\mathcal{M}$,

$$\frac{1}{\delta}\sum_{i=1,2} dq^1_{i,t+1} = (Q^*+H^*)\frac{1}{\mathbb{E}_{t+1}[V^*_{a,t+2}]}(\mathcal{R}+\mathcal{M})\mu^{t+1}d\overline{X}.$$

The expectations of marginal value of wealth can be replaced in the last equation noting that $\mathbb{E}_{t+1}[V^*_{a,t+2}] = \mathbb{E}_{t+1}[(Y^*_{t+2})^{-\gamma}] = \overline{Y}^{-\gamma}\mathbb{E}_{t+1}[e^{\epsilon_{t+2}}] = \overline{Y}^{-\gamma}e^{\rho}/(1+r)$, where the last equality used equation (A.37). As a result,

$$\frac{1}{\delta}\sum_{i=1,2} dq^1_{i,t+1} = (Q^*+H^*)(1+r)\overline{Y}^{\gamma}e^{-\rho}(\mathcal{R}+\mathcal{M})\mu^{t+1}d\overline{X}. \tag{A.47}$$

Finally, we find an expression for the infinite sum of expected changes in future income. As we noted in Step 1 of this proof, output in REE is $Y^*_{t+s+1} = \overline{Y}e^{-\frac{1}{\gamma}\epsilon_{t+s+1}}$. Thus, using (A.42), we obtain

$$\mathbb{E}_{t+1}\sum_{s=1}^{\infty} m_{t+1,t+s+1}dY^1_{t+s+1} = \frac{1}{\gamma}(1+r)e^{-\rho}(\mathcal{R}+\mathcal{M})\mathbb{E}_{t+1}\sum_{s=1}^{\infty} e^{\sum_{j=1}^{s}\epsilon_{t+1+j}-\rho\cdot s}Y^*_{t+s+1}\mu^{t+s+1}d\overline{X}.$$

From the law of iterated expectations and the fact that shocks are i.i.d. over time,

$$\mathbb{E}_{t+1}\sum_{s=1}^{\infty} m_{t+1,t+s+1}dY^1_{t+s+1} = \mathbb{E}\left[\exp(\epsilon_{t+1}-\rho)Y^*_{t+1}\right]\mathbb{E}_{t+1}\left[\sum_{s=1}^{\infty} m_{t+1,t+s}\right]\mu^{t+s+1}d\overline{X},$$

Using the law of iterated expectations together with the fact that the SDF satisfies $(1+r)\mathbb{E}_{t+s-1}[m_{t+1,t+s}] =$

$m_{t+1,t+s-1}$, $s > 1$, and $m_{t+1,t+1} = 1$, we obtain

$$\begin{aligned}\mathbb{E}_{t+1}[m_{t+1,t+s}] &= \mathbb{E}_{t+1}[\mathbb{E}_{t+s-1}[m_{t+1,t+s-1}]] \\ &= (1+r)^{-s+1}.\end{aligned}$$

As a result, assuming that $1 + r > \mu$, we have

$$\mathbb{E}_{t+1} \sum_{s=1}^{\infty} m_{t+1,t+s+1} dY^1_{t+s+1} = \frac{1}{\gamma}(1+r)e^{-\rho}(\mathcal{R} + \mathcal{M})\mathbb{E}\left[\exp(\epsilon_{t+1} - \rho)Y^*_{t+1}\right] \frac{\mu(1+r)}{1+r-\mu}\mu^{t+1} d\overline{X}.$$

We can further simplify the expression by noticing that, since labor income is perfectly correlated with capital income, it must be priced by the SDF, that is,

$$\mathbb{E}_t[m_{t,t+1}(Y^*_{t+1} - r(Q^* + H^*))] = 0$$

or using $m_{t,t+1} = e^{\epsilon_{t+1}-\rho}$,

$$r(Q^* + H^*) = \frac{\mathbb{E}_t[\exp(\epsilon_{t+1} - \rho)Y^*_{t+1}]}{\mathbb{E}_t[\exp(\epsilon_{t+1} - \rho)]} = \frac{\mathbb{E}_t[\overline{Y} e^{\epsilon_{t+1}(1-\frac{1}{\gamma})}]}{\mathbb{E}_t[e^{\epsilon_{t+1}}]}. \tag{A.48}$$

Finally, since shocks are i.i.d., conditional expectations coincide with unconditional ones, thus, by (A.37), we have

$$Q^* + H^* = \frac{\mathbb{E}[(1+r)e^{\epsilon_{t+1}-\rho}Y^*_{t+1}]}{r}. \tag{A.49}$$

We conclude that

$$\mathbb{E}_{t+1} \sum_{s=1}^{\infty} m_{t+1,t+s+1} dY^1_{t+s+1} = \frac{1}{\gamma}(1+r)e^{-\rho}(\mathcal{R} + \mathcal{M})\frac{r(Q^* + H^*)}{1+r}\mu^{t+2} d\overline{X}\frac{1}{1 - \frac{\mu}{1+r}}. \tag{A.50}$$

It is useful to compare the last formula to the following expression $m_{t,t+1} dY^1_{t+1} = \frac{1}{\gamma}(1+r)e^{-\rho}(\mathcal{R} + \mathcal{M})[m_{t,t+1}Y^*_{t+1}]\mu^{t+1} d\overline{X}$. Note that the term μ^{t+2} appears in (A.50) because the sum on the left-hand side of this equation starts from period $t + 2$.

Combining (A.42), (A.47) and (A.50), we obtain

$$\begin{aligned}-d\widetilde{V}^2_{a,t+1} = -V^*_{aa,t+1}\Bigg[&\frac{1}{\gamma}\overline{Y}^{\gamma}(1+r)e^{-\rho}Y^*_{t+1} + \frac{1}{\gamma}(Q^* + H^*)(1+r)\overline{Y}^{\gamma}e^{-\rho} \\ &+ \left(1 - \frac{1}{\gamma}\right)\frac{r}{\gamma}(Q^* + H^*)e^{-\rho}\frac{\mu(1+r)}{1+r-\mu}\Bigg](\mathcal{R} + \mathcal{M})\mu^{t+1} d\overline{X}\end{aligned}$$

or, using $V^*_{aa,t+1} = -\frac{\gamma\overline{Y}^{-\gamma}e^{\epsilon_{t+1}}}{Y^*_{t+1} + Q^* + H^*}$,

$$\begin{aligned}-d\widetilde{V}^2_{a,t+1} &= \left[e^{\epsilon_{t+1}}(1+r)e^{-\rho} - V^*_{aa,t+1}\left(1 - \frac{1}{\gamma}\right)\frac{r}{\gamma}(Q^* + H^*)e^{-\rho}\frac{\mu(1+r)}{1+r-\mu}\right](\mathcal{R} + \mathcal{M})\mu^{t+1} d\overline{X} \\ &= \left(e^{\epsilon_{t+1}}(1+r)e^{-\rho} + \frac{-V^*_{aa,t+1}}{E}\varphi\right)(\mathcal{R} + \mathcal{M})\mu^{t+1} d\overline{X}. \end{aligned} \tag{A.51}$$

The latter proves (A.43) for $k = 2$, if we set $\varphi_2 = \left(1 - \frac{1}{\gamma}\right) \varphi$, where

$$\varphi \equiv E\frac{1}{\gamma}(1+r)e^{-\rho}\frac{r(Q^* + H^*)}{1+r} \cdot \frac{1}{1 - \frac{\mu}{1+r}}\mu > 0. \tag{A.52}$$

Also, from (A.45), output is

$$dY_t^2 = \frac{1}{\gamma}(1+r)e^{-\rho}Y_t^* \left(1 + \left(1 - \frac{1}{\gamma}\right)\varphi\right)(\mathcal{R} + \mathcal{M})\mu^{t+1}d\overline{X}. \tag{A.53}$$

Comparing equations (A.42) and (A.53), we see that

$$\frac{dY_t^2}{Y_t^*} = \frac{dY_{t+1}^1}{Y_{t+1}^*} + \varphi\frac{dY_{t+1}^1}{Y_{t+1}^*} - \frac{1}{\gamma}\varphi\frac{dY_{t+1}^1}{Y_{t+1}^*}. \tag{A.54}$$

The last equation is instructive. It states that the percentage change in period-t output in level-2 equilibrium equals the percentage change in period-$(t+1)$ output in level-1 equilibrium plus the same output change times the coefficient $\varphi(1 - 1/\gamma)$. The first term is the standard New Keynesian effect. Specifically, level-2 agents think that their income will increase by dY_{t+1}^1/Y_{t+1}^* in the next period, as a result they increase their consumption and saving in period t. But when prices are sticky, an increase in consumption in period t, increases their income in period t so that the percentage change in output in period t identically equals the percentage change in period $t+1$ output. In fact, saving do not change in equilibrium because the supply of saving is unchanged. The second and third terms in equation (A.54) arise from the fact that level-2 agents expect changes in their more distant future consumption due to changes in distant future income (starting from period $t+2$). As before, an increase in future income has two effects: it increases consumption and saving. However, unlike in period t, market forces do not insure that their plans about their future saving choice satisfy market equilibrium conditions. As a result, we have the second and third terms with the opposing signs. Note that $1/\gamma$ indexes the strength of the effect on saving. In the special case, when agents have log utility, $\varphi(1 - 1/\gamma) = 0$ implying that distant future income changes do not affect consumption choice in period t.

Step 3: from level-k to level-$(k+1)$. We now proceed by induction. Assume that $k \geq 2$. Also assume that equation (A.43) holds for all l such that $2 \leq l \leq k$. Next, we compute $d\widetilde{V}_{a,t+1}^{k+1}$, which, by equation (A.46), equals

$$d\widetilde{V}_{a,t+1}^{k+1} = V_{aa,t+1}^* \cdot \left[dY_{t+1}^k + \frac{1}{\gamma\delta}\sum_{i=1,2} dq_{i,t+1}^k + \left(1 - \frac{1}{\gamma}\right)\mathbb{E}_{t+1}\sum_{s=1}^{\infty} m_{t+1,t+s+1}dY_{t+s+1}^k\right]. \tag{A.55}$$

We derive each term inside the square bracket of the last equation separately.

The first term has already been computed in equation (A.45). Second, we compute the change in the asset prices sum $\sum_{i=1,2} dq_{i,t}^k$. Using Lemma 2 again (equation (25)), the asset prices sum satisfies

$$\frac{1}{\delta}\sum_{i=1,2} dq_{i,t}^k = \frac{1}{\delta} \cdot \frac{\mathbb{E}_t\left[d\widetilde{V}_{a,t+1}^k(\delta Y_{t+1}^* - rQ^*)\right]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]} + \frac{\mathbb{E}_t\left[V_{a,t+1}^*(dY_{t+1}^{k-1} + \frac{1}{\delta}\sum_{i=1,2} dq_{i,t+1}^{k-1})\right]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]}.$$

In Section A.2.8, we showed that $H^* = Q^*(1-\delta)/\delta$ or $Q^*/\delta = Q^* + H^*$. Thus, using equation (A.43) for

$l = k$, we rewrite the change in the asset prices sum as

$$\frac{1}{\delta}\sum_{i=1,2} dq_{i,t}^k = \frac{\mathbb{E}_t\left[-e^{\epsilon_{t+1}}(Y_{t+1}^* - r(Q^*+H^*))\right]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]}(1+r)e^{-\rho}(1+\varphi_{k-1})(\mathcal{R}+\mathcal{M})\mu^{t+k-1}d\overline{X}$$
$$+\frac{\mathbb{E}_t\left[-V_{aa,t+1}^*(Y_{t+1}^* - r(Q^*+H^*))\right]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]}\cdot\frac{1}{E}(\varphi_{k-1}-\varphi_k)(\mathcal{R}+\mathcal{M})\mu^{t+k-1}d\overline{X}$$
$$+\frac{\mathbb{E}_t\left[V_{a,t+1}^*(dY_{t+1}^{k-1}+\frac{1}{\delta}\sum_{i=1,2}dq_{i,t+1}^{k-1})\right]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]}$$

From $V_{a,t+1}^* = \overline{Y}^{-\gamma}e^{\epsilon_{t+1}}$and equation (A.49), the first line in the previous expression is zero. Hence,

$$\frac{1}{\delta}\sum_{i=1,2} dq_{i,t}^k = \frac{\mathbb{E}_t\left[-V_{aa,t+1}^*(Y_{t+1}^* - r(Q^*+H^*))\right]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]}\cdot\frac{1}{E}(\varphi_{k-1}-\varphi_k)(\mathcal{R}+\mathcal{M})\mu^{t+k-1}d\overline{X}$$
$$+\frac{\mathbb{E}_t\left[V_{a,t+1}^*(dY_{t+1}^{k-1}+\frac{1}{\delta}\sum_{i=1,2}dq_{i,t+1}^{k-1})\right]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]}.$$

Adding and subtracting Q^*+H^* to $Y_{t+1}^* - r(Q^*+H^*)$ and using $V_{aa,t+1}^* = -\frac{\gamma\overline{Y}^{-\gamma}e^{\epsilon_{t+1}}}{Y_{t+1}^*+Q^*+H^*}$ gives

$$\frac{1}{\delta}\sum_{i=1,2} dq_{i,t}^k = \left[\frac{\gamma}{1+r}\cdot\frac{1}{E}+\frac{\mathbb{E}_t[V_{aa,t+1}^*]}{\mathbb{E}_t[V_{a,t+1}^*]}(Q^*+H^*)\frac{1}{E}\right](\varphi_{k-1}-\varphi_k)(\mathcal{R}+\mathcal{M})\mu^{t+k-1}d\overline{X}$$
$$+\frac{\mathbb{E}_t\left[V_{a,t+1}^*(dY_{t+1}^{k-1}+\frac{1}{\delta}\sum_{i=1,2}dq_{i,t+1}^{k-1})\right]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]}.$$

Since shocks are i.i.d., $\mathbb{E}_t[V_{a,t+1}^*] = \mathbb{E}[\overline{Y}^{-\gamma}e^{\epsilon_{t+1}}]$ and

$$\frac{1}{\delta}\sum_{i=1,2} dq_{i,t}^k = \left[\frac{\gamma}{1+r}\cdot\frac{1}{E}-\frac{\overline{Y}^{\gamma}}{\mathbb{E}[e^{\epsilon_{t+1}}]}(Q^*+H^*)\right](\varphi_{k-1}-\varphi_k)(\mathcal{R}+\mathcal{M})\mu^{t+k-1}d\overline{X}$$
$$+\frac{\mathbb{E}_t\left[V_{a,t+1}^*(dY_{t+1}^{k-1}+\frac{1}{\delta}\sum_{i=1,2}dq_{i,t+1}^{k-1})\right]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]}. \tag{A.56}$$

The last terms of equation (A.56) can be simplified further by using (A.45) for time period $t+1$ and the level of sophistication $k-1$:

$$\frac{\mathbb{E}_t\left[V_{a,t+1}^*(dY_{t+1}^{k-1}+\frac{1}{\delta}\sum_{i=1,2}dq_{i,t+1}^{k-1})\right]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]} = \frac{\mathbb{E}_t\left[V_{a,t+1}^*dY_{t+1}^{k-1}\right]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]}+\frac{\mathbb{E}_t\left[V_{a,t+1}^*\frac{1}{\delta}\sum_{i=1,2}dq_{i,t+1}^{k-1}\right]}{(1+r)\mathbb{E}_t[V_{a,t+1}^*]}$$
$$=\frac{\mathbb{E}_t[V_{a,t+1}^*Y_{t+1}^*]}{\mathbb{E}_t[V_{a,t+1}^*]}\cdot\frac{1}{\gamma}e^{-\rho}(1+\varphi_{k-1})(\mathcal{R}+\mathcal{M})\mu^{t+k-1}d\overline{X}$$
$$+\frac{1}{(1+r)\delta}\sum_{i=1,2}dq_{i,t+1}^{k-1}.$$

Since shocks are i.i.d., $\mathbb{E}_t[V_{a,t+1}^*Y_{t+1}^*]/\mathbb{E}_t[V_{a,t+1}^*] = \mathbb{E}[\overline{Y}^{-\gamma}\exp(\epsilon_{t+1})Y_{t+1}^*]/\mathbb{E}[\overline{Y}^{-\gamma}\exp(\epsilon_{t+1})] = r(Q^*+H^*)$.

