Ambiguity and Information Processing in a Model of Intermediary Asset Pricing[*]

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Abstract

This paper incorporates ambiguity and information processing constraints into a model of intermediary asset pricing proposed by He and Krishnamurthy (2012), and examine their implications for equilibrium asset prices. Specifically, we assume that financial intermediaries (specialists) possess greater information processing capacity than households; consequently, households optimally choose to delegate their investment decisions to specialists. In addition, both households and specialists face model uncertainty due to their preference for robustness, reflecting ambiguity about the risky asset return. The amount of model uncertainty due to robustness is endogenously determined by the pessimistic drift distortions. When the fundamental volatility increases, so do the drift distortions and the amount of model uncertainty. These distortions produce heterogeneous beliefs because specialists become relatively pessimistic when volatility increases, which tightens the capital constraint and accelerates the onset of a financial crisis.

Keywords: Ambiguity, Information Processing, Asset Pricing, Financial Crisis.

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1 Introduction

In a pair of influential papers, He and Krishnamurthy (2012, 2013; henceforth HK12, HK13, respectively) argue that for many assets it is misleading to characterize prices using household Euler equations. This is because many assets are not held by households. They are held by leveraged financial intermediaries. Although these intermediaries may be investing on behalf of households, the contractual relationships between them are plagued by a variety of frictions. In HK12, asymmetric information produces a moral hazard problem that leads to a capital constraint, requiring the intermediary to maintain a minimum degree of "skin in the game". HK12 and HK13 show that the effective stochastic discount factor becomes much more volatile, and that the nonlinearity induced by the constraint can account for observed state-dependent risk premia.[1]

Although the work of He and Krishnamurthy has been influential, it has not gone unquestioned. The key premise of HK12,13 is that some securities are too "complex" for households to understand, so they delegate investment in these securities to specialists, whose actions cannot be precisely monitored. Cochrane (2017) questions how widespread and insurmountable this complexity problem really is:[2]

> "*Furthermore, if there is such an extreme agency problem, that delegated managers were selling during the buying opportunity of a generation, why do fundamental investors put up with it? Why not invest directly, or find a better contract?...So, in my view, institutional finance and small arbitrages are surely important frosting on the macro-finance cake, needed to get a complete description of financial markets in times of crisis...But are they also the cake?...Or can we understand the big picture of macro-finance without widespread frictions, and leave the frictions to understand the smaller puzzles, much as we conventionally leave the last 10 basis points to market microstructure.*" *by* Cochrane (2017, pp. 963-64)

In this paper, we argue that intermediary asset pricing is indeed "the cake". We operationalize complexity by assuming that agents face limits on their ability to process information, giving rise to so-called Rational Inattention (RI) (Sims 2003). Although there have been many applications of rational inattention to financial markets, these applications either abstract from heterogeneity in information-processing capacity, or assume that any differences are fixed and immutable.[3] In contrast, we argue that trade in information-processing capacity is the raison d'etre of financial

[1] Brunnermeier and Sannikov (2014) develop a macro model with the financial sector. See He and Krishnamurthy (2018) for a recent survey on "intermediary asset pricing"

[2] Perhaps in anticipation of this critique, HK13 confines their analysis to the market for mortgage-backed securities.

[3] Sims (2006) criticizes applications of RI in finance, arguing that in most financial applications information is scarce and costly, so the relevant constraint is on the supply-side, not the demand-side. Kacperczyk et. al. (2018) argue that differences in information-processing capacity contribute to wealth inequality, but do not allow agents to buy and sell this information-processing capacity.

markets, and that when this trade is combined with the monitoring frictions of HK12, the scope of intermediary asset pricing models is greatly expanded. Although most households could manage their portfolios themselves, most *choose* not to do so.[4]

Another key ingredient of our analysis is the assumption that investment is subject to Knightian Uncertainty, or equivalently, ambiguity. Of course, this is not a new idea. Besides Knight (1921), Keynes (1936) argued that financial markets are by their very nature mechanisms for intermediating differences of opinion about ambiguous investment opportunities. However, it took many decades before this idea became operationalized in formal mathematical models. Our particular approach is based on the work of Hansen and Sargent (2008). Agents are assumed to have a (correctly specified) benchmark model of asset returns, which they distrust in a way that cannot be captured by a conventional finite-dimensional Bayesian prior. Rather than commit to a single model/prior, agents entertain a *set* of unstructured alternative models, and then optimize against the worst-case model. Since the worst-case model depends on an agent's own actions, agents view themselves as being immersed in a dynamic zero-sum game. Solutions of this game produce 'robust' portfolio policies. To prevent agents from being unduly pessimistic, in the sense that they attempt to hedge against empirically implausible alternatives, the hypothetical 'evil agent' who selects the worst-case model is required to pay a penalty that is proportional to the relative entropy between the benchmark model and the worst-case model.