Where the last equality used equation (A.48). Therefore,

$$\frac{\mathbb{E}_t\left[V^*_{a,t+1}(dY^{k-1}_{t+1}+\frac{1}{\delta}\sum_{i=1,2}dq^{k-1}_{i,t+1})\right]}{(1+r)\mathbb{E}_t[V^*_{a,t+1}]}=\frac{r}{\gamma}e^{-\rho}(1+\varphi_{k-1})(Q^*+H^*)(\mathcal{R}+\mathcal{M})\mu^{t+k-1}d\overline{X}+\frac{1}{(1+r)\delta}\sum_{i=1,2}dq^{k-1}_{i,t+1}.$$

Combining the latter with (A.56) gives

$$\begin{aligned}\frac{1}{\delta}\sum_{i=1,2}dq^k_{i,t}=&\frac{\gamma}{1+r}\cdot\frac{1}{E}(\varphi_{k-1}-\varphi_k)(\mathcal{R}+\mathcal{M})\mu^{t+k-1}d\overline{X}\\&+\left(\frac{r}{\gamma}(1+\varphi_{k-1})+\overline{Y}^{\gamma}(1+r)(\varphi_k-\varphi_{k-1})\right)e^{-\rho}(Q^*+H^*)(\mathcal{R}+\mathcal{M})\mu^{t+k-1}d\overline{X}+\frac{1}{1+r}\cdot\frac{1}{\delta}\sum_{i=1,2}dq^{k-1}_{i,t+1}.\end{aligned}$$

Iterating forward and using the initial condition (A.47), after a few steps of straightforward algebra, we obtain

$$\begin{aligned}\frac{1}{\delta}\sum_{i=1,2}dq^k_{i,t}=&\frac{\gamma}{E}\left(\frac{r}{1+r}\sum_{j=1}^{k-1}\frac{\varphi_{k-j}}{(1+r)^j}-\frac{1}{1+r}\varphi_k\right)(\mathcal{R}+\mathcal{M})\mu^{t+k-1}d\overline{X}\\&+\frac{\gamma\varphi}{E}\left(1-\frac{1}{\gamma}\right)\left(\frac{1+r}{r}\cdot\frac{1}{\gamma-1}+\frac{1}{r(1+r)^{k-2}}-\sum_{j=1}^{k-1}\frac{\varphi_{k-j}}{(1+r)^{j-1}}+\frac{\gamma}{\gamma-1}\cdot\frac{1+r}{r}\varphi_k\right)\\&\cdot\frac{1+r-\mu}{\mu(1+r)}(\mathcal{R}+\mathcal{M})\mu^{t+k-1}d\overline{X}.\end{aligned}\tag{A.57}$$

Finally, we are left to compute the infinite sum $\mathbb{E}_{t+1}\sum_{s=1}^{\infty}m_{t+1,t+s+1}dY^k_{t+s+1}$ in equation (A.55). From equations (A.42) and (A.45),

$$\begin{aligned}dY^k_{t+s+1}&=\frac{1}{\gamma}(1+r)e^{-\rho}Y^*_{t+s+1}(\mathcal{R}+\mathcal{M})(1+\varphi_k)\mu^{t+k-1+s+1}d\overline{X}\\&=(1+\varphi_k)\mu^{t+k-1}dY^1_{t+s+1}.\end{aligned}$$

We can then repeat the same steps as for case with $k=2$ to obtain

$$\mathbb{E}_{t+1}\sum_{s=1}^{\infty}m_{t+1,t+s+1}dY^k_{t+s+1}=\frac{r}{\gamma}e^{-\rho}(\mathcal{R}+\mathcal{M})(Q^*+H^*)\frac{\mu(1+r)}{1+r-\mu}(1+\varphi_k)\mu^{t+k}d\overline{X}.\tag{A.58}$$

We now out equations (A.45), (A.57), and (A.58) together to compute (A.55)

$$\begin{aligned}\frac{d\widetilde{V}^{k+1}_{a,t+1}}{V^*_{aa,t+1}}=&\left\{\frac{1}{\gamma}(1+r)e^{-\rho}(1+\varphi_k)Y^*_{t+1}+\frac{1}{E}\cdot\frac{r}{1+r}\sum_{j=1}^{k-1}\frac{\varphi_{k-j}}{(1+r)^j}-\frac{1}{E}\cdot\frac{1}{1+r}\varphi_k\right.\\&+\frac{\varphi}{E}\left(1-\frac{1}{\gamma}\right)\left(\frac{1+r}{r}\cdot\frac{1}{\gamma-1}+\frac{1}{r(1+r)^{k-2}}-\sum_{j=1}^{k-1}\frac{\varphi_{k-j}}{(1+r)^{j-1}}+\frac{\gamma}{\gamma-1}\cdot\frac{1+r}{r}\varphi_k\right)\frac{1+r-\mu}{\mu(1+r)}\\&\left.+\frac{\varphi}{E}(1+\varphi_k)\right\}(\mathcal{R}+\mathcal{M})\mu^{t+k}d\overline{X}.\end{aligned}$$

Adding and subtracting $Q^* + H^*$ and using $-V^*_{aa,t+1} = \frac{\gamma \overline{Y}^{-\gamma} e^{\epsilon_{t+1}}}{Y^*_{t+1}+Q^*+H^*}$ yields

$$
\begin{aligned}
d\widetilde{V}^{k+1}_{a,t+1} = &- e^{\epsilon_{t+1}}(1+r)e^{-\rho}(1+\varphi_k)(\mathcal{R}+\mathcal{M})\mu^{t+k}d\overline{X} + V^*_{aa,t+1}\frac{1}{E}\Bigg\{\frac{r}{1+r}\sum_{j=1}^{k-1}\frac{\varphi_{k-j}}{(1+r)^j} - \frac{1}{1+r}\varphi_k \\
&+ \varphi\left(1-\frac{1}{\gamma}\right)\left(-\frac{1+r}{r} + \frac{1}{r(1+r)^{k-2}} - \sum_{j=1}^{k-1}\frac{\varphi_{k-j}}{(1+r)^{j-1}}\right)\frac{1+r-\mu}{\mu(1+r)} + \varphi\left(1-\frac{1}{\gamma}\right)(1+\varphi_k)\Bigg\}(\mathcal{R}+\mathcal{M})\mu^{t+k}d\overline{X},
\end{aligned}
$$

where we also used the definition of φ in (A.52). After a few steps of straightforward algebra, we finally obtain

$$
\begin{aligned}
d\widetilde{V}^{k+1}_{a,t+1} = &- e^{\epsilon_{t+1}}(1+r)e^{-\rho}(1+\varphi_k)(\mathcal{R}+\mathcal{M})\mu^{t+k}d\overline{X} \\
&+ V^*_{aa,t+1}\frac{1}{E}\Bigg\{\left(\frac{r}{1+r} - \varphi\left(1-\frac{1}{\gamma}\right)\frac{1+r-\mu}{\mu}\right)\sum_{j=1}^{k-1}\frac{\varphi_{k-j}}{(1+r)^j} + \frac{\varphi\left(1-\frac{1}{\gamma}\right)}{(1+r)^{k-1}}\cdot\frac{1+r-\mu}{r\mu} \\
&- \varphi_k\left(\frac{1}{1+r} - \varphi\left(1-\frac{1}{\gamma}\right)\right) - \varphi\left(1-\frac{1}{\gamma}\right)\frac{(1-\mu)(1+r)}{r\mu}\Bigg\}(\mathcal{R}+\mathcal{M})\mu^{t+k}d\overline{X}. \qquad \text{(A.59)}
\end{aligned}
$$

Equation (A.59) is identical to our conjecture (A.43), when φ_{k+1} satisfies the difference equation (A.44). Instead of solving equation (A.44) explicitly, we only compute the output effect in reflective equilibrium in the next Step 4.

Step 4 (reflective equilibrium). Multiply both sides of (A.44) by $f(k+1)\mu^k$ and sum over $k \geq 2$:

$$
\begin{aligned}
&\sum_{k=2}^{\infty} f(k+1)\mu^k\varphi_{k+1} - \sum_{k=2}^{\infty} f(k+1)\mu^k\varphi_k \\
&= \left(\frac{r}{1+r} - \varphi\left(1-\frac{1}{\gamma}\right)\frac{1+r-\mu}{\mu}\right)\sum_{k=2}^{\infty} f(k+1)\mu^k\sum_{j=1}^{k-1}\frac{\varphi_{k-j}}{(1+r)^j} \\
&\quad + \frac{1+r-\mu}{r\mu}\sum_{k=2}^{\infty} f(k+1)\mu^k\frac{\varphi\left(1-\frac{1}{\gamma}\right)}{(1+r)^{k-1}} \\
&\quad - \left(\frac{1}{1+r} - \varphi\left(1-\frac{1}{\gamma}\right)\right)\sum_{k=2}^{\infty} f(k+1)\mu^k\varphi_k - \varphi\left(1-\frac{1}{\gamma}\right)\frac{(1-\mu)(1+r)}{r\mu}\sum_{k=2}^{\infty} f(k+1)\mu^k.
\end{aligned}
$$

Using $f(k) = (1-\lambda)\lambda^{k-1}$ and letting $S \equiv \sum_{k=1}^{\infty} f(k)\mu^{k-1}\varphi_k$ (we assume that this sum exists), after a few steps of algebra, we obtain

$$
\begin{aligned}
&S - f(2)\mu\varphi\left(1-\frac{1}{\gamma}\right) - \lambda\mu S \\
&= \left(\frac{r}{1+r} - \varphi\left(1-\frac{1}{\gamma}\right)\frac{1+r-\mu}{\mu}\right)\frac{\lambda^2\mu^2}{1+r-\lambda\mu}S + \varphi\left(1-\frac{1}{\gamma}\right)\frac{1+r-\mu}{r\mu}(1-\lambda)\frac{\lambda^2\mu^2}{1+r-\lambda\mu} \\
&\quad - \left(\frac{1}{1+r} - \varphi\left(1-\frac{1}{\gamma}\right)\right)\lambda\mu S - \varphi\left(1-\frac{1}{\gamma}\right)\frac{(1-\mu)(1+r)}{r\mu}\cdot\frac{(1-\lambda)\lambda^2\mu^2}{1-\lambda\mu}.
\end{aligned}
$$

This equation has solution

$$S = \frac{(1-\lambda)^2\lambda\mu\left(1-\frac{1}{\gamma}\right)\varphi}{(1-\lambda\mu)(1-\lambda\mu(1+(1-\lambda)\left(1-\frac{1}{\gamma}\right)\varphi))}.$$

Also, if we multiply both sides of (A.59) by $f(k+1)$ and sum over $k \geq 2$, we have

$$\begin{aligned}\sum_{k=2}^{\infty} f(k+1)d\widetilde{V}_{a,t+1}^{k+1} =& - e^{\epsilon_{t+1}}(1+r)e^{-\rho}\left(\frac{(1-\lambda)\lambda^2\mu^2}{1-\lambda\mu} + \lambda\mu S\right)(\mathcal{R}+\mathcal{M})\mu^t d\overline{X} \\ &+ V_{aa,t+1}^{*}\frac{1}{E}\left(S - f(2)\mu\left(1-\frac{1}{\gamma}\right)\varphi - \lambda\mu S\right)(\mathcal{R}+\mathcal{M})\mu^t d\overline{X}\end{aligned}$$

or, adding the term (A.51) multiplied by $f(2)$,

$$\begin{aligned}\sum_{k=1}^{\infty} f(k+1)d\widetilde{V}_{a,t+1}^{k+1} =& - e^{\epsilon_{t+1}}(1+r)e^{-\rho}\left(\frac{1-\lambda}{1-\lambda\mu}\lambda\mu + \lambda\mu S\right)(\mathcal{R}+\mathcal{M})\mu^t d\overline{X} \\ &+ V_{aa,t+1}\frac{1}{E}S(1-\lambda\mu)(\mathcal{R}+\mathcal{M})\mu^t d\overline{X}.\end{aligned}$$

Finally, taking expectations of both sides, we obtain

$$\begin{aligned}\mathbb{E}_t\left[-\sum_{k=1}^{\infty} f(k+1)d\widetilde{V}_{a,t+1}^{k+1}\right] &= \left(\frac{1-\lambda}{1-\lambda\mu}\lambda\mu + S\right)(\mathcal{R}+\mathcal{M})\mu^t d\overline{X} \\ &= \frac{(1-\lambda)\lambda\mu(1+(1-\lambda)\left(1-\frac{1}{\gamma}\right)\varphi(\mu))}{1-\lambda\mu(1+(1-\lambda)\left(1-\frac{1}{\gamma}\right)\varphi(\mu))}(\mathcal{R}+\mathcal{M})\mu^t d\overline{X}.\end{aligned}$$

Therefore, the effect of asset purchases on output in the reflective equilibrium is

$$\begin{aligned}dY_t^{RE} &= \frac{1}{\gamma}(1+r)e^{-\rho}Y_t^{*}\left(f(1)d\overline{X} + \mathbb{E}_t\left[-\sum_{k=1}^{\infty} f(k+1)d\widetilde{V}_{a,t+1}^{k+1}\right]\right) \\ &= \frac{1}{\gamma}(1+r)e^{-\rho}Y_t^{*}\frac{1-\lambda}{1-\lambda\mu(1+(1-\lambda)\left(1-\frac{1}{\gamma}\right)\varphi(\mu))}(\mathcal{R}+\mathcal{M})\mu^t d\overline{X}.\end{aligned}$$

Equation (30) then follows from the fact that $\Delta Y_t^{RE} = dY_t^{RE} + o(\overline{X})$ and $\overline{X} = d\overline{X}$.

We are left to prove that dY_t^{RE} has the same sign as $\mathcal{R}+\mathcal{M}$. First, notice that, when $\gamma \leq 1$, we immediately have

$$\Gamma(\mu) \equiv \frac{1-\lambda}{1-\lambda\mu(1+(1-\lambda)\left(1-\frac{1}{\gamma}\right)\varphi(\mu))} \geq 0,$$

where, with a slight abuse of notation, we have made the dependence of φ on μ explicit. Now suppose $\gamma > 1$. We first show that $\Gamma(1) > 0$. To see this, notice that, when $\mu = 1$, the solution to the recursion (A.44) is simply $\varphi_k = \sum_{j=1}^{k-1}\left(1-\frac{1}{\gamma}\right)^j \varphi(1)^j$. As a result,

$$S \equiv \sum_{k=1}^{\infty} f(k)\varphi_k\mu^{k-1} = \sum_{k=2}^{\infty} f(k)\left(\sum_{j=1}^{k-1}\left(1-\frac{1}{\gamma}\right)^j\varphi(1)^j\right) = (1-\lambda)\sum_{k=1}^{\infty}\left(1-\frac{1}{\gamma}\right)^k\lambda^k\varphi(1)^k.$$

This sum exists if and only if $\left(1-\frac{1}{\gamma}\right)\lambda\varphi(1) < 1$. The latter implies $\Gamma(1) > 0$. Finally, differentiating $\Gamma(\cdot)$ yields

$$\Gamma'(\mu) = (1-\lambda)\frac{\lambda + \lambda(1-\lambda)\left(1-\frac{1}{\gamma}\right)[\varphi(\mu) + \mu\varphi'(\mu)]}{(1-\lambda\mu(1+(1-\lambda)\left(1-\frac{1}{\gamma}\right)\varphi(\mu)))^2},$$

which is positive since, from (A.52), $\varphi'(\mu) > 0$. We conclude that $\Gamma(\mu) > 0$ for all μ and, therefore, dY_t^{RE} has the same sign as $\mathcal{R} + \mathcal{M}$.

B Predictability of Forecast Errors in the Data

In this appendix, we provide details of the empirical exercise that we only briefly described in the main text. As we mention in the main text, we follow Fieldhouse, Mertens and Ravn (2018). The authors provide a comprehensive description of the institutional details of the operations of the GSEs. Here, we briefly describe some of the details that are relevant for understanding our empirical results.

The GSEs have been routinely buying mortgages from mortgage issuers since their incorporation in the 1960s. They finance their purchases with debt securities that command a "liquidity and safety" premium similar to the one of Treasury securities. Although most of these purchases are motivated by the cyclical developments in the mortgage market (e.g., stimulating housing starts in recessions), some purchases are related to non-cyclical regulatory events (e.g., those invoked by a desire to increase homeownership among lower-income households or by concerns regarding structural budget deficits). FMR use narrative records to identify the motivation behind any considerable change in the GSEs' mortgage purchasing behavior and construct a list of major regulatory events that are not related to cyclical considerations (the paper contains a detailed discussion of the construction of these narrative events). We call these events "exogenous."

To quantify the impact of these exogenous events, FMR use various sources to obtain an estimate of the projected impact, denoted by m_t, of the agencies' capacity to purchase mortgages during the first year following the moment when a policy is publicly announced. Therefore, m_t can be thought of as news about future purchases by the GSEs following the exogenous events. The approach in FMR is similar to the one used in the literature on the effects of fiscal policies (see, for example, Ramey and Zubairy (2018), who used news about military spending as an instrument for government spending). We take m_t directly from FMR.

Empirical strategy. To estimate the effect of the asset purchases by the GSEs, we follow FMR and use the Jordà (2005) local projections method, implemented by two-stage least squares (2SLS). Specifically, in the first stage, we project the cumulative commitments $\sum_{j=0}^{h} p_{t+j}$ to purchase mortgages by the GSEs over $h+1$ months, expressed in constant dollars, on the non-cyclical narrative instrument m_t, also expressed in constant dollars, and a host of controls:

$$\frac{\sum_{j=0}^{h} p_{t+j}}{X_t} = \alpha_h^{(1)} + \gamma_h^{(1)}\frac{m_t}{X_t} + \varphi_h^{(1)}(L)Z_{t-1} + u_{t+h}^{(1)}. \tag{B.1}$$

We express the left-hand side variable as well as m_t on the right-hand side as ratios of X_t, a deterministic trend in real personal income obtained by fitting a third-degree polynomial of time to the log of personal income, deflated by the core personal consumption expenditures (PCE) price index. In equation (B.1), we also control for lagged values of the left-hand side variable, lagged growth rates of the core PCE price index, a nominal house price index, total mortgage debt, the log level of real mortgage originations, housing starts,

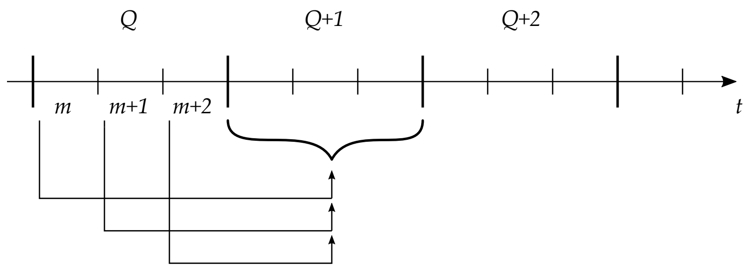

Figure B.1: This figure depicts the definition of the monthly Blue Chip forecasts. The diagram shows that, when the participants are asked to forecast a certain variable over quarter $Q+1$ in months m and $m+1$, the forecast horizon varies.

and lags of several interest rate variables: the 3-month T-bill rate, the 10-year Treasury rate, the conventional mortgage interest rate, and the BAA-AAA corporate bond spread. The superscript (1) denotes first-stage regression coefficients and errors.

In the second stage, we estimate

$$y_{t+h} = \alpha_h^{(2)} + \gamma_h^{(2)} \left(\frac{12}{8} \times \frac{\sum_{j=0}^{7} p_{t+j}}{\widetilde{X}_t} \right) + \varphi_h^{(2)} (L) Z_{t-1} + u_{t+h}^{(2)}, \tag{B.2}$$

where y_{t+h} is any variable of interest in month $t+h$—such as the realized mortgage rate, or the mortgage rate forecast error—and $\widetilde{X}_t$ is a long-run trend in annualized mortgage originations. Since, in the first stage, we estimate the reaction of the GSEs' cumulative commitments at various horizons, we pick a specific horizon of eight months to use as an indicator of policy actions. The reason for this choice is that the F-statistics of the first stage is maximized at this horizon. By doing this, we again follow FMR. We estimate $\gamma_h^{(2)}$ by 2SLS, i.e., we replace the term multiplying $\gamma_h^{(2)}$ in (B.2) with its predicted value in the first stage (B.1). The regressions on both stages include twelve lags of the dependent variables.

Data. We use data from October 1982 to December 2006. The choice of the starting date is dictated by the availability of the forecast data. The choice of the end date avoids using data from the Great Recession when the GSEs faced a particularly turbulent experience, which culminated in their conservatorship by the government in September 2008. All data sources, except for data on forecasts, are identical to those used in FMR. We list them in Section B.1 of this Appendix.

To measure mortgage rate forecasts, we use a survey of expectations by major financial institutions collected in the Blue Chip Financial Forecast (BCFF) database. The Blue Chip Financial Forecasts dataset is proprietary; it can either be purchased directly from the official website or obtained through the institutions subscribed to this dataset. The BCFF contains monthly surveys of around forty financial institutions that forecast major financial indicators, including mortgage rates at horizons up to six quarters. The surveys are usually conducted in the last few days of a month and released on the first date of the following month. We focus on the median forecast across forecasters at any point in time.

Importantly, the Blue Chip survey asks participants to forecast the average value of a variable over the current and future calendar quarters. As a result, there is no fixed forecast horizon at a monthly frequency. For example, a January forecast of the mortgage rate r_t over the second quarter of a particular year is a three-/five-month-ahead forecast, while a February forecast of the same variable is a two-/four-month-ahead forecast. This property of the Blue Chip forecasts is illustrated in Figure B.1.

We thus employ the following definition of forecast errors about next-quarter average mortgage rate:

$$\widetilde{u}_{t,t+\text{“}1:3\text{”}} \equiv \frac{r_{t+3-mod(t+2,3)} + r_{t+4-mod(t+2,3)} + r_{t+5-mod(t+2,3)}}{3} - f_t^{\text{“}1:3\text{”}},$$

where $f_t^{\text{“}1:3\text{”}}$ is the median forecast of next-quarter mortgage rate at time t in the BCFF database. Note that $t = 1$ corresponds to January 1982, $t = 2$ to February 1982, and so on. The notation "1:3" emphasizes the fact that the horizon of this forecast varies from one to three months and $mod(t + 2, 3)$ is the remainder of the division of $t + 2$ by 3. Similarly, we define forecast errors of mortgage rates in the subsequent quarters as

$$\widetilde{u}_{t,t+\text{“}(3n-2):3n\text{”}} \equiv \frac{\sum_{i=0}^{2} r_{t+3n+i-mod(t+2,3)}}{3} - f_t^{\text{“}(3n-2):3n\text{”}}, \tag{B.3}$$

where $n = 1, 2, 3, 4$.

Null hypothesis. As long as forecasters working for financial institutions are aware of significant purchases by the GSEs and hold rational expectations, the forecast errors $\widetilde{u}_{t,t+\text{“}(3n-2):3n\text{”}}$ should not be predictable by such purchases. To test this, we can simply use $\widetilde{u}_{t+1,t+1+\text{“}(3n-2):3n\text{”}}$ in place of y_{t+h} in equation (B.2) and verify whether the coefficient $\gamma_1^{(2)}$ is statistically different from zero. Note that, by regressing the forecast errors based on the information available to forecasters at the beginning of month $t + 1$ on the GSEs' purchases in month t, we avoid the possibility that these interventions were not implemented before forecasters were asked to predict future prices.

Results. We begin by estimating the effects of the GSEs' exogenous mortgage purchases on mortgage yields in our sample from October 1982 to December 2006. In doing this, we confirm that the main conclusion in FMR does not change much when we use our restricted data sample. The left panel of Figure 2 shows the impulse response function of the conventional mortgage rate—i.e., the coefficients $\gamma_h^{(2)}$ in the second stage equation (B.2) when the dependent variable is r_{t+h}—following an exogenous increase in the GSEs' purchases by 1 percent of trend originations. Our results are only slightly different from those in FMR. One notable difference between our results and those reported in FMR is the value of the first-stage F-statistics. While the authors estimate the F-statistics to be higher than ten in their longer sample, the value of F-statistics is just slightly above five in our smaller sample. However, quantitatively, the results reported in Figure 2 are close to those presented in Figure VII of FMR, suggesting that the weak instrument bias is small.

Next, we turn to the estimation of the response of mortgage-rate forecast errors to purchases by GSEs. The right panel of Figure 2 presents the estimates of coefficients $\gamma_1^{(2)}$ in equation (B.2) when the dependent variable is $\widetilde{u}_{t+1,t+1+\text{“}(3n-2):3n\text{”}}$, $n = 1, 2, 3, 4$, along with one- and two-standard-error confidence intervals. Consistently with the predictions of our model, forecast errors react negatively and significantly to the GSEs' mortgage purchases, which suggests that forecasters tend to under-react to news about such interventions. Moreover, under the additional assumption that forecasters working for financial institutions are aware of significant purchases by the GSEs, imperfect information models would fail to predict the under-reaction in the forecast errors.

We repeat our analysis for the "nowcast" error. We define the "nowcast" error using equation (B.3) where n is set to zero and $f_t^{\text{”}-2:0\text{”}}$ denotes the "nowcast"—the current-calendar-quarter average forecast of mortgage rates. It is clear that, when the nowcast is released in the beginning of the first month of a quarter, it is effectively a forecast of the mortgage rate during the whole quarter ahead. On the other hand, the nowcast released in the last month of the quarter is likely to depend on the data that has become available during the first part of the quarter that is being nowcasted. As a result, our measure of the nowcast error $\widetilde{u}_{t+1,t+1+"-2:0"}$ is an average between a true nowcast and a forecast at a short horizon. Hence, we expect our nowcast error to still be predictable, but perhaps to a smaller degree than the forecast errors at more distant horizons. Consistent with this logic, we find that the point estimate of $\gamma_1^{(2)}$ is -0.8 basis points with the standard deviation of 1.3 basis points.

Finally, to deduce the average level of sophistication of forecasters in the data, we first estimate the responses of the moving average of the realized mortgage rates, defined as $\sum_{i=0}^{2} r_{t+3n+i-mod(t+2,3)}/3$, which corresponds to the first term on the right-hand side of equation (B.3), at horizons $n = 1,2,3,4$ (the results are not explicitly shown here). Then, in accordance with the formula for the average level of sophistication in equation (32) of the main text, we divide the responses of forecast errors, depicted in the right panel of Figure 2, by the responses of the moving average of the realized mortgage rates. Note that equation (32) also holds for the variables that are defined similarly to the monthly forecasts in the BCFF data. Table B.1 presents the results. The average sophistication different over horizons, but in all of the cases it is between 1 and 1.4. The average value over four horizons is 1.17.

	Forecast horizon in months				
	1-3	4-6	7-9	10-12	mean
$\widehat{\bar{k}}$	1.23	1.03	1.35	1.03	1.17

Table B.1: The average level of sophistication of agents in the economy obtained from equation (32), for different horizons, and the mean value across all horizons. Because we use the BCFF dataset where the participants are asked to forecast variables for future *calendar* quarters, we do not have fixed forecasting horizons when we use monthly data. As a result, we introduce the notation where "1-3" denotes the forecast for the next calendar quarter, "4-6" denotes the forecast for the quarter after the next calendar quarter, and so on.

B.1 Data Sources

All variables used in the empirical part of the paper are monthly and identical to those in Fieldhouse et al. (2018) (FMR) except for forecasts of mortgage returns. For convenience we list all of the data sources here.

- **Agency purchase commitments** are computed by FMR in summing purchases by Fannie Mae, Freddie Mac, and the Federal Reserve.
- **The noncyclical narrative policy indicator** m_t is computed in FMR.
- **Personal income** is from NIPA (series PI in the FRED database).
- **The core PCE price index** is from NIPA (series PCEPILFE in the FRED database).
- **Nominal house price index** is the Freddie Mac house price index.
- **Total mortgage debt** are from the Financial Accounts of the United States and additional computations in FMR.
- **Residential mortgage originations** are computed by FMR from various sources and available from the authors.
- **Housing starts** are from the Census Bureau (series HOUST in the FRED database).

- **The 3-month T-bill rate** is from the Federal Reserve Release (FRSR) H.15 (series TB3MS in the FRED database).
- **The 10-year Treasury rate** is from the Federal Reserve Release (FRSR) H.15 (series GS10 in the FRED database).
- **The BAA-AAA corporate bond spread** is obtained by taking the difference in the Moody's seasoned BAA and AAA yields (series BAA and AAA in the FRED database).
- **The conventional mortgage rate** is the 30-year fixed-rate conventional conforming mortgage rate. It is measured as monthly average commitment rate from the Freddie Mac primary mortgage market survey.
- **Mortgage rate forecast** is the Blue Chip Forecasts of home mortgage rate which is defined as the 30-year fixed-rate conventional conforming mortgage rate. The Blue Chip reports note that "Interest rate definitions are the same as those in FRSR H.15."

Online Appendix

C Extensions of the Simple Model

In this online appendix, we extend the simple model in four ways. First, we explicitly introduce public long-term nominal bonds and show that balance sheet interventions, such as the so-called "Operation Twist", affect bond prices in the presence of inflation risk. Second, we extend the simple model to a two-country setting and discuss sterilized FX interventions. Third, we derive the consequences of the presence of the fraction of agents who form their expectations rationally. Finally, we study a simple learning mechanism.

C.1 A Model with Public Long-term Nominal Bonds

To study the effects of long-term public bonds purchases, we extend the simple model of Section 2 in two ways. First, we add a nominal friction in the form of a utility service from money balances, which is necessary to generate a demand for money. Second, we introduce nominal long-term bonds. There are thus four assets in the economy: *(i)* a riskless real asset, which pays a net return $r > 0$ and is available in perfectly elastic supply; *(ii)* money, which is issued by the central bank; *(iii)* a one-period nominal bond, which pays a continuously compounded nominal interest rate i_t and is issued by the treasury; and *(iv)* a nominal long-term bond, which pays one unit of currency every period, trades at price q_t in real terms, and is issued by the treasury. We assume that each long-term bond is a perpetuity that matures with probability $\delta \in [0,1]$ in every period, independently of the other bonds. The expected time to maturity of a long-term bond is thus equal to $1/\delta$ in every period. Finally, for the sake of simplicity, we do not consider private risky assets in this extension (none of the results are affected by this simplification).

The only source of aggregate risk in the economy is given by shocks to the money supply. We abstract from default in this extension. In particular, we assume that money supply follows the stochastic process $\log M_{t+1} = \log \overline{M} + \epsilon_t^m - \epsilon_{t-1}^m (1+v)/v$, where v and $\overline{M}$ are positive parameters. The disturbances $\{\epsilon_t^m\}$ are assumed to be i.i.d. and normally distributed with zero mean and standard deviation σ_m. The specific form of the money supply—i.e., the presence of the lagged shock ϵ_{t-1}^m and the parameter v that also appears in the household preferences—allows us to streamline the analysis. Under these assumptions, in fact, there is no inflation risk between two consecutive periods, making one-period nominal bonds riskless and long-term bonds risky in real terms. It is straightforward to solve the model under alternative processes for money supply that lead to a one-period-ahead inflation risk. However, this clutters the exposition without adding any important economic insights.

Households. Household preferences are

$$-\frac{1}{\gamma}\mathbb{E}_0 \sum_{t=0}^{\infty} \exp\left[-\rho t - \gamma\left(c_t - \frac{m_t\left[\log\left(m_t/\overline{m}\right) - 1\right]}{v}\right)\right], \tag{C.1}$$

where $\overline{m}$ is a positive constant. Preferences are assumed to depend on real money balances m_t, which is a standard way of introducing the demand for money in macroeconomic models. The particular functional form assumed here simplifies the analysis by making money demand independent of the consumption choice. Note that utility is increasing in m_t, for $m_t \leq \overline{m}$, and decreasing in m_t, for $m_t > \overline{m}$. We thus restrict our analysis to the case with $m_t \leq \overline{m}$.

Each household chooses safe real assets s_{t+1}, short-term nominal bonds b_{t+1} (expressed in units of period-t consumption), long-term nominal bonds d_{t+1} (expressed in units of period-t consumption), real money balances m_{t+1}, and consumption c_t, so as to maximize (C.1), subject to the budget constraint

$$\begin{aligned}&P_t c_t + P_t s_{t+1} + P_t b_{t+1} + P_t q_t d_{t+1} + P_t m_{t+1} \\ \leq &P_t \left(W_t - T_t\right) + P_t(1+r)s_t + e^{i_{t-1}} P_{t-1} b_t + [1 + (1-\delta) P_{t-1} q_t] d_t + P_{t-1} m_t,\end{aligned} \tag{C.2}$$

where T_t are real per capita taxes and P_t is the nominal price level.

Beliefs are defined as in the simple model in Section 2. Similarly, we let $\widetilde{\mathcal{Z}}_t^k$, $\widetilde{\mathcal{Z}}_t^{SQ}$, and $\widetilde{\mathcal{Z}}_t^*$ denote level-k beliefs, rational-expectations beliefs before ("status quo") and after the intervention, respectively. Here, $Z_t \equiv (p_t, i_t, q_t, \mathcal{T}_t, Tr_t, R_{t+1})$, where $p_t \equiv \log P_t$, R_{t+1} and $\mathcal{T}_t$ are part of government policies, which we define below.

Government. The government consists again of the treasury and the central bank. The former sets real taxes, the *real* amount of one-period (short-term) nominal bonds, the *real* amount of nominal long-term bonds, and the real safe short-term bonds (those that pay the interest rate r). Without any loss of generality and to simplify notation, we assume that the outstanding real amounts of short- and long-term bonds are held constant at $\overline{B}$ and $\overline{D}$, thus, the fiscal authority simply replaces maturing bonds with newly issued bonds. In addition, the fiscal authority receives transfers $\{Tr_t\}$ from the monetary authority in every period. The treasury's per-period budget constraint is thus

$$\overline{D} + e^{i_{t-1}} P_{t-1} \overline{B} + (1+r) S_t P_t = P_t T_t + P_t q_t \delta \overline{D} + P_t \overline{B} + S_{t+1} P_t + P_t Tr_t. \tag{C.3}$$

The left-hand side represents payments on long-term and two types of short-term bonds. Recall that each unit of long-term bonds $\overline{D}$ pays one unit of currency every period. The right-hand side sums up all sources of revenue: taxes, replacement of matured long-term bonds (i.e., issuance of new long-term bonds), issuance of nominal and real short-term bonds, and transfers from the monetary authority. Finally, note that we are implicitly assuming that the original quantity $\overline{D}$ of long-term bonds was issued at some date before period t.

The central bank controls the nominal money supply $\{M_{t+1}\}$, the *real* amount of one-period interest-paying reserves $\{R_{t+1}\}$, and purchases of long-term public bonds $\{D_{t+1}\}$. Since reserves and short-term public bonds will be perfect substitutes in equilibrium, they will pay the same interest rate i_t. For simplicity, we assume that only cash M_t, which we refer to as "money," provides utility benefits to households.The budget constraint of the monetary authority is

$$M_t + e^{i_{t-1}} P_{t-1} R_t + P_t q_t D_{t+1} + P_t Tr_t = M_{t+1} + P_t R_{t+1} + [1 + (1-\delta) P_t q_t] D_t.$$

The left-hand side represents outlays consisting of repayment to money holders, payments on reserves, purchases of long-term bonds, and transfers to the fiscal authority. The right-hand side represents central bank revenues consisting of issuance of money, creation of reserves, and income from coupons and sales of long-term bonds.

To save on notation and without loss of generality, we assume that central bank bond holdings before the intervention are zero. Moreover, again without loss of generality, we consider only balance sheet policies consisting of purchases of long-term bonds entirely financed by creation of reserves. We again refer to such policies as "quantitative easing." Formally, we require

$$R_{t+1} = q_t D_{t+1}, \tag{C.4}$$

which implies that the central bank's budget constraint simplifies into

$$P_t Tr_t = M_{t+1} - M_t + [1 + (1-\delta) P_t q_t - e^{i_{t-1}} P_{t-1} q_{t-1}] D_t. \tag{C.5}$$

Finally, note that, in this section, $\overline{D}$ and D_t denote total supply and purchases of long-term bonds, respectively, and not average and realized dividends, as in the baseline model of Section 2.