Incorporating robustness into intermediary asset pricing models is important for a couple of reasons. First, it delivers a natural source of heterogeneous beliefs. In contrast to Maenhout (2004), we do *not* scale the entropy penalty parameter by the value function. Even with log preferences, a constant entropy penalty produces horizon effects in portfolio choice. In particular, the effective degree of ambiguity aversion depends on an agent's rate of time preference. Agents with a low rate of time preference are endogenously more ambiguity averse, since they care more about the future. Following HK12, we assume specialists are more patient than households, which in our model makes them more ambiguity averse. As a result, their pessimistic drift distortions are greater. This is important because it allows households to survive in the long-run, despite their greater impatience. In contrast, the model in HK12 does not possess a nondegenerate stationary equilibrium, which makes it difficult to evaluate empirically.[5]

The second reason robustness is important is that it tightens the specialist's capital constraint, making crisis episodes more likely. The constraint binds when households want to invest in the risky asset, but specialists do not. We assume throughout that differences in channel capacity

[4] Pagel (2018) also bases portfolio delegation on inattention. However, in her model inattention is not based on information processing limits, but rather on 'information avoidance' (Golman et. al. (2017)), which arises from from loss aversion.

[5] HK13 remedies this defect by introducing nontradeable labor income. However, to keep the analysis tractable, they assume households live for a single-period and have a rather implausible bequest motive. HK12 note that when households are relatively impatient, their model can capture 'liquidation effects', in which asset values fall in response to financial disintermediation.

are sufficiently great that households choose to remain in the contract. This imposes an upper bound on the fee the specialist can charge.[6] Because specialists are relatively ambiguity averse, they want to invest less in the risky asset. As a result, the constraint binds at higher levels of specialist wealth than without ambiguity. We inject cyclicality into this mechanism by assuming that dividend volatility is stochastic, and follows a 2-state jump process. In robust control models, pessimistic drift distortions "hide behind" objective risk. When volatility increases, it becomes more difficult to discriminate among models, and this endogenously makes ambiguity increase as well. Since specialists have a higher degree of ambiguity aversion, their *relative* pessimism increases during volatile periods, thus making it more likely that the economy will hit the capital constraint.

The remainder of the paper is organized as follows. Section 2 presents an ambiguity version of the He-Krishnamurthy model with information processing constraints. Section 3 solves the model, and discusses the theoretical implications of ambiguity for the risk-free rate, the risk premium, and the market price of risk in general equilibrium. Section 4 calibrates the model parameters and examines the model's quantitative implications. Section 4 concludes.

2 An Intermediary Asset Pricing Model with Ambiguity and Information-Processing

2.1 Model Specifications

Our benchmark model in this paper is based on HK12. Specifically, we assume that the dividend of the risky asset is governed by a geometric Brownian motion with stochastic growth rate g_t and constant volatility σ:

$$\frac{dD_t}{D_t} = g_t dt + \sigma dZ_t, \tag{1}$$

where Z_t is a standard Brownian motion. The return of the risky asset is defined as:

$$dR_t = \frac{D_t dt + dP_t}{P_t} = \mu_{R,t} dt + \sigma_{R,t} dZ_t, \tag{2}$$

where P_t is the risky asset price, $\mu_{R,t}$ is the expected return, and $\sigma_{R,t}$ is the volatility of the risky asset. The riskless asset is in zero-net supply, and has an interest rate r_t. We define the risk premium as: $\pi_{R,t} \equiv \mu_{R,t} - r_t$.

The key assumption in this paper is that the expected growth rate g_t is unobservable to the agents, but follows a (known) mean reverting process:

$$dg_t = \rho_g \left(\bar{g} - g_t\right) dt + \sigma_g dZ_t^u \tag{3}$$

[6] In contrast, in HK12, where households have no ability to opt out, intermediation fees actually *increase* during crises.

where Z_t^u captures innovations to the growth rate that are not correlated with the dividend process. Furthermore, we assume that the agents only have finite channel capacity κ when learning the stochastic drift g_t. Specifically, following Peng (2005), Kasa (2006), and Luo (2017), we adopt the noisy-information specification and assume that the investor observes only a noisy signal containing imperfect information about g_t:

$$ds_t = g_t dt + \sigma_s dZ_t^s, \tag{4}$$

where Z_t^s is the noise shock and is a standard Brownian motion. The Kalman-Bucy filter for the above learning problem can be written as:

$$d\hat{g}_t = \rho_g \left(\bar{g} - \hat{g}_t\right) dt + \frac{\Sigma_t}{\sigma} d\hat{Z}_t + \frac{\Sigma_t}{\sigma_s} d\hat{Z}_t^s, \tag{5}$$