Household optimization. We begin by writing the Bellman equation for the household problem:

$$V(a_t, m_t; \widetilde{\mathcal{Z}}_t) = \max_{\widehat{c}_t, \widehat{b}_{t+1}, \widehat{d}_{t+1}, \widehat{m}_{t+1}} \left\{ -\frac{1}{\gamma} \exp\left[-\gamma \left(\widehat{c}_t - \frac{m_t \left[\log\left(m_t/\overline{m}\right) - 1\right]}{v} \right) \right] + e^{-\rho} \mathbb{E}_t V(\widehat{a}_{t+1}, \widehat{m}_{t+1}; \widetilde{\mathcal{Z}}_{t+1}) \right\},$$

subject to

$$\begin{aligned} \widetilde{P}_{t+1} \widehat{a}_{t+1} =& \widetilde{P}_{t+1} (1+r) \left(a_t - \widehat{c}_t - \widehat{b}_{t+1} - \widehat{m}_{t+1} - q_t \widehat{d}_{t+1} \right) + e^{i_t} P_t \widehat{b}_{t+1} + P_t \widehat{m}_{t+1} \\ &+ [1 + (1-\delta) \widetilde{P}_{t+1} \widetilde{q}_{t+1}] \widehat{d}_{t+1} + \widetilde{P}_{t+1} (1+r) \left(W_{t+1} - \widetilde{T}_{t+1} \right), \end{aligned} \tag{C.6}$$

where a_t is individual net worth. We let $\widetilde{H}_t \equiv \mathbb{E}_t \sum_{j=1}^{\infty} (W_{t+j} - \widetilde{T}_{t+j})/(1+r)^j$ be household's human capital. We also let $\widetilde{\mathcal{T}}_t \equiv \mathbb{E}_t \sum_{s=0}^{\infty} \widetilde{T}_{t+s}/(1+r)^s$ denote the present discounted value of future taxes.

We first rewrite (C.6) in real units by dividing both sides by $\widetilde{P}_{t+1}$:

$$\begin{aligned} \widehat{a}_{t+1} =& (1+r)(a_t - \widehat{c}_t) + \left[e^{i_t} \frac{P_t}{\widetilde{P}_{t+1}} - (1+r) \right] \widehat{b}_{t+1} \\ &+ \left[\frac{1}{\widetilde{P}_{t+1}} + (1-\delta) \widetilde{q}_{t+1} - (1+r) q_t \right] \widehat{d}_{t+1} + \left[\frac{P_t}{\widetilde{P}_{t+1}} - (1+r) \right] \widehat{m}_{t+1} + W_{t+1} - \widetilde{T}_{t+1}. \end{aligned}$$

We then take a first-order Taylor expansion around $(i_t, \widetilde{\pi}_{t+1}, \widetilde{p}_{t+1}) = (r, 0, 0)$:

$$\widehat{a}_{t+1} = (1+r)(a_t - \widehat{c}_t) + \begin{pmatrix} \widehat{b}_{t+1}, & \widehat{d}_{t+1}, & \widehat{m}_{t+1} \end{pmatrix} \mathcal{R}_{t+1} + W_{t+1} - \widetilde{T}_{t+1}.$$

where $\widetilde{\mathcal{R}}_{t+1} \equiv (i_t - \widetilde{\pi}_{t+1} - r, 1 - \widetilde{p}_{t+1} + (1-\delta)\widetilde{q}_{t+1} - (1+r)q_t, -\widetilde{\pi}_{t+1} - r)'$ is the vector of excess returns. Our strategy of log-linearizing the budget constraint and treating it as exact follows Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006). This approach yields an approximation to the true solution that allows for an analytic characterization.

We conjecture that beliefs $\widetilde{\mathcal{Z}}_t$ are such that the vector Z_{t+1} is linear in the underlying shocks of the economy. This conjecture will be verified in all the equilibria we consider below. Specifically,

$$x_{t+1} = \alpha_{x,t} + \beta_{x,t} \epsilon_{t+1}^m + \xi_{x,t} \epsilon_t^m, \tag{C.7}$$

for some, possibly time-varying, coefficients $\alpha_{x,t}$, $\beta_{x,t}$, and $\xi_{x,t}$, for all $x \in \{p, q, i, Tr, \mathcal{T}\}$.

We guess and verify that

$$V(a_t, m_t; \widetilde{\mathcal{Z}}_t) = -\frac{1}{\gamma} e^{-\gamma\left[A(a+\widetilde{H}_t+\widetilde{\vartheta}_t) - A_m \frac{m[\log(m/\overline{m})-1]}{v}\right]},$$

where $\widetilde{\vartheta}_t$ is a deterministic function of time, which summarizes a number of endogenous variables taken as given by the households.

Standard properties of Normal distributions imply

$$\begin{aligned} V(a_t, m_t; \widetilde{\mathcal{Z}}_t) = \max_{\substack{\widehat{c}_t, \widehat{m}_{t+1}, \\ \widehat{b}_{t+1}, \widehat{d}_{t+1}}} & -\frac{1}{\gamma} \exp\left(-\gamma\left\{\widehat{c}_t - \frac{m_t[\log(m_t/\overline{m}) - 1]}{v}\right\}\right) \\ & -\frac{1}{\gamma} \exp\bigg[-\rho - \gamma A \mathbb{E}_t[\widehat{a}_{t+1} + \widetilde{H}_{t+1}] - \gamma A \widetilde{\vartheta}_{t+1} + \frac{1}{2}\gamma^2 A^2 \mathbb{V}_t[\widehat{a}_{t+1} + \widetilde{H}_{t+1}] \\ & + A_m \gamma \frac{\widehat{m}_{t+1}[\log(\widehat{m}_{t+1}/\overline{m}) - 1]}{v}\bigg], \end{aligned}$$

where the conditional moments are given by

$$\begin{aligned} \mathbb{E}_t[\widehat{a}_{t+1} + \widetilde{H}_{t+1}] =& (1+r)(a_t - \widehat{c}_t) + (i_t - \alpha_{p,t} - \xi_{p,t}\epsilon_t^m + p_t - r)\widehat{b}_{t+1} \\ & + [1 - \alpha_{p,t} - \xi_{p,t}\epsilon_t^m + (1-\delta)(\alpha_{q,t} + \xi_{q,t}\epsilon_t^m) - (1+r)q_t]\widehat{d}_{t+1} \\ & + (-\alpha_{p,t} - \xi_{p,t}\epsilon_t^m + p_t - r)\widehat{m}_{t+1} + W_{t+1} - \mathbb{E}_t\widetilde{T}_{t+1} + \mathbb{E}_t\widetilde{H}_{t+1}, \\ \mathbb{V}_t[\widehat{a}_{t+1} + \widetilde{H}_{t+1}] =& \Big(-\beta_{p,t}\widehat{b}_{t+1} - [\beta_{p,t} - (1-\delta)\,\beta_{q,t}]\widehat{d}_{t+1} - \beta_{p,t}\widehat{m}_{t+1} - \beta_{\mathcal{T},t}\Big)^2 \sigma_m^2. \end{aligned}$$

The solution to the optimization problem at time t gives realized choices $c_t = c(a_t, m_t; \widetilde{\mathcal{Z}}_t)$, $a_{t+1} = a(a_t, m_t; \widetilde{\mathcal{Z}}_t)$, $b_{t+1} = b(a_t, m_t; \widetilde{\mathcal{Z}}_t)$, $d_{t+1} = d(a_t, m_t; \widetilde{\mathcal{Z}}_t)$, $m_{t+1} = m(a_t, m_t; \widetilde{\mathcal{Z}}_t)$, as well as planned choices, that is, the choices that the agent expects to implement in the future. Due to bounded rationality, the latter may differ from the former. We use a "hat" for planned choices.

The solution to the problem satisfies the first-order condition for consumption

$$e^{-\gamma\left\{c_t - \frac{m_t[\log(m_t/\overline{m})-1]}{v}\right\}} = A(1+r)e^{-\rho-\gamma A\mathbb{E}_t[a_{t+1}+\widetilde{H}_{t+1}]-\gamma A\widetilde{\vartheta}_{t+1}+\frac{1}{2}\gamma^2A^2\mathbb{V}_t[a_{t+1}+\widetilde{H}_{t+1}]+A_m\gamma\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}} \tag{C.8}$$

and the first-order conditions with respect to $\widehat{m}_{t+1}$, $\widehat{b}_{t+1}$, and $\widehat{d}_{t+1}$

$$\widetilde{\Sigma}_t \cdot \begin{pmatrix} b_{t+1} \\ d_{t+1} \\ m_{t+1} \end{pmatrix} = \frac{1}{A\gamma}\mathbb{E}_t\widetilde{\mathcal{R}}_{t+1} + \frac{1}{A\gamma}\begin{pmatrix} 0 \\ 0 \\ -\frac{A_m}{Av}\log\left(\frac{m_{t+1}}{\overline{m}}\right) \end{pmatrix} + cov_t\left(\widetilde{\mathcal{T}}_{t+1}, \widetilde{\mathcal{R}}_{t+1}\right). \tag{C.9}$$

The variance-covariance matrix $\widetilde{\Sigma}_t \equiv \mathbb{V}_t(\widetilde{\mathcal{R}}_{t+1})$ is such that $(\widetilde{\Sigma}_t)_{1,1} = (\widetilde{\Sigma}_t)_{3,3} = (\widetilde{\Sigma}_t)_{1,3} = (\widetilde{\Sigma}_t)_{3,1} = (\beta_{p,t})^2\sigma_m^2$, $(\widetilde{\Sigma}_t)_{2,2} = [(1-\delta)\beta_{q,t} - \beta_{p,t}]^2\sigma_m^2$, $(\widetilde{\Sigma}_t)_{1,2} = (\widetilde{\Sigma}_t)_{2,1} = (\widetilde{\Sigma}_t)_{3,2} = (\widetilde{\Sigma}_t)_{2,3} = -\beta_{p,t}[(1-\delta)\,\beta_{q,t} - \beta_{p,t}]\sigma_m^2$. We use $(\widetilde{\Sigma}_t)_{m,n}$ to denote the (m,n)'th element of matrix $\widetilde{\Sigma}_t$. Note that $\widetilde{\Sigma}_t$ is not invertible because the return on money, the return on short-term bonds, and the one-period ahead return on long-term bonds have the same risk profile. Also, the covariance in (C.9) equals

$$cov_t\left(\widetilde{\mathcal{T}}_{t+1}, \widetilde{\mathcal{R}}_{t+1}\right) = \begin{pmatrix} -\beta_{p,t} \\ (1-\delta)\,\beta_{q,t} - \beta_{p,t} \\ -\beta_{p,t} \end{pmatrix} (\beta_{\mathcal{T},t})\,\sigma_m^2.$$

The first and the third raw of equations (C.9) imply

$$m_{t+1} = \overline{m} e^{-\frac{A}{A_m} v i_t}.$$

We now verify our guess for the value function. To do this, we evaluate the Bellman equation at the optimum and check if it holds for every values of state variables a_t and m_t. At the optimum, the Bellman equation is

$$-\frac{1}{\gamma} e^{-\gamma\left[A(a_t+\widetilde{H}_t)-A_m\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right]+\widetilde{\vartheta}_t} = -\frac{1}{\gamma} e^{-\gamma\left\{c_t-\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right\}}$$
$$-\frac{1}{\gamma} e^{-\rho-\gamma A\mathbb{E}_t[a_{t+1}+\widetilde{H}_{t+1}]-\gamma A\widetilde{\vartheta}_{t+1}+\frac{1}{2}\gamma^2 A^2\mathbb{V}_t[a_{t+1}+\widetilde{H}_{t+1}]+A_m\gamma\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}}.$$

Using the first-order condition for consumption, we can rewrite the latter as

$$-\frac{1}{\gamma} e^{-\gamma\left[A(a_t+\widetilde{H}_t+\widetilde{\vartheta}_t)-(A_m-1)\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right]} = -\frac{1}{\gamma} e^{-\gamma c_t}\frac{1+A(1+r)}{A(1+r)}.$$

Optimal consumption is obtained from equation (C.8):

$$\begin{aligned}
[1+A(1+r)]\,c_t =& \frac{m_t[\log(m_t/\overline{m})-1]}{v} - \frac{1}{\gamma}\log\left[A(1+r)\right] + \frac{\rho}{\gamma} + A(1+r)(a_t+\widetilde{H}_t) \\
&+ A\begin{pmatrix} b_{t+1}, & d_{t+1}, & m_{t+1}\end{pmatrix}\mathbb{E}_t\widetilde{\mathcal{R}}_{t+1} + A\widetilde{\vartheta}_{t+1} - \frac{1}{2}\gamma A^2\mathbb{V}_t[a_{t+1}+\widetilde{H}_{t+1}] \\
&- \frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v},
\end{aligned}$$

where we used $(1+r)\widetilde{H}_t = W_{t+1} - \mathbb{E}_t\widetilde{T}_{t+1} + \mathbb{E}_t\widetilde{H}_{t+1}$. Combining the last two equations, we get

$$\begin{aligned}
&[1+A(1+r)]\left\{A(a_t+\widetilde{H}_t) - (A_m-1)\frac{m_t[\log(m_t/\overline{m})-1]}{v} - \frac{1}{\gamma}\widetilde{\vartheta}_t\right\} \\
=&\frac{m_t[\log(m_t/\overline{m})-1]}{v} - \frac{1}{\gamma}\log\left[A(1+r)\right] + \frac{\rho}{\gamma} + A(1+r)(a_t+\widetilde{H}_t) + A\begin{pmatrix} b_{t+1}, & d_{t+1}, & m_{t+1}\end{pmatrix}\mathbb{E}_t\widetilde{\mathcal{R}}_{t+1} \\
&+ A\widetilde{\vartheta}_{t+1} - \frac{1}{2}\gamma A^2\mathbb{V}_t[a_{t+1}+\widetilde{H}_{t+1}] - \frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v} - [1+A(1+r)]\frac{1}{\gamma}\log\frac{1+A(1+r)}{A(1+r)}.
\end{aligned}$$

For our conjecture to be true, the coefficients multiplying a_t and m_t must be identical, that is,

$$\begin{aligned}
a_t+\widetilde{H}_t &: [1+A(1+r)]\,A = A(1+r), \\
\frac{m_t[\log(m_t/\overline{m})-1]}{v} &: -[1+A(1+r)]\,(A_m-1) = 1,
\end{aligned}$$

which is true if and only if

$$A = A_m = \frac{r}{1+r}.$$

Finally, we can express $\widetilde{\vartheta}_t$ as follows

$$\widetilde{\vartheta}_t = \frac{1}{r\gamma}\left[\rho + \log\frac{r^r}{(1+r)^{1+r}}\right] + \frac{\widetilde{\vartheta}_{t+1} + (\ b_{t+1},\ \ d_{t+1},\ \ m_{t+1}\ )\mathbb{E}_t\widetilde{\mathcal{R}}_{t+1}}{1+r} - \frac{\gamma r}{2\,(1+r)^2}\mathbb{V}_t[a_{t+1} + \widetilde{H}_{t+1}]$$
$$-\frac{1}{r}\cdot\frac{m_{t+1}[\log(m_{t+1}/\overline{m}) - 1]}{v}.$$

Temporary equilibrium. First, the market-clearing conditions in the asset markets in period t are

$$\overline{B} + q_t D_{t+1} = b_{t+1},$$
$$D - D_{t+1} = d_{t+1},$$
$$\frac{\overline{M}}{P_t}e^{\epsilon_t^m - \epsilon_{t-1}^m\frac{1+v}{v}} = \overline{m}e^{-vi_t}.$$

The money-market equilibrium condition implies the following relationship between the price level and the short-term interest rate:

$$p_t = \log\left(\overline{M}/\overline{m}\right) + vi_t + \epsilon_t^m - \epsilon_{t-1}^m\frac{1+v}{v}. \tag{C.10}$$

To streamline the analysis, we assume that the demand and supply of money are negligibly small. Specifically, we let $\overline{m}$ and $\overline{M}$ approach zero so that the ratio $\overline{m}/\overline{M}$ approaches one. This "cashless limit" is a standard assumption employed in, for example, the New Keynesian literature to eliminate the real effects of money supply above and beyond its effects on inflation and the nominal interest rate. We can thus abstract from money holdings when computing equilibria.

In the cashless limit, the market clearing conditions together with optimal choice of bonds imply

$$\left(\widetilde{\Sigma}_t\right)_{1:2,1:2}\cdot\begin{pmatrix}\overline{B} + q_t D_{t+1}\\ D - D_{t+1}\end{pmatrix} = \frac{1}{\frac{r}{1+r}\gamma}\mathbb{E}_t\left(\widetilde{\mathcal{R}}_{t+1}\right)_{1:2} + cov_t\left(\widetilde{\mathcal{T}}_{t+1}, \left(\widetilde{\mathcal{R}}_{t+1}\right)_{1:2}\right), \tag{C.11}$$

where $(\widetilde{\Sigma}_t)_{1:2,1:2}$ is the upper-left sub-matrix of $\widetilde{\Sigma}_t$, and $(\widetilde{\mathcal{R}}_{t+1})_{1:2}$ is the vector containing the first two elements of $\widetilde{\mathcal{R}}_{t+1}$. The second line of this equation can be solved for price q_t:

$$q_t = \frac{1 - \alpha_{p,t} - \xi_{p,t}\epsilon_t^m + (1-\delta)\left(\alpha_{q,t} + \xi_{q,t}\epsilon_t^m\right)}{(1+r)\left\{1 - \frac{r\gamma\sigma_m^2}{(1+r)^2}D_{t+1}\beta_{p,t}\left[(1-\delta)\,\beta_{q,t} - \beta_{p,t}\right]\right\}}$$
$$-\frac{r\gamma\sigma_m^2}{(1+r)^2}\cdot\frac{\left[-\beta_{p,t}\overline{B} + [(1-\delta)\,\beta_{q,t} - \beta_{p,t}](\overline{D} - D_{t+1}) - \beta_{\mathcal{T},t}\right]\left[(1-\delta)\,\beta_{q,t} - \beta_{p,t}\right]}{1 - \frac{r\gamma\sigma_m^2}{(1+r)^2}D_{t+1}\beta_{p,t}\left[(1-\delta)\,\beta_{q,t} - \beta_{p,t}\right]}. \tag{C.12}$$

Similarly, the nominal interest rate on short-term bonds is obtained from the first line of (C.11):

$$i_t = r + \alpha_{p,t} + \xi_{p,t}\epsilon_t^m - p_t + RP_t, \tag{C.13}$$

where, for convenience, we let

$$RP_t \equiv \frac{r}{1+r}\gamma\left[\beta_{p,t}(\overline{B}+q_t D_{t+1}) - [(1-\delta)\,\beta_{q,t} - \beta_{p,t}](D - D_{t+1}) + \beta_{\mathcal{T},t})\right]\beta_{p,t}\sigma_m^2.$$

The price level p_t is obtained by combining C.10, taking into account that $\overline{M}/\overline{m} = 1$, and (C.13):

$$\begin{aligned} p_t =& v i_t + \epsilon_t^m - \frac{1+v}{v}\epsilon_{t-1}^m \\ =& \frac{v}{1+v}(r+\alpha_{p,t}) + \frac{1+v\xi_{p,t}}{1+v}\epsilon_t^m - \frac{1}{v}\epsilon_{t-1}^m + \frac{v}{1+v}RP_t. \end{aligned} \tag{C.14}$$

Next, we consider fiscal policy. The realized transfers from the central bank to the treasury can be computed using equation (C.5) at the cashless limit. We obtain

$$\begin{aligned} Tr_t &= \left[e^{-p_t} + (1-\delta)\,q_t - e^{i_{t-1}-\pi_t}q_{t-1}\right]D_t \\ &= \left[1 - p_t + (1-\delta)\,q_t - (1+i_{t-1}-\pi_t)q_{t-1}\right]D_t, \end{aligned} \tag{C.15}$$

where the second line log-linearized around $(i_{t-1}, \pi_t, p_t) = (r, 0, 0)$. The treasury's budget constraint (C.3) implies the following realized taxes in period t:

$$T_t = \frac{\overline{D}}{P_t} + e^{i_{t-1}-\pi_t}\overline{B} - \overline{B} + (1+r)S_t - S_{t+1} - q_t\delta\overline{D} - Tr_t.$$

The expected discounted sum of taxes is then

$$\widetilde{\mathcal{T}}_t = (1+r)S_t - \mathbb{E}_t\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s\left[\widetilde{Tr}_{t+s} + \left(\widetilde{q}_{t+s}\delta - e^{-\widetilde{p}_{t+s}}\right)\overline{D} + \left(1 - e^{\widetilde{i}_{t-1+s}-\widetilde{\pi}_{t+s}}\right)\overline{B}\right]. \tag{C.16}$$

Finally,

$$\begin{aligned} \widetilde{\vartheta}_t =& \frac{1}{r\gamma}\left[\rho + \log\frac{r^r}{(1+r)^{1+r}}\right] + \frac{\widetilde{\vartheta}_{t+1} + \left(\overline{B}+q_t D_{t+1}, \overline{D} - D_{t+1}\right)\mathbb{E}_t\left(\widetilde{\mathcal{R}}_{t+1}\right)_{1:2}}{1+r} \\ & - \frac{\gamma r}{2\,(1+r)^2}\widetilde{\mathbb{V}}_t[a_{t+1} + \widetilde{H}_t]. \end{aligned} \tag{C.17}$$

Rational expectations equilibrium. In the REE, the equilibrium distribution of endogenous variables must be equal to the agents' beliefs about these variables.

Specifically, for the price level, we need to make sure that the coefficients $\alpha_{p,t}^*$, $\beta_{p,t}^*$, and $\xi_{p,t}^*$ satisfy

$$\begin{aligned} p_{t+1} =& \frac{v}{1+v}(r+\alpha_{p,t+1}) + \frac{1+v\xi_{p,t+1}^*}{1+v}\epsilon_{t+1}^m - \frac{1}{v}\epsilon_t^m + \frac{v}{1+v}RP_t \\ \overset{REE}{=}& \alpha_{p,t}^* + \beta_{p,t}^*\epsilon_{t+1}^m + \xi_{p,t}^*\epsilon_t^m, \end{aligned}$$

for all realizations of the shocks. The latter is satisfied if and only if

$$\begin{aligned}
\xi^*_{p,t} &= -\frac{1}{v}, \\
\beta^*_{p,t} &= \frac{1 + v\xi^*_{p,t+1}}{1+v} = 0.
\end{aligned}$$

Similarly, for the price of the perpetuity,

$$\begin{aligned}
\xi^*_{q,t} &= 0, \\
\beta^*_{q,t} &= \frac{-\xi^*_{p,t+1} + (1-\delta)\,\xi^*_{q,t+1}}{1+r} = \frac{1}{(1+r)v}.
\end{aligned}$$

The short-term nominal interest rate must satisfy

$$\begin{aligned}
i_{t+1} &= r + \alpha^*_{p,t+1} + \xi^*_{p,t+1}\epsilon^m_{t+1} - (\alpha^*_{p,t} + \xi^*_{p,t}\epsilon^m_t) + RP_{t+1} \\
&= \alpha^*_{i,t} + \beta^*_{i,t}\epsilon^m_{t+1} + \xi^*_{i,t}\epsilon^m_t,
\end{aligned}$$

implying

$$\begin{aligned}
\xi^*_{i,t} &= -\xi^*_{p,t} = \frac{1}{v}, \\
\beta^*_{i,t} &= \xi^*_{p,t+1} - \beta^*_{p,t} = -\frac{1}{v}.
\end{aligned}$$

Note also that $RP_{t+1} = 0$ because $\beta^*_{p,t} = 0$. This automatically implies that $\alpha^*_{i,t} = r$ and $\alpha^*_{p,t} = vr$.

In addition, the sensitivities of the central bank transfers are

$$\begin{aligned}
\beta^*_{Tr,t} &= \frac{1-\delta}{(1+r)v} D_{t+1} \\
\xi^*_{Tr,t} &= 0.
\end{aligned}$$

while the sensitivity of the expected sum of taxes are

$$\begin{aligned}
\beta^*_{\mathcal{T},t} &= \frac{1-\delta}{(1+r)v}(\overline{D} - D_{t+1}), \\
\xi^*_{\mathcal{T},t} &= \frac{1}{v}(\overline{D} + \overline{B}).
\end{aligned}$$

To simplify notation, we omit time subscripts from those coefficients that are constant over time. We thus have

$$\begin{aligned}
\left(\xi^*_p, \xi^*_i, \xi^*_q, \xi^*_{Tr}, \xi^*_{\mathcal{T}}\right) &= \left(-\frac{1}{v}, \frac{1}{v}, 0, 0, \frac{\overline{D} + \overline{B}}{v}\right), \\
\left(\beta^*_p, \beta^*_i, \beta^*_q, \beta^*_{Tr,t}, \beta^*_{\mathcal{T}}\right) &= \left(0, -\frac{1}{v}, \frac{1}{(1+r)v}, \frac{1-\delta}{(1+r)v} D_{t+1}, \frac{1-\delta}{(1+r)v}(\overline{D} - D_{t+1})\right).
\end{aligned}$$

In the REE, the top left entry of the variance-covariance matrix of $\mathcal{R}_{t+1}$, which we denote with Σ^*, is given by

$$(\Sigma^*)_{1,1} = \frac{(1-\delta)^2}{(1+r)^2 v^2}\sigma^2_m.$$

Finally, for the non-random part of the endogenous variables, we have

$$\alpha^*_{p,t} = \frac{v}{1+v}\left(\alpha^*_{p,t+1} + r\right),$$
$$\alpha^*_{i,t} = r + \alpha^*_{p,t} - \frac{v}{1+v}(r + \alpha^*_{p,t+1}),$$
$$\alpha^*_{q,t} = \frac{1 - vr + (1-\delta)\,\alpha^*_{q,t+1}}{1+r},$$
$$\alpha^*_{Tr,t} = \left[1 - vr + (1-\delta)\,\alpha^*_{q,t} - \alpha^*_{q,t-1}(1+r)\right] D_{t+1},$$
$$\alpha^*_{\mathcal{T},t} = (1+r)S_{t+1} - \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s \left[\alpha^*_{Tr,t+s} - \overline{D} + \left(\alpha^*_{q,t+s}\delta + \alpha^*_{p,t+s}\right)\overline{D} - r\overline{B}\right].$$

We can then solve the system of equations above as follows:

$$\alpha^*_p = vr,$$
$$\alpha^*_i = r,$$
$$\alpha^*_q = \frac{1-vr}{r+\delta}.$$
$$\alpha^*_{Tr,t} = 0,$$
$$\alpha^*_{\mathcal{T},t} = (1+r)S_{t+1} - \left[\frac{vr-1}{r+\delta} r\overline{D} - r\overline{B}\right]\frac{1+r}{r}.$$

Finally, using market-clearing conditions and taking the cashless limit, we can rewrite the recursion for ϑ^*_t as

$$\vartheta^*_t = \frac{1}{r\gamma}\left[\rho + \log\frac{r^r}{(1+r)^{1+r}}\right] + \frac{\vartheta^*_{t+1}}{1+r} + \frac{(\overline{D} - D_{t+2})\left(\frac{\delta vr - 1 + vr + vr^2}{r+\delta}\right)}{1+r}.$$

This expression shows that ϑ^*_t is a deterministic function time, thus, our initial conjecture is verified.

We summarize these results with the following proposition.

Proposition C.1. *In the cashless limit of the REE, the price level p^*_t*, the nominal interest rate i^*_t, *and the price of the long-term bond q^*_t are given by*

$$p^*_t = rv - \frac{1}{v}\epsilon^m_{t-1},$$
$$i^*_t = r - \frac{1}{v}\left(\epsilon^m_t - \epsilon^m_{t-1}\right),$$

and

$$q^*_t = \frac{1-vr}{r+\delta} + \frac{1}{v(1+r)}\epsilon^m_t.$$

In particular, they are all independent of balance sheet policies.

Level-k beliefs. Equations C.4, (C.12), (C.13)-(C.17) define a mapping $\Psi(\cdot)$ from beliefs into equilibrium variables.

By iterating $\Psi(\cdot)$, starting from $\widetilde{\mathcal{Z}}^{SQ}_t$, we can compute the beliefs of level-k agents, for any $k \geq 1$. It is easy to see that some β's and ϑ's coincide with their REE counterparts: $\beta^k_{p,t} = \beta^*_p$, $\beta^k_{i,t} = \beta^*_i$, $\beta^k_{q,t} = \beta^*_q$, $\zeta^k_{p,t} = \zeta^*_p$, $\zeta^k_{i,t} = \zeta^*_i$, $\zeta^k_{q,t} = \zeta^*_q$, for all $k \geq 1$. In particular, the latter imply that $(\widetilde{\Sigma}^k_t)_{1,1} = (\Sigma^*)_{1,1}$, for all $k \geq 1$.

We are left to derive $\beta^k_{Tr,t}$, $\beta^k_{\mathcal{T},t}$, $\xi^k_{Tr,t}$, $\xi^k_{\mathcal{T},t}$, and the α^k's, for all $k \geq 1$. For $k = 1$, since we assume that the expectations of level-1 households coincide with the distributions of REE variables before the intervention by the central bank, the sensitivity of transfers and taxes to the current shock coincides with its REE counterpart *in the absence* of asset purchases:

$$\begin{aligned}
\beta^1_{Tr,t} =& 0, \\
\xi^1_{Tr,t} =& 0, \\
\beta^1_{\mathcal{T},t} =& \frac{1-\delta}{(1+r)v}\overline{D}, \\
\xi^1_{\mathcal{T},t} =& \frac{1}{v}(\overline{D}+\overline{B}).
\end{aligned}$$

Proceeding recursively for $k > 1$, we have

$$\begin{aligned}
\beta^k_{Tr,t} =& \frac{1-\delta}{(1+r)v}D_{t+1}, \\
\xi^k_{Tr,t} =& 0 \\
\beta^k_{\mathcal{T},t} =& \frac{1-\delta}{(1+r)v}\left(\overline{D}-D_{t+1}\right), \\
\xi^k_{\mathcal{T},t} =& \frac{1}{v}(\overline{D}+\overline{B})
\end{aligned}$$

From equation (C.14) and $\beta^k_{q,t} = \beta^*_q = 0$, the intercept of the price level p_t remains

$$\alpha^k_{p,t} = vr,$$

for any $k \geq 1$. For the intercept of i_t, we get $\alpha^k_{i,t} = r$. For the intercept of q_t, from equation (C.12), we have

$$\alpha^k_{q,t} = \begin{cases} \frac{1-vr}{r+\delta}, & k=1, \\ \frac{1-vr+(1-\delta)\alpha^{k-1}_{q,t+1}}{1+r} - \frac{r\gamma\sigma_m^2}{(1+r)^2}\left[\frac{1-\delta}{(1+r)v}\left(\overline{D}-D_{t+2}\right)-\beta^{k-1}_{\mathcal{T},t+1}\right]\frac{1-\delta}{(1+r)v}, & k>1, \end{cases}$$

where the expression for $k = 1$ follows from the fact that, following the policy announcement, level-1 thinkers do not change their expectations about the future. Thus, $\alpha^k_{q,t}$ can be rewritten as

$$\alpha^k_{q,t} = \begin{cases} \frac{1-vr}{R-1+\delta}, & k=1, \\ \frac{1-vr}{r+\delta} + \frac{r}{1+r}\cdot\frac{\gamma(1-\delta)^2}{(1+r)^3v^2}D_{t+2}\sigma_m^2, & k=2, \\ \frac{1-vr+(1-\delta)\alpha^{k-1}_{q,t+1}}{R}, & k\geq 3. \end{cases}$$

A few simple steps of algebra let us rewrite the line corresponding to $k \geq 3$ as follows:

$$\begin{aligned}
\alpha^k_{q,t} - \frac{1-vr}{r+\delta} &= \frac{1}{1+r}\left((1-\delta)\,\alpha^{k-1}_{q,t+1} + 1 - vr - (1+r)\frac{1-vr}{r+\delta}\right) \\
&= \frac{1-\delta}{1+r}\left(\alpha^{k-1}_{q,t+1} - \frac{1-vr}{r+\delta}\right),
\end{aligned}$$

which, after applying the "diagonal iteration", becomes

$$\alpha_{q,t}^{k} - \frac{1-vr}{r+\delta} = \left(\frac{1-\delta}{1+r}\right)^{k-2}\left(\alpha_{q,t+k-2}^{2} - \frac{1-vr}{r+\delta}\right)$$
$$= \left(\frac{1-\delta}{1+r}\right)^{k-2}\frac{r}{1+r}\cdot\frac{\gamma(1-\delta)^2}{(1+r)^3 v^2}D_{t+k}\sigma_m^2,$$

for $k \geq 3$. As a result, $\alpha_{q,t}^{k}$ satisfies

$$\alpha_{q,t}^{k} - \frac{1-vr}{r+\delta} = \begin{cases} 0, & k=1, \\ \left(\frac{1-\delta}{1+r}\right)^{k-2}\frac{r}{1+r}\cdot\frac{\gamma(1-\delta)^2}{(1+r)^3v^2}D_{t+k}\sigma_m^2, & k\geq 2. \end{cases} \tag{C.18}$$

Reflective equilibrium. The market-clearing condition for long-term bonds is

$$\sum_{k=1}^{\infty} f(k)(\Sigma^*)_{1,1}(\overline{D} - D_{t+1}) = \frac{1}{\frac{r}{1+r}\gamma}\sum_{k=1}^{\infty} f(k)(1 - \mathbb{E}_t\tilde{p}_{t+1}^{k} + (1-\delta)\,\mathbb{E}_t\tilde{q}_{t+1}^{k} - (1+r)q_t^{RE})$$
$$+ \sum_{k=1}^{\infty} f(k)cov_t(\widetilde{\mathcal{T}}_{t+1}, \widetilde{\mathcal{R}}_{d,t+1}),$$

which, after substituting out $cov_t(\widetilde{\mathcal{T}}_{t+1}, \widetilde{\mathcal{R}}_{d,t+1}) = [(1-\delta)\,\beta_{q,t}^{k} - \beta_{p,t}^{k}]\beta_{\mathcal{T},t}^{k}\sigma_m^2$, $\mathbb{E}_t\tilde{p}_{t+1}^{k} = \alpha_{p,t}^{k} + \xi_p^k\epsilon_t^m$, and $\mathbb{E}_t\tilde{q}_{t+1}^{k} = \alpha_{q,t}^{k} + \xi_q^k\epsilon_t^m$, can be rewritten as

$$q_t^{RE} = \frac{1-vr}{r+\delta} + \frac{1}{(1+r)v}\epsilon_t^m + \frac{r\gamma}{(1+r)^2}\cdot\frac{(1-\delta)^2}{(1+r)^2v^2}\sigma_m^2\sum_{k=1}^{\infty} f(k)\left(\frac{1-\delta}{R}\right)^{k-1}D_{t+k}$$
$$= q_t^* + \frac{r\gamma}{(1+r)^2}\cdot\frac{(1-\delta)^2}{(1+r)^2v^2}\sigma_m^2\sum_{k=1}^{\infty} f(k)\left(\frac{1-\delta}{R}\right)^{k-1}D_{t+k}.$$

We summarize the outcomes of the reflective equilibrium in the following proposition.

Proposition C.2. *Consider a sequence of quantitative easing policies $\{D_{t+1}\}$. In the cashless limit of the reflective equilibrium, the short-term nominal interest rate and the price level coincide with their REE counterparts, while the long-term bond price satisfies*

$$q_t^{RE} = q_t^* + \frac{r\gamma}{(1+r)^2}\cdot\frac{(1-\delta)^2}{(1+r)^2v^2}\sigma_m^2\sum_{k=1}^{\infty} f(k)\left(\frac{1-\delta}{R}\right)^{k-1}D_{t+k},$$

where q_t^* denotes the *long-term bond price* in the REE defined in Proposition *C.1.*

In particular, q_t^{RE} is increasing in the amount of long-term bonds purchased by the central bank.