and the Riccati equation is:

$$d\Sigma_t = \left[\sigma_g^2 - 2\rho_g \Sigma_t - \Sigma_t^2 \left(\frac{1}{\sigma^2} + \frac{1}{\sigma_s^2}\right)\right] dt \tag{6}$$

where $\hat{g}_t = \mathbb{E}_t \left[g_t | \mathcal{I}_t\right]$ and $\Sigma_t = \mathbb{E}_t \left[\left(g_t - \hat{g}_t\right)^2 | \mathcal{I}_t\right]$ are the conditional mean and variance, respectively, and $d\hat{Z}_t$ and $d\hat{Z}_t^s$ are innovations corresponding to (1) and (3).

To model rational inattention (RI) due to finite capacity, we follow Sims (2003) and impose the following constraint on the investor's information-processing ability:

$$\mathcal{H}\left(g_{t+\Delta t} | \mathcal{I}_t\right) - \mathcal{H}\left(g_{t+\Delta t} | \mathcal{I}_{t+\Delta t}\right) \leq \kappa \Delta t,$$

where κ is the investor's information channel capacity; $\mathcal{H}\left(g_{t+\Delta t} | \mathcal{I}_t\right)$ denotes the entropy of the state prior to observing the new signal at $t + \Delta t$; and $\mathcal{H}\left(g_{t+\Delta t} | \mathcal{I}_{t+\Delta t}\right)$ is the entropy after observing the new signal. κ imposes an upper bound on the signal/noise ratio – that is, the change in the entropy – that can be transmitted in any given period. The Kalman gain is constrained by the agent's channel capacity, which limits the rate of learning.

$$\frac{1}{2} \frac{\Sigma_t^i}{\sigma_s^2} \leq \kappa^i, \ i = \{s, h\}. \tag{7}$$

where the superscripts s and h denote the specialist and the household throughout the paper. Since the constraint will always be binding, we can rewrite the agent-specific Kalman filter as:

$$d\hat{g}_t^i = \rho_g \left(\bar{g} - \hat{g}_t^i\right) dt + \frac{\Sigma_t^i}{\sigma} d\hat{Z}_t + \frac{\Sigma_t^i}{\sigma_s} d\hat{Z}_t^s, \tag{8}$$

where Σ_t^i is governed by:

$$d\Sigma_t^i = \left(\sigma_g^2 - 2\rho_g \Sigma_t^i - \frac{\Sigma_t^{i2}}{\sigma^2} - 2\kappa^i \Sigma_t^i\right) dt. \tag{9}$$

2.2 The Agent's Problem under Full-Information Rational Expectations

We consider an infinite horizon continuous-time Lucas (1978)-type model in which the economy is populated by two types of agents, specialists and households. There are two assets in the economy: one risky asset and one risk-free asset. The risky asset represents complex assets that require some expertise and information processing capacity. We assume the market is incomplete due to limited market participation as Basak and Cuoco (1998), where only specialists who own the intermediaries can invest into the risky asset. Households can purchase channel capacity from specialists and make investments through intermediaries. Households thus face the decision to allocate portfolio between purchasing equity from intermediaries and the riskless short term bond. Figure 1 shows the market structure of the economy where the intermediary sector is indicated in the middle block.