C.2 FX Interventions in a Two-country Version of the Simple Model

We now present the second extension of the simple model that we use to investigate the effects of international balance sheet policies, such as sterilized FX interventions. Relative to the simple model in Section 2, this extension features both a nominal friction in the form of the demand for money and an international dimension, which borrows elements of the open-economy models of Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006).

There are two countries: home and foreign. Foreign-country variables will bear an asterisk. Both countries produce the same good, which is traded freely across borders. As a result, the law of one price necessarily holds in equilibrium and we have $P_t = E_t P_t^*$, where P_t and P_t^* are the nominal price levels in the home and foreign countries, respectively, and E_t is the nominal exchange rate. The fact that the law of one price holds is a necessary no-arbitrage equilibrium condition: without it, the household problem does not have a solution. The exchange rate is defined as the quantity of home currency bought by one unit of foreign currency. Consequently, an increase in E_t corresponds to a depreciation of home currency. For convenience, we let $e_t \equiv \log E_t$, $p_t \equiv \log P_t$, and $p_t^* \equiv \log P_t^*$.

There are several assets in this world. Households in the home country can hold money issued by their own country's central bank, one-period nominal bonds/reserves created by both countries—which pay, respectively, continuously compounded interest rates i_t and i_t^*—and riskless real assets, available in perfectly elastic supply, that pay off a real net return $r > 0$. Similarly, households in the foreign country can hold money issued by their own country, nominal bonds/reserves created by both countries, and the riskless real assets. For simplicity, we do not consider private risky assets and long-term public bonds. Note that we follow the international economics literature and make the simplifying assumption that households in a country can only hold the money of the country they live in. It is easy to extend the analysis to the case where households can hold money of both countries.

There are two sources of risk in the world economy. Both home- and foreign-country money supplies follow processes given by $\log M_{t+1} = \log \overline{M} + \epsilon_t^m$ and $\log M_{t+1}^* = \log \overline{M}^* + \epsilon_t^{m*}$, where the disturbances ϵ_t^m and ϵ_t^{m*} are assumed to be independent from each other, independent over time, and normally distributed with zero mean and standard deviations σ_m, and σ_{m^*}, respectively. Note that these money-supply processes differ from the one assumed in Section C.1, where, for simplicity of exposition, the money-supply process eliminated any risk in one-period nominal bonds. Since, for simplicity, we do not introduce long-term bonds in this section, it is essential to ensure that short-term bonds are risky.

We focus on home-country households, foreign-country households are symmetric. The size of the home and foreign country are, respectively, ω and $1-\omega$, $\omega \in (0,1)$. Households maximize preferences (C.1) by choosing safe assets s_{t+1}, nominal home bonds $b_{H,t+1}$ (expressed in period-t consumption goods), nominal foreign bonds $b_{F,t+1}$ (expressed in period-t consumption goods), home-country real money balances m_{t+1}, and consumption c_{t+1}, subject to the budget constraint

$$\begin{aligned} & P_t c_t + P_t s_{t+1} + P_t b_{H,t+1} + P_t b_{F,t+1} + P_t m_{t+1} \\ \leq & P_t (W_t - T_t) + P_t(1+r)s_t + e^{i_{t-1}} P_{t-1} b_{H,t} + E_t e^{i_{t-1}^*} P_{t-1}^* b_{F,t} + P_{t-1} m_t. \end{aligned} \tag{C.19}$$

We now specify the behavior of the central bank and the treasury in each country. The home-country government controls real taxes $\{T_{t+1}\}$, the *real* amount of one-period nominal bonds $\{B_{H,t+1}\}$, the amount of one-period real bonds $\{S_{H,t+1}\}$. Without loss of generality we set $B_{H,t+1} = 0$. Notice, however, that the central bank in the home country will still create reserves, which are equivalent to short-term nominal bonds. The home-country central bank controls the nominal money supply $\{M_{t+1}\}$, the *real* amount of one-period interest-paying reserves $\{R_{t+1}\}$, real transfers to the treasury $\{Tr_{t+1}\}$, and the *real* amount of purchases of foreign-currency one-period bonds $\{B_{F,t+1}\}$. As before, we assume that the purchases are fully financed by creating reserves bonds. Using the law of one price, the latter implies $R_{t+1} = B_{F,t+1}$. A policy of foreign-bond purchases financed with the creation of reserves will be referred to as "(sterilized) FX intervention."

The home-country treasury's per-period budget constraint is

$$P_t(1+r)S_{H,t} = \omega P_t T_t + P_t Tr_t + P_t S_{H,t+1}. \tag{C.20}$$

Similarly, the home-country central bank's per-period budget constraint is

$$P_t Tr_t + e^{i_{t-1}} P_{t-1} R_t + M_t + E_t P_t^* B_{F,t+1} = E_t e^{i_{t-1}^*} P_{t-1}^* B_{F,t} + P_t R_{t+1} + M_{t+1}$$

or, using the law of one price and the requirement that FX interventions are financed entirely with reserves,

$$P_t Tr_t + e^{i_{t-1}} P_{t-1} B_{F,t} + M_t = E_t e^{i_{t-1}^*} P_{t-1}^* B_{F,t} + M_{t+1}. \tag{C.21}$$

We will only consider the case in which balance sheet policies are implemented by the central bank in the home country. The analysis for the foreign country is symmetric. Without loss of generality, we assume that the government in the foreign-country sets money supply and taxes so as to keep a constant level of real bonds B^*.

Beliefs are defined as in the simple model in Section 2. Similarly, we let $\widetilde{\mathcal{Z}}_t^k$, $\widetilde{\mathcal{Z}}_t^{SQ}$, and $\widetilde{\mathcal{Z}}_t^*$ denote level-k beliefs, rational-expectations beliefs before ("status quo") and after the intervention, respectively. Here, $Z_t \equiv (p_t, i_t, \mathcal{T}_t, Tr_t, p_t^*, i_t^*, \mathcal{T}_t^*, e_t)$, where $p_t \equiv \log P_t$, and $\mathcal{T}_t$ and $\mathcal{T}_t^*$ are defined as in Section C.1. Note that, since in the foreign country transfers from central bank to the treasury are zero, we do not need to consider beliefs about them.

It is straightforward to adapt all the definitions of equilibria to this environment.

Household optimization. As before, we let a_t represent individual net worth and $\widetilde{H}_t \equiv \mathbb{E}_t \sum_{j=0}^{\infty}(W_{t+j} - \widetilde{T}_{t+j})/(1+r)^j$ be human capital. The Bellman equation for the household problem is

$$V(a_t, m_t; \widetilde{\mathcal{Z}}_t) = \max_{\widehat{c}_t, \widehat{b}_{H,t+1}, \widehat{b}_{F,t+1}, \widehat{m}_{t+1}} \left\{ -\frac{1}{\gamma} \exp\left[-\gamma \left(\widehat{c}_t - \frac{m_t \left[\log\left(m_t/\overline{m}\right) - 1\right]}{v} \right) \right] + e^{-\rho} \mathbb{E}_t V(\widehat{a}_{t+1}, \widehat{m}_{t+1}; \widetilde{\mathcal{Z}}_{t+1}) \right\},$$

subject to

$$\begin{aligned} \widetilde{P}_{t+1}\widehat{a}_{t+1} =& \widetilde{P}_{t+1}(1+r)\left(a_t - \widehat{c}_t - \widehat{b}_{H,t+1} - \widehat{b}_{F,t+1} - \widehat{m}_{t+1}\right) \\ &+ e^{i_t} P_t \widehat{b}_{H,t+1} + \widetilde{E}_{t+1} e^{i_t^*} P_t^* \widehat{b}_{F,t+1} + P_t \widehat{m}_{t+1} + \widetilde{P}_{t+1}(W_{t+1} - \widetilde{T}_{t+1}). \end{aligned} \tag{C.22}$$

We first rewrite the budget constraint (C.22) in real units by dividing both sides by $\widetilde{P}_{t+1}$ and then take a first-order Taylor expansion around $\left(i_t, i_t^*, \widetilde{\pi}_{t+1}, \widetilde{\pi}_{t+1}^*\right) = (r, r, 0, 0)$:

$$\widehat{a}_{t+1} = (1+r)(a_t - \widehat{c}_t) + \left(\widehat{b}_{H,t+1}, \widehat{b}_{F,t+1}, \widehat{m}_{t+1}\right) \widetilde{\mathcal{R}}_{t+1} + (W_{t+1} - \widetilde{T}_{t+1}).$$

where $\widetilde{\mathcal{R}}_{t+1} \equiv \left(i_t - \widetilde{\pi}_{t+1} - r, i_t^* - \widetilde{\pi}_{t+1}^* - r, -\widetilde{\pi}_{t+1} - r\right)'$ is the vector of excess returns. Our strategy of log-linearizing the budget constraint and treating it as exact follows Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006). This approach yields an approximation to the true solution that allows for an analytic characterization.

We conjecture that beliefs $\widetilde{\mathcal{Z}}_t$ are such that the vector of endogenous variables Z_{t+1} is linear in the underlying shocks of the economy. This conjecture will be verified in all the equilibria we consider below. Specifically,

$$x_{t+1} = \alpha_{x,t} + \beta_{x,t}\epsilon^{m}_{t+1} + \xi_{x,t}\epsilon^{m*}_{t+1}, \tag{C.23}$$

for some, possibly time-varying, coefficients $\alpha_{x,t}$, $\beta_{x,t}$, and $\xi_{x,t}$, $x \in \{p, i, \mathcal{T}, Tr, p^*, i^*, \mathcal{T}^*, e\}$.

We guess and verify that

$$V(a_t, m_t; \widetilde{\mathcal{Z}}_t) = -\frac{1}{\gamma} e^{-\gamma\left[A\left(a+\widetilde{H}_t+\widetilde{\vartheta}_t\right)-A_m\frac{m[\log(m/\overline{m})-1]}{v}\right]},$$

where $\widetilde{\vartheta}_t$ is a deterministic function of time, which summarizes a number of endogenous variables taken as given by the households. Standard properties of Normal distributions imply

$$\begin{aligned} V(a_t, m_t; \widetilde{\mathcal{Z}}_t) = \max_{\widehat{c}_t, \widehat{b}_{H,t+1}, \widehat{b}_{F,t+1}, \widehat{m}_{t+1}} \Bigg\{ & -\frac{1}{\gamma} \exp\left(-\gamma\left(\widehat{c}_t - \frac{m_t\left(\log\left(m_t/\overline{m}\right)-1\right)}{v}\right)\right) \\ & -\frac{1}{\gamma}\exp\Bigg(-\rho - \gamma A \mathbb{E}_t[\widehat{a}_{t+1} + \widetilde{H}_{t+1}] - \gamma A \vartheta_{t+1} \\ & + \frac{1}{2}\gamma^2 A^2 \mathbb{V}_t[\widehat{a}_{t+1} + \widetilde{H}_t] + A_m \gamma \frac{m_{t+1}[\log(\widehat{m}_{t+1}/\overline{m}) - 1]}{v}\Bigg)\Bigg\}, \end{aligned}$$

where

$$\begin{aligned} \mathbb{E}_t[\widehat{a}_{t+1} + \widetilde{H}_{t+1}] =& (1+r)(a_t - \widehat{c}_t) + (i_t - \alpha_{p,t} + p_t - r)\widehat{b}_{H,t+1} \\ & + \left(i^*_t - \alpha_{p^*,t} + p^*_t - r\right)\widehat{b}_{F,t+1} + (-\alpha_{p,t} + p_t - r)\widehat{m}_{t+1} \\ & + \mathbb{E}_t \widetilde{H}_{t+1} \end{aligned}$$

and

$$\begin{aligned} \mathbb{V}_t[\widehat{a}_{t+1} + \widetilde{H}_t] =& \sigma^2_m \left(-\beta_{p,t}\widehat{b}_{H,t+1} - \beta_{p^*,t}\widehat{b}_{F,t+1} - \beta_{p,t}\widehat{m}_{t+1} - \beta_{\mathcal{T},t}\right)^2 \\ & + (\sigma^*_m)^2\left(-\xi_{p,t}\widehat{b}_{H,t+1} - \xi_{p^*,t}\widehat{b}_{F,t+1} - \xi_{p,t}\widehat{m}_{t+1} - \xi_{\mathcal{T},t}\right)^2. \end{aligned}$$

The solution to the optimization problem at time t gives realized choices $c_t = c(a_t, m_t; \widetilde{\mathcal{Z}}_t)$, $a_{t+1} = a(a_t, m_t; \widetilde{\mathcal{Z}}_t)$, $b_{H,t+1} = b_H(a_t, m_t; \widetilde{\mathcal{Z}}_t)$, $b_{F,t+1} = d(a_t, m_t; \widetilde{\mathcal{Z}}_t)$, $m_{t+1} = m(a_t, m_t; \widetilde{\mathcal{Z}}_t)$, as well as planned choices, that is, the choices that the agent expects to implement in the future. Due to bounded rationality, the latter may differ from the former. We use a "hat" for planned choices.

The solution to the problem satisfies the first-order condition for consumption

$$e^{-\gamma\left(c_t - \frac{m_t(\log(m_t/\overline{m})-1)}{v}\right)} = A(1+r)e^{-\rho-\gamma A\mathbb{E}_t\left[a_{t+1}+\widetilde{H}_{t+1}\right]-\gamma A\vartheta_{t+1}+\frac{1}{2}\gamma^2A^2\mathbb{V}_t\left[a_{t+1}+\widetilde{H}_{t+1}\right]+A_m\gamma\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}} \tag{C.24}$$

and the first-order conditions with respect to $b_{H,t+1}$, $b_{F,t+1}$, and m_{t+1}

$$\widetilde{\Sigma}_t \cdot \begin{pmatrix} b_{H,t+1} \\ b_{F,t+1} \\ m_{t+1} \end{pmatrix} = \frac{1}{A\gamma}\mathbb{E}_t\widetilde{\mathcal{R}}_{t+1} + \frac{1}{A\gamma}\begin{pmatrix} 0 \\ 0 \\ -\frac{A_m}{Av}\log\left(\frac{m_{t+1}}{\overline{m}}\right) \end{pmatrix} + cov_t\left(\widetilde{\mathcal{T}}_{t+1}, \widetilde{\mathcal{R}}_{t+1}\right), \tag{C.25}$$

where the variance-covariance matrix $\widetilde{\Sigma}_t \equiv \mathbb{V}_t(\widetilde{\mathcal{R}}_{t+1})$ is such that $(\widetilde{\Sigma}_t)_{1,1} = (\widetilde{\Sigma}_t)_{3,3} = (\widetilde{\Sigma}_t)_{1,3} = (\widetilde{\Sigma}_t)_{3,1} =$

$\left(\beta_{p,t}\right)^2\sigma_m^2+\left(\xi_{p,t}\right)^2\sigma_{m^*}^2$, $(\widetilde{\Sigma}_t)_{2,2}=\left(\beta_{p^*,t}\right)^2\sigma_m^2+\left(\xi_{p^*,t}\right)^2\sigma_{m^*}^2$, $(\widetilde{\Sigma}_t)_{1,2}=(\widetilde{\Sigma}_t)_{2,1}=(\widetilde{\Sigma}_t)_{3,2}=(\widetilde{\Sigma}_t)_{2,3}=\beta_{p,t}\beta_{p^*,t}\sigma_m^2+\xi_{p,t}\xi_{p^*,t}\sigma_{m^*}^2$. We use $(\widetilde{\Sigma}_t)_{m,n}$ to denote the (m,n)'th element of matrix $\widetilde{\Sigma}_t$. Note that the matrix $\widetilde{\Sigma}_t$ is not invertible because the return on money, the return on short-term bonds, and the one-period ahead return on long-term bonds have the same risk profile. Also, the covariance in (C.25) equals

$$cov_t\left(\widetilde{\mathcal{T}}_{t+1},\widetilde{\mathcal{R}}_{t+1}\right)=-\begin{pmatrix}\beta_{p,t}\\ \beta_{p^*,t}\\ \beta_{p,t}\end{pmatrix}\beta_{\mathcal{T},t}\sigma_m^2-\begin{pmatrix}\xi_{p,t}\\ \xi_{p^*,t}\\ \xi_{p,t}\end{pmatrix}\xi_{\mathcal{T},t}\sigma_{m^*}^2.$$

Combining the first and the third raw of equations (C.25) yields

$$m_{t+1}=\overline{m}e^{-\frac{A}{A_m}vi_t}.$$

An analogous equation holds for the foreign country.

We now verify our guess for the value function. To do this, we evaluate the Bellman equation at the optimum and check if it holds for every values of state variables a_t and m_t. At the optimum, the Bellman equation is

$$\begin{aligned}-\frac{1}{\gamma}e^{-\gamma\left[A\left(a_t+\widetilde{H}_t+\widetilde{\vartheta}_t\right)-A_m\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right]}=&-\frac{1}{\gamma}e^{-\gamma\left\{c_t-\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right\}}\\&-\frac{1}{\gamma}e^{-\rho-\gamma A\mathbb{E}_t[a_{t+1}+\widetilde{H}_{t+1}]+\widetilde{\vartheta}_{t+1}+\frac{1}{2}\gamma^2A^2\mathbb{V}_t[a_{t+1}+\widetilde{H}_{t+1}]+A_m\gamma\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}},\end{aligned}$$

Using the first order condition for consumption, we write

$$-\frac{1}{\gamma}e^{-\gamma\left[A\left(a_t+\widetilde{H}_t+\widetilde{\vartheta}_t\right)-(A_m-1)\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right]}=-\frac{1}{\gamma}e^{-\gamma c_t}\frac{1+A(1+r)}{A(1+r)}.$$

Optimal consumption is obtained from equation (C.24):

$$\begin{aligned}\left[1+A(1+r)\right]c_t=&\frac{m_t[\log(m_t/\overline{m})-1]}{v}-\frac{1}{\gamma}\log\left[A(1+r)\right]+\frac{\rho}{\gamma}+A(1+r)(a_t+\widetilde{H}_t)\\&+A\left(b_{H,t+1},b_{F,t+1},m_{t+1}\right)\mathbb{E}_t\widetilde{\mathcal{R}}_{t+1}+A\widetilde{\vartheta}_{t+1}-\frac{1}{2}\gamma A^2\mathbb{V}_t[a_{t+1}+\widetilde{H}_{t+1}]\\&-\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v},\end{aligned}$$

where we used $(1+r)\widetilde{H}_t=W_{t+1}-\mathbb{E}_t\widetilde{T}_{t+1}+\mathbb{E}_t\widetilde{H}_{t+1}$. Combining the last two equations, we get

$$\begin{aligned}&\left[1+A(1+r)\right]\left\{A\left(a_t+\widetilde{H}_t+\widetilde{\vartheta}_t\right)-(A_m-1)\frac{m_t[\log(m_t/\overline{m})-1]}{v}\right\}\\=&\frac{m_t[\log(m_t/\overline{m})-1]}{v}-\frac{1}{\gamma}\log\left[A(1+r)\right]+\frac{\rho}{\gamma}+A(1+r)(a_t+\widetilde{H}_t)+A\left(b_{H,t+1},b_{F,t+1},m_{t+1}\right)\mathbb{E}_t\widetilde{\mathcal{R}}_{t+1}\\&+A\widetilde{\vartheta}_{t+1}-\frac{1}{2}\gamma A^2\mathbb{V}_t[a_{t+1}+\widetilde{H}_{t+1}]-\frac{m_{t+1}[\log(m_{t+1}/\overline{m})-1]}{v}-\left[1+A(1+r)\right]\frac{1}{\gamma}\log\frac{1+A(1+r)}{A(1+r)}.\end{aligned}$$

For our conjecture to be true, the coefficients multiplying a_t and m_t should be identical, that is,

$$a_t + \widetilde{H}_t : [1 + A(1+r)]\, A = A(1+r),$$
$$\frac{m_t[\log(m_t/\overline{m}) - 1]}{v} : -\,[1 + A(1+r)]\,(A_m - 1) = 1.$$

As a result,

$$A = A_m = \frac{r}{1+r}.$$

Finally, we can express $\widetilde{\vartheta}_t$ as follows

$$\widetilde{\vartheta}_t = \frac{1}{r\gamma}\left[\rho + \log\frac{r^r}{(1+r)^{1+r}}\right] + \frac{\alpha_{\vartheta,t} + (b_{H,t+1}, b_{F,t+1}, m_{t+1})\,\mathbb{E}_t\widetilde{\mathcal{R}}_{t+1}}{1+r} - \frac{\gamma r}{2\,(1+r)^2}\mathbb{V}_t[a_{t+1} + \widetilde{H}_{t+1}]$$
$$-\,\frac{1}{r}\cdot\frac{m_{t+1}[\log(m_{t+1}/\overline{m}) - 1]}{v}.$$

Temporary equilibrium. First, the market-clearing conditions in assets markets in period t are

$$B_{F,t+1} = \omega b_{H,t+1} + (1-\omega) b^*_{H,t+1},$$
$$B^* - B_{F,t+1} = \omega b_{F,t+1} + (1-\omega) b^*_{F,t+1},$$
$$\frac{\overline{M}}{P_t} e^{\epsilon^m_t} = \overline{m} e^{-v i_t},$$
$$\frac{\overline{M}^*}{P^*_t} e^{\epsilon^{m*}_t} = \overline{m} e^{-v i^*_t}.$$

The money-market equilibrium condition implies the following relationship between the price level and the short-term interest rate:

$$p_t = \log\left(\overline{M}/\overline{m}\right) + v i_t + \epsilon^m_t, \tag{C.26}$$

with an analogous equation for the foreign country.

As we did in Section C.1, to streamline the analysis, we focus on the "cashless limit" of our model in which the demand and supply of money are negligibly small. Specifically, we let $\overline{m}$, $\overline{M}$, and $\overline{M}^*$ approach zero, so that the ratios $\overline{m}/\overline{M}$ and $\overline{m}/\overline{M}^*$ approaches one.

In the cashless limit, the market-clearing conditions together with optimal choice of bonds imply

$$\left(\widetilde{\Sigma}_t\right)_{1:2,1:2} \cdot \begin{pmatrix} B_{F,t+1} \\ B^* - B_{F,t+1} \end{pmatrix} = \frac{1}{\frac{r}{1+r}\gamma}\mathbb{E}_t\left(\widetilde{\mathcal{R}}_{t+1}\right)_{1:2} + cov_t\left(\omega\widetilde{\mathcal{T}}_{t+1} + (1-\omega)\,\widetilde{\mathcal{T}}^*_{t+1}, \left(\widetilde{\mathcal{R}}_{t+1}\right)_{1:2}\right), \tag{C.27}$$

where $(\widetilde{\Sigma}_t)_{1:2,1:2}$ is the upper-left sub-matrix of $\widetilde{\Sigma}_t$, and $(\widetilde{\mathcal{R}}_{t+1})_{1:2}$ is the vector containing the first two elements of $\widetilde{\mathcal{R}}_{t+1}$. The nominal interest rate on domestic bonds is obtained from the first line of (C.27):

$$i_t = r + \alpha_{p,t} - p_t + RP_t, \tag{C.28}$$

where, for convenience, we let

$$
\begin{aligned}
RP_t \equiv & \frac{r}{1+r}\gamma\left[\left(\left(\beta_{p,t}\right)^2\sigma_m^2+\left(\xi_{p,t}\right)^2\sigma_{m^*}^2\right)B_{F,t+1}+\left(\beta_{p,t}\beta_{p^*,t}\sigma_m^2+\xi_{p,t}\xi_{p^*,t}\sigma_{m^*}^2\right)\left(B^*-B_{F,t+1}\right)\right.\\
&\left.+\beta_{p,t}\left(\omega\beta_{\mathcal{T},t}+(1-\omega)\beta_{\mathcal{T}^*,t}\right)\sigma_m^2+\xi_{p,t}\left(\omega\xi_{\mathcal{T},t}+(1-\omega)\xi_{\mathcal{T}^*,t}\right)\sigma_{m^*}^2\right].
\end{aligned}
$$

The price level p_t is obtained by combining C.26 and (C.28), taking into account the cashless limit:

$$
\begin{aligned}
p_t =& v i_t+\epsilon_t^m\\
=& \frac{v}{1+v}\left(r+\alpha_{p,t}+RP_t\right)+\frac{1}{1+v}\epsilon_t^m.
\end{aligned} \tag{C.29}
$$

Next, we consider fiscal policy. The realized transfers from the central bank to the treasury can be computed by log-linearizing equation (C.21) around $\left(i_t, i_t^*, \pi_{t+1}, \pi_{t+1}^*\right)=(r,r,0,0)$ and evaluating it at the cashless limit. We obtain

$$
Tr_t=\left(i_{t-1}^*-\pi_t^*-i_{t-1}+\pi_t\right)B_{F,t}. \tag{C.30}
$$

Also, by iterating forward the treasury's budget constraint (C.20) and taking expectations, we have that the expected discounted sum of taxes satisfies

$$
\omega\widetilde{\mathcal{T}}_t=(1+r)S_{H,t}-\mathbb{E}_t\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s\widetilde{Tr}_{t+s}. \tag{C.31}
$$

Since only the central bank in the home country trades assets, transfers to the treasury in the foreign country are zero. The treasury in the foreign country targets a constant supply B^* of bonds, therefore, taxes in the foreign country must satisfy

$$
(1-\omega)\widetilde{\mathcal{T}}_t^*=(1+r)S_{F,t}+B^*\mathbb{E}_t\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s\left(\widetilde{i}_{t+s-1}^*-\widetilde{\pi}_{t+s}^*\right) \tag{C.32}
$$

Finally, to verify our initial conjecture, we need to show that, in the cashless equilibrium, $\widetilde{\vartheta}_t$ is a deterministic function of time. We do this in two steps. First, we compute foreign demand $b_{H,t+1}^*$ and $b_{F,t+1}^*$ using the analogue of (C.25):[25]

$$
\begin{aligned}
\left(\widetilde{\Sigma}_t\right)_{1:2,1:2}\cdot\begin{pmatrix} b_{H,t+1}^* \\ b_{F,t+1}^* \end{pmatrix} &= \frac{1}{\frac{r}{1+r}\gamma}\mathbb{E}_t(\widetilde{\mathcal{R}}_{t+1})_{1:2}+cov_t\left(\widetilde{\mathcal{T}}_{t+1}^*,\widetilde{\mathcal{R}}_{t+1}\right)\\
&= \frac{1}{\frac{r}{1+r}\gamma}\begin{pmatrix} RP_t \\ RP_t^* \end{pmatrix}+cov_t\left(\widetilde{\mathcal{T}}_{t+1}^*,\widetilde{\mathcal{R}}_{t+1}\right),
\end{aligned}
$$

where the last line uses (C.28). Since RP_t, RP_t^*, and all the conditional moments are independent of shocks, so are asset demands. Second, we use the demands just computed together with market-clearing conditions

[25] With a slight abuse of notation, we use $b_{H,t+1}^*$ and $b_{F,t+1}^*$ to denote the foreign-country household optimal choices.

to obtain

$$\widetilde{\vartheta}_t = \frac{1}{r\gamma}\left[\rho + \log\frac{r^r}{(1+r)^{1+r}}\right] + \frac{\widetilde{\vartheta}_{t+1} + \left(\overline{B}_{H,t+1}, \overline{B}_{F,t+1}\right)(RP_t, RP_t^*)'}{1+r} - \frac{\gamma r}{2(1+r)^2}\mathbb{V}_t[a_{t+1} + \widetilde{H}_{t+1}], \tag{C.33}$$

where, $\overline{B}_{H,t+1} \equiv (B_{F,t+1} - (1-\omega)b^*_{H,t+1})/\omega$ and $\overline{B}_{F,t+1} \equiv (B^* - B_{F,t+1} - (1-\omega)b^*_{F,t+1})/\omega$ are, respectively, the supply of home-country and foreign-country bonds after netting out net demand from the central bank and the foreign country. Importantly, since RP_t, RP_t^*, and all the conditional moments are independent of shocks, (C.33) implies that ϑ_t is a deterministic function of time, therefore, our initial conjecture is verified.

Rational expectations equilibrium In the REE, the equilibrium distribution of endogenous variables must be equal to the agents' beliefs about these variables. We use the superscript "REE" instead of a star so as to avoid confusion with foreign-country variables. Specifically, for the price level, we need to make sure that the coefficients $\alpha^{REE}_{p,t}$, $\beta^{REE}_{p,t}$, and $\xi^{REE}_{p,t}$ satisfy

$$\begin{aligned} p_{t+1} &= \frac{v}{1+v}\left(r + \alpha^{REE}_{p,t+1} + RP_{t+1}\right) + \frac{1}{1+v}\epsilon^m_{t+1} \\ &\overset{REE}{=} \alpha^{REE}_{p,t} + \beta^{REE}_{p,t}\epsilon^m_{t+1} + \xi^{REE}_{p,t}\epsilon^{m*}_{t+1}, \end{aligned}$$

for all realizations of the shocks. The latter is satisfied if and only if

$$\begin{aligned} \beta^{REE}_{p,t} &= \frac{1}{1+v}, \\ \xi^{REE}_{p,t} &= 0. \end{aligned}$$

By simmetry, the price level in the foreign country satisfies $\beta^{REE}_{p^*,t} = 0$ and $\xi^{REE}_{p^*,t} = 1/(1+v)$. The short-term nominal interest rate must satisfy

$$\begin{aligned} i_{t+1} &= r + \alpha^{REE}_{p,t+1} - p_{t+1} + RP_{t+1} \\ &\overset{REE}{=} \alpha^{REE}_{i,t} + \beta^{REE}_{i,t}\epsilon^m_{t+1} + \xi^{REE}_{i,t}\epsilon^{m*}_{t+1}, \end{aligned}$$

implying

$$\begin{aligned} \beta^{REE}_{i,t} &= -\frac{1}{1+v}, \\ \xi^{REE}_{i,t} &= 0, \end{aligned}$$

with symmetric expressions for the foreign country. In addition, from (C.30), central bank transfers are such that

$$\begin{aligned} \beta^{REE}_{Tr,t} &= \frac{1}{1+v}B_{F,t+1}, \\ \xi^{REE}_{Tr,t} &= -\frac{1}{1+v}B_{F,t+1}. \end{aligned}$$

and, from (C.31), the expected sum of taxes are such that

$$\omega\beta_{\mathcal{T},t}^{REE} = -\frac{1}{1+v}B_{F,t+1},$$
$$\omega\xi_{\mathcal{T},t}^{REE} = \frac{1}{1+v}B_{F,t+1}.$$

Similarly, from (C.32), taxes in the foreign country satisfy

$$(1-\omega)\,\beta_{\mathcal{T}^*,t}^{REE} = 0,$$
$$(1-\omega)\,\xi_{\mathcal{T}^*,t}^{REE} = -\frac{1}{1+v}B^*.$$

To simplify notation, we omit time subscripts from those coefficients that are constant over time. We thus have

$$\left(\beta_p^{REE},\beta_i^{REE},\beta_{p^*}^{REE},\beta_{i^*}^{REE},\beta_{Tr,t}^{REE},\beta_{\mathcal{T},t}^{REE},\beta_{\mathcal{T}^*,t}^{REE}\right) = \frac{1}{1+v}\left(1,-1,0,0,B_{F,t+1},-B_{F,t+1}/\omega,0,0\right),$$
$$\left(\xi_p^{REE},\xi_i^{REE},\xi_{p^*}^{REE},\xi_{i^*}^{REE},\xi_{Tr,t}^{REE},\xi_{\mathcal{T},t}^{REE},\xi_{\mathcal{T}^*,t}^{REE}\right) = \frac{1}{1+v}\left(0,0,1,-1,-B_{F,t+1},B_{F,t+1}/\omega,0,-B^*\right).$$

Finally, for the non-random part of the endogenous variables, we have

$$\begin{aligned}
\alpha_{p,t}^{REE} &= \frac{v}{1+v}\left(r+\alpha_{p,t+1}^{REE}+RP_{t+1}\right),\\
\alpha_{i,t}^{REE} &= r+\alpha_{p,t+1}^{REE}-\alpha_{p,t}^{REE}+RP_{t+1},\\
\alpha_{Tr,t}^{REE} &= \left(\alpha_{i^*,t-1}^{REE}-\alpha_{p^*,t}^{REE}+\alpha_{p^*,t-1}^{REE}-\alpha_{i,t-1}^{REE}+\alpha_{p,t}^{REE}-\alpha_{p,t-1}^{REE}\right)B_{F,t+1},\\
\alpha_{\mathcal{T},t}^{REE} &= (1+r)S_{H,t+1}-\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s\alpha_{Tr,t+s}^{REE}.
\end{aligned}$$

Combining the expressions above, we immediately obtain that $RP_t = RP_t^* = 0$. We can then solve the system of equations above as follows:

$$\begin{aligned}
\alpha_p^{REE} &= vr,\\
\alpha_i^{REE} &= r,\\
\alpha_{Tr,t}^{REE} &= 0,\\
\alpha_{\mathcal{T},t}^{REE} &= (1+r)S_{H,t+1}.
\end{aligned}$$

with analogous expressions for the foreign-country interest rate and price level.

We summarize these results with the following proposition.

Proposition C.3. *In the cashless limit of the rational expectations equilibrium, the price level and the short-term nominal interest rate in the home country are, respectively, given by*

$$i_t = r-\frac{1}{1+v}\epsilon_t^m,\;\; p_t = vr+\frac{1}{1+v}\epsilon_t^m.$$

For the foreign country,

$$i_t^* = r-\frac{1}{1+v}\epsilon_t^{m*},\;\; p_t^* = vr+\frac{1}{1+v}\epsilon_t^{m*}.$$

Finally, the nominal exchange rate satisfies

$$e_t = \frac{1}{1+v}\left(\epsilon_t^m - \epsilon_t^{m*}\right).$$

In particular, they are all independent of balance sheet policies.

As in all the other settings considered in the paper, in the benchmark with rational expectations, the interest rate, the price level, and the exchange rate are independent of balance sheet policies. First, the interest rate is given by the risk-free real rate minus the shock to the money supply. Intuitively, to stimulate agents to hold more money, the opportunity cost of holding money, that is, the nominal interest rate i_t, must go down. Second, in this economy, the nominal interest rate i_t equals the constant real interest rate r plus expected inflation $\mathbb{E}_t p_{t+1} - p_t$. As a result, the expected inflation must decline following a shock to the money supply. Since the economy is stationary and the future expected price level is constant, the drop in expected inflation is achieved through an increase in the current price level p_t. Third, the law of one price requires that nominal exchange rate depreciates (i.e., e_t goes up, after a positive shock to the home-country money supply). Note also that inflation risk reduces the demand for nominal government bonds, but at the same time, this risk makes future real taxes positively correlated with real bond returns. The last effect increases the demand for nominal bonds. In the REE, the two effects cancel each other out and the nominal interest rate equals the real rate plus expected inflation.

Level-k beliefs. Equations (C.29)-(C.33) define a mapping $\Psi(\cdot)$ from beliefs into equilibrium variables.

By iterating $\Psi(\cdot)$, starting from $\widetilde{\mathcal{Z}}_t^{SQ}$, we can compute the beliefs of level-k agents, for any $k \geq 1$. It is easy to see that some β's and ξ's coincide with their REE counterparts: $\beta_x^k = \beta_p^{REE}$, $\xi_x^k = \xi_x^{REE}$, $x \in \{p, i, p^*, i^*, \mathcal{T}^*, e\}$, for all $k \geq 1$. In particular, the latter imply that the conditional moments of the excess returns coincide with their REE counterparts for all $k \geq 1$.

We are left to derive $\beta_{Tr,t}^k$, $\beta_{\mathcal{T},t}^k$, $\xi_{Tr,t}^k$, $\xi_{\mathcal{T}^*,t}^k$, and the α^k's, for all $k \geq 1$. For $k = 1$, since we assume that the expectations of level-1 households coincide with the distributions of REE variables before the intervention by the central bank, the sensitivity of transfers and taxes to the current shock coincides with its REE counterpart *in the absence* of asset purchases:

$$\beta_{Tr,t}^1 = \xi_{Tr,t}^1 = \beta_{\mathcal{T},t}^1 = \xi_{\mathcal{T},t}^1 = 0.$$

Proceeding recursively for $k > 1$, we have

$$\begin{aligned}
\beta_{Tr,t}^k &= \frac{1}{1+v} B_{F,t+1}, \\
\xi_{Tr,t}^k &= -\frac{1}{1+v} B_{F,t+1}, \\
\beta_{\mathcal{T},t}^k &= -\frac{1}{1+v} B_{F,t+1}, \\
\xi_{\mathcal{T},t}^k &= \frac{1}{1+v} B_{F,t+1}.
\end{aligned}$$

We are left to compute the intercepts. From (C.29),

$$\alpha_{p,t}^k - vr = \begin{cases} 0, & k = 1, \\ \frac{r}{1+r}\gamma\sigma_m^2 \left(\frac{v}{1+v}\right)^k \frac{1}{v(1+v)} B_{F,t+k}, & k > 1. \end{cases} \tag{C.34}$$

Similarly, from equation (C.28),

$$\alpha_{i,t}^{k}-r=\begin{cases}0, & k=1,\\ \frac{r}{1+r}\gamma\sigma_m^2\left(\frac{v}{1+v}\right)^k\frac{1}{v^2(1+v)}B_{F,t+k}, & k>1.\end{cases} \tag{C.35}$$

Note that the expressions for $k=1$ follow from the fact that, upon observing the policy announcement, level-1 thinkers do not change their expectations about the future.

Similarly, for the foreign country,

$$\alpha_{p^*,t}^{k}-vr=\begin{cases}0, & k=1,\\ -\frac{r}{1+r}\gamma\left(\sigma_{m^*}\right)^2\left(\frac{v}{1+v}\right)^k\frac{1}{v(1+v)}B_{F,t+k}, & k>1,\end{cases} \tag{C.36}$$

and

$$\alpha_{i^*,t}^{k}-r=\begin{cases}0, & k=1,\\ -\frac{r}{1+r}\gamma\left(\sigma_{m^*}\right)^2\left(\frac{v}{1+v}\right)^k\frac{1}{v^2(1+v)}B_{F,t+k}, & k>1.\end{cases} \tag{C.37}$$

Reflective equilibrium. The market-clearing conditions for the nominal bonds in both countries are

$$\sum_{k=1}^{\infty}f(k)(\Sigma_t^{REE})_{1:2,1:2}\begin{pmatrix}B_{F,t+1}\\ B^*-B_{F,t+1}\end{pmatrix}=\frac{1}{\frac{r}{1+r}\gamma}\sum_{k=1}^{\infty}f(k)\mathbb{E}_t\left(\widetilde{\mathcal{R}}_{t+1}^k\right)_{1:2}$$
$$+\sum_{k=1}^{\infty}f(k)cov_t\left(\omega\widetilde{\mathcal{T}}_{t+1}^k+(1-\omega)\widetilde{\mathcal{T}}_{t+1}^{*,k},\left(\widetilde{\mathcal{R}}_{t+1}^k\right)_{1:2}\right).$$

After substituting out for the expressions of the conditional moments, we can rewrite the last equation as

$$\left(\frac{1}{1+v}\right)^2\begin{pmatrix}\sigma_m^2B_{F,t+1}\\ \sigma_{m^*}^2\left(B^*-B_{F,t+1}\right)\end{pmatrix}=\frac{1}{\frac{r}{1+r}\gamma}\sum_{k=1}^{\infty}f(k)\mathbb{E}_t\left(\widetilde{\mathcal{R}}_{t+1}^k\right)_{1:2}+f\left(1\right)\left(\frac{1}{1+v}\right)^2\begin{pmatrix}0\\1\end{pmatrix}B^*\sigma_{m^*}^2$$
$$+\left(1-f\left(1\right)\right)\left(\frac{1}{1+v}\right)^2\left(\begin{pmatrix}1\\0\end{pmatrix}\sigma_m^2B_{F,t+1}+\begin{pmatrix}0\\1\end{pmatrix}\left(B^*-B_{F,t+1}\right)\sigma_{m^*}^2\right).$$

Finally, using (C.26) together with the intercepts (C.34)-(C.37), we can solve the equation above for the nominal interest rate, both in the home and in the foreign country:

$$\begin{pmatrix}i_t-i_t^{REE}\\ i_t^*-i_t^{*REE}\end{pmatrix}=\frac{r}{1+r}\gamma\begin{pmatrix}\sigma_m^2\\ -\sigma_{m^*}^2\end{pmatrix}\frac{1}{v\left(1+v\right)^2}\sum_{k=1}^{\infty}f(k)\left(\frac{v}{1+v}\right)^kB_{F,t+k}.$$

Analogous steps lead to a solution for p_t, p_t^*, and the nominal exchange rate e_t. We summarize the reflective equilibrium outcomes in the following proposition.

Proposition C.4. *Consider a sequence of (sterilized) FX intervention $\{B_{F,t+1}\}$. Let p_t^{REE}, i_t^{REE}, p_t^{*REE}, i_t^{*REE},* and e_t^{REE} denote the price levels, the nominal interest rates, and the nominal exchange rate in the rational expectations equilibrium defined in Proposition *C.3. Also, let*

$$\Gamma_t\equiv\frac{r}{1+r}\gamma\frac{1}{v\left(1+v\right)^2}\sum_{k=1}^{\infty}f(k)\left(\frac{v}{1+v}\right)^kB_{F,t+k}.$$

In the cashless limit of the reflective equilibrium, the price level and the short-term nominal interest rate in the home country are, respectively, given by

$$i_t = i_t^{REE} + \sigma_m^2 \Gamma_t, \;\; p_t = p_t^{REE} + \upsilon \sigma_m^2 \Gamma_t.$$

For the foreign country,

$$i_t^* = i_t^{*REE} - \sigma_{m^*}^2 \Gamma_t, \;\; p_t^* = p_t^{*REE} - \upsilon \sigma_{m^*}^2 \Gamma_t.$$

Finally, the nominal exchange rate satisfies

$$e_t = e_t^{REE} + \upsilon \left(\sigma_m^2 + \sigma_{m^*}^2 \right) \Gamma_t.$$

Proposition C.4 contains the main result of this extension. It shows that, in the reflective equilibrium, balance sheet policies affect asset prices, including the exchange rate. In particular, the nominal interest rate and the price level—and, therefore, the exchange rate—are now functions of the entire path of bond purchases. In the reflective equilibrium, some households fail to anticipate that future taxes will now be risky in real terms since bonds promise a risk-free *nominal* payment. In particular, since the FX intervention is sterilized, the tax risk in the home country will be a combination of the shocks to the money supplies in the two countries. As before, if households do not hold rational expectations and, hence, fail to hedge the tax risk, asset prices will be affected.

C.3 A Model with Learning

In this section, we modify our simple model and allow for statistical learning. Specifically, we focus on level-1 thinkers and assume that their forecasts, at time t, about the asset price and transfers at time $s > t$ are

$$\tilde{z}^1_{s+1|t-1} = \alpha^1_{s|t-1} + \beta^1_{s|t-1} \epsilon^x_{s+1}.$$

The notation emphasizes that these forecasts, which are the analogue of equation (10) in the main text, now take into account that expectations depend on the information available at the time the forecast is made. Note that we have not assumed any particular statistical process for the updating of α's and β's.

We follow the literature on statistical learning and assume that, at any given time t, the agent considers $(\alpha^1_{s|t-1}, \beta^1_{s|t-1})$ to be the "true" parameters. That is, the agent does not take into account that she will update her beliefs over time as more data becomes available. In turn, this feature implies that the agent's problem is exactly the same as in the simple model without learning. In particular, the arguments in the proof of Lemma 1 extend directly to this setting, with the only exception that (α^1_s, β^1_s) are replaced with $(\alpha^1_{s|t-1}, \beta^1_{s|t-1})$. As a result, household asset demand is

$$x_{t+1} = \frac{\mathbb{E}_t[D_{t+1} + \tilde{q}^1_{t+1|t-1}] - (1+r) q_t}{\gamma \frac{r}{1+r} \sigma_x^2} - \beta^1_{Tr,t|t-1}, \tag{C.38}$$

which is the analogue of (11) in the main text. By imposing market clearing (6), we derive the asset price in the temporary equilibrium with level-1 thinkers:

$$q^1_t = \frac{\overline{D} + \alpha^1_{q,t|t-1} - \frac{r}{1+r} \gamma \sigma_x^2 (1 - X_{t+1} + \beta^1_{Tr,t|t-1})}{1+r},$$

which is the analogue of (12) in the main text.

C.4 A Model with Unraveling

In this section, we modify our simple model and allow for a simple version "dynamic equilibrium unraveling." In particular, we assume that the level of sophistication of agents changes over time. We introduce this assumption in the simplest possible way to highlight a number of qualitative results. Specifically, we assume that a current level-k thinker becomes level-$(k+h)$ in the subsequent period, where h is a constant non-negative integer number. One interpretation of this assumption is as follows. At the time a new policy is announced, an agent can compute k deductive iterations to form the expectations about future endogenous variables. In every subsequent period, the agent uses the already computed beliefs and performs h additional deductive iterations.

Formally, we assume that the distribution of levels of sophistication changes over time according to

$$f_t(k) = \begin{cases} f(k-ht), & k \geq 1+ht, \\ 0, & k < 1+ht, \end{cases} \tag{C.39}$$

where $f(k)$ is the distribution at the time the policy is announced.

We can compute the price effect of the intervention by evaluating the results in Proposition 2 with the distribution (C.39). Specifically, if we focus, for simplicity, on a permanent intervention of the size $\overline{X}$, we obtain

$$q_t = q^* + \frac{r}{1+r}\gamma\sigma_x^2\overline{X}\sum_{k=1}^{\infty} f_t(k)\frac{1}{(1+r)^k}.$$

Now the distribution of level-k agents changes over time. Using (C.39) and assuming that the initial distribution is exponential, we get

$$q_t = q^* + \frac{r}{1+r}\gamma\sigma_x^2\frac{\overline{X}}{\bar{k}r+1}(1+r)^{-ht}. \tag{C.40}$$

Equation (C.40) shows that, over time, the price approaches q^* at rate $1/(1+r)^h$. Thus, h determines the speed of convergence. The key implication of equation (C.40) is that a central bank cannot stimulate the economy forever by keeping the size of its balance sheet at a constant level.

To counteract the dampening forces coming from equilibrium unraveling, it is crucial that the size of the intervention increases over time. In the specific example, to keep the price q_t at a constant higher level, the central bank needs to increase asset purchases exponentially at the rate $\mu = (1+r)^{h/(1+h)} > 1$. Formally, with $X_{t+1} = \mu^t\overline{X}$, where $\mu \leq ((1+r)/\mu)^h$ (to keep the price bounded), we get

$$q_t = q^* + \frac{r}{1+r}\gamma\sigma_x^2 \cdot \frac{1}{\bar{k}\,(1+r-\mu)+\mu}\overline{X}\left(\frac{\mu^{1+h}}{(1+r)^h}\right)^t,$$

so that, if $\mu^{1+h}/(1+r)^h = 1$, then q_t does not depend on time.

An empirical prediction of equilibrium unraveling is that new rounds of balance sheet policies tend to be weaker than initial rounds. For example, after controlling for the size of the intervention, the first round of quantitative easing implemented by the Federal Reserve in 2009 should have had stronger effects than the second round implemented in 2010.

C.5 A Model with Rational Expectation Agents

To investigate whether the presence of households with rational expectations can undo the non-neutrality result of balance sheet policies, we add a mass of rational-expectations agents to the simple model. Specifically, we assume that fraction $\tau \in [0,1]$ of agents form their expectations rationally, while the remaining fraction $1-\tau$ uses the level-k thinking process. Moreover, the fraction $1-\tau$ is split into groups with different levels k, where groups have mass given by the distribution function $f(k)$.

In the reflective equilibrium augmented with rational-expectations agents, market-clearing in the risky asset market requires

$$\tau\left(\frac{r^x+q_{t+1}-q_t(1+r)}{\gamma\frac{r}{1+r}\sigma_x^2}-X_{t+1}\right)+(1-\tau)\sum_{k=1}^{\infty}f(k)\left(\frac{r^x+\alpha_{q,t}^k-q_t(1+r)}{\gamma\frac{r}{1+r}\sigma_x^2}-\beta_{Tr,t}^k\right)=1-X_{t+1}.$$

As in the simple model, the price q_t is deterministic. The first term on the left-hand side is the demand for risky assets by the rational-expectations agents, while the second term represents the demand by level-k thinkers. Conditional on the beliefs of level-k thinkers that that we computed in Section 2, we solve the equation above for price q_t:

$$q_t-q^*=\tau\frac{q_{t+1}-q^*}{1+r}+(1-\tau)\frac{r}{1+r}\gamma\sigma_x^2\sum_{k=1}^{\infty}f(k)\frac{X_{t+k}}{(1+r)^k}. \tag{C.41}$$

To solve for price q_t, we introduce the following new variables:

$$G_t\equiv\frac{r}{1+r}\gamma\sigma_x^2\sum_{k=1}^{\infty}f(k)\frac{X_{t+k}}{(1+r)^k},$$
$$\Delta q_t\equiv q_t-q^*.$$

We can then rewrite equation (C.41) as follows:

$$\Delta q_t=\tau\frac{\Delta q_{t+1}}{1+r}+(1-\tau)\,G_t,$$

and iterate it forward (imposing a non-bubble condition) to get

$$\Delta q_t=(1-\tau)\sum_{s=0}^{\infty}\left(\frac{\tau}{1+r}\right)^s G_{t+s},$$

or, using the definitions of G_t and Δq_t,

$$q_t=q^*+(1-\tau)\frac{r}{1+r}\gamma\sigma_x^2\sum_{k=1}^{\infty}\frac{f(k)}{(1+r)^k}\sum_{s=0}^{\infty}\left(\frac{\tau}{1+r}\right)^s X_{t+s+k}. \tag{C.42}$$

Comparing (C.42) to the price of risky assets in the reflective equilibrium in the simple model, equation (17), we note the following. First, the presence of the term $(1-\tau)$ implies that a higher fraction of rational-expectations agents must reduce the effectiveness of balance sheet policies. At the same time, however, the term $(\tau/(1+r))^s$ implies that a higher fraction of rational-expectations agents leads to a lower discounting of future policies and, thus, to a stronger effect of future purchases. These two opposing effects echo the discussion on the implications for balance sheet policies of a higher average level of sophistication in Section 2.5. To make this more explicit, we compute the price q_t in two special cases.

Example 1. Consider an example in which $f(k) = (1-\lambda)\lambda^{k-1}$ and $X_{t+k} = X_{t+1}\mu^{k-1}$. In this case,

$$\begin{aligned} G_t &= \frac{r}{1+r}\gamma\sigma_x^2 X_{t+1} \sum_{k=1}^{\infty} (1-\lambda)\lambda^{k-1} \frac{\mu^{k-1}}{(1+r)^k} \\ &= \frac{r}{1+r}\gamma\sigma_x^2 X_{t+1} \frac{1-\lambda}{1+r-\lambda\mu}. \end{aligned}$$

Thus, the asset price is

$$q_t = q^* + \frac{r}{1+r}\gamma\sigma_x^2 \frac{1-\lambda}{1+r-\lambda\mu} X_{t+1} \frac{(1+r)(1-\tau)}{1+r-\tau\mu}.$$

The last expression has its maximum at $\tau = 0$ and monotonically declines to zero at $\tau = 1$. In this example, the first effect dominates and a higher fraction of rational-expectations agents makes balance sheet policies weaker.

Example 2. Consider now the following path of risky assets purchases $\{X_{t+1}\} = \{0, X_{t+2}, 0, 0, \ldots\}$. In this case,

$$\begin{aligned} G_t &= \frac{r}{1+r}\gamma\sigma_x^2 f(2) \frac{X_{t+2}}{(1+r)^2}, \\ G_{t+1} &= \frac{r}{1+r}\gamma\sigma_x^2 f(1) \frac{X_{t+2}}{(1+r)^1}, \\ G_{t+2} &= 0, \end{aligned}$$

and the asset price is

$$q_t = q^* + \frac{r}{1+r}\frac{\gamma\sigma_x^2}{(1+r)^2}(1-\tau)(\lambda+\tau)(1-\lambda) X_{t+2}.$$

The price is now a non-monotonic function of τ. The derivative of the price with respect to τ is:

$$\frac{dq_t}{d\tau} = \frac{r}{1+r}\frac{\gamma\sigma_x^2}{(1+r)^2}(1-\lambda) X_{t+2}[1-\lambda-2\tau],$$

we have that, when the fraction of rational-expectations agents is low enough, $\tau < (1-\lambda)/2$, then $dq_t/d\tau > 0$. That is, the second effect dominates and balance sheet policies become stronger as τ grows and as long as $\tau < (1-\lambda)/2$.