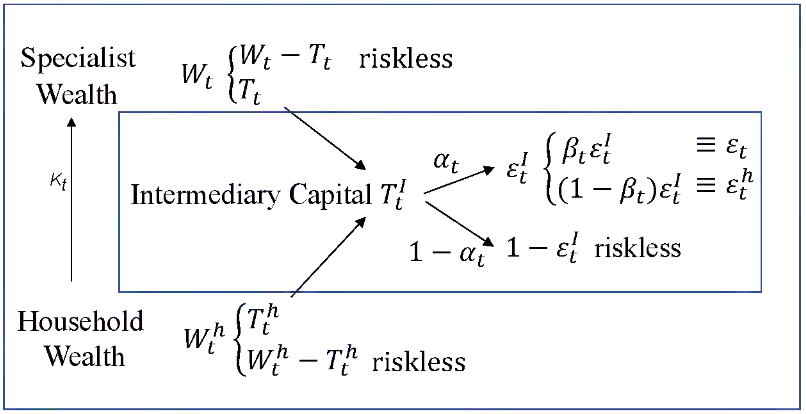

Figure 1: Market Structure and Intermediation Relationship

The total wealth of the specialist is W_t and the household wealth is W_t^h. Households choose to buy the channel capacity from the specialist by paying an intermediation fee K_t. Households allocate T_t^h to purchase intermediary equities and the remaining fraction is used to buy riskless bonds. Intermediaries absorb in sum T_t^I funds from households T_t^h and specialists T_t, allocate a fraction α_t to the risky asset and $1 - \alpha_t$ to the riskless bond. Assuming there is no short-selling constraint for the intermediary, we expect α_t to be larger than 1, i.e., specialists use leverage. In this case, specialists invest more than total intermediary capital into risky equity and borrow $(\alpha_t - 1)\, T_t^I$ from the bond market. The total risky asset position or intermediary's dollar exposure in risky asset is ε_t^I. Through an affine contract developed by HK12, $\beta_t \in [0, 1]$ is the share of returns going to specialists and $1 - \beta_t$ to households. Thus, at time t, the specialist bears a total risk exposure of $\varepsilon_t = \beta_t \varepsilon_t^I$ and the household is offered an exposure of $(1 - \beta t)\varepsilon_t^I$ to excess return.

We assume that the measure of households and specialists are normalized to one. Both are infinitely lived and have log preferences over consumption. Denote households (specialists) consumption rate as C_t^h (C_t). The household's objective is to:

$$\max_{\{C_t,\varepsilon_t^h\}} \mathbb{E}\left[\int_0^\infty e^{-\rho^h t}\ln C_t^h dt\right], \tag{10}$$

while the specialist's objective is to:

$$\max_{\{C_t,\varepsilon_t,\beta_t\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\ln C_t dt\right], \tag{11}$$

where ρ^h and ρ denote the time discount rates for households and specialists, respectively. The dynamic budget constraints are

$$dW_t^h = \varepsilon_t^h(dR_t - r_t dt) - k_t\varepsilon_t^h dt + W_t^h r_t dt - C_t^h dt, \tag{12}$$

and

$$dW_t = \varepsilon_t(dR_t - r_t dt) + \max_{\beta_t\in[\frac{1}{1+m},1]}\left(\frac{1-\beta_t}{\beta_t}\right)k_t\varepsilon_t^* + W_t r_t dt - C_t dt, \tag{13}$$

where k_t is the exposure price that clears the intermediation market and $q_t \equiv \frac{K_t}{W_t} = \left(\frac{1-\beta_t^*}{\beta_t^*}\right)k_t\frac{\pi_{R,t}}{\sigma_{R,t}}$. Households obtain an exposure ε_t^h from the intermediary with an excess return indicated as the first term in the budget constraint, i.e., $\varepsilon_t^h(dR_t - r_t dt)$. Specialists bear a risky exposure ε_t by putting their own wealth into the intermediary. In order to use the intermediation service, households pay an intermediation fee $k_t\varepsilon_t^h \equiv K_t$. The second term denotes the transfer from households to intermediary. The specialist chooses the optimal contract share β_t to maximize the intermediation fee. The third term is the risk-free interest earns by the household (specialist) on his own wealth. The last term is the consumption expense. The optimal exposure supply schedule is $\beta_t^* = \frac{1}{1+m}$ if $k_t > 0$ and $\beta_t^* \in \left[\frac{1}{1+m}, 1\right]$ if $k_t = 0$. The full-information rational expectations solutions for the above two maximization problems are:

$$C_t^{h*} = \rho^h W_t^h \text{ and } \varepsilon_t^{h*} = \frac{\pi_{R,t} - k_t}{\sigma_{R,t}^2}W_t^h,$$

and

$$C_t^* = \rho W_t \text{ and } \varepsilon_t^* = \frac{\pi_{R,t}}{\sigma_{R,t}^2}W_t.$$

Now define the scaled specialist wealth as an aggregate state variable in the economy:

$$x_t \equiv \frac{W_t}{D_t}. \tag{14}$$

It is governed by the following stochastic process:

$$\frac{dx_t}{x_t} = \mu_{x,t}dt + \sigma_{x,t}dZ_t, \tag{15}$$

where $\mu_{x,t}$ and $\sigma_{x,t}$ are the endogenously determined growth rate and volatility.

2.3 Information-Processing and Ambiguity

The novel feature of this paper is that we will explore an intermediation relationship between households and specialists when they have heterogeneous information processing capacities and a preference for robustness. Specifically, we assume the household and specialist have channel capacities κ^h and κ, where $\kappa^h \leq \kappa$. The corresponding signal/noise ratios of households and specialists are Σ_t^h and Σ_t, respectively. Furthermore, assume agents in our economy do not know the true model governing the evolution of the economy, and incorporate model uncertainty due to robustness into their decision problems. Following Hansen and Sargent (2007), we assume that when agents face model misspecifications, they take Equation (12) as the approximating model which is generated by the probability measure $\mathbb{P}$. Assume the probability distribution in the distorted problem $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$. All the random variables and expectation operators for the robust problem below are defined on $\mathbb{Q}$ and $\mathcal{F}_t$ measurable. As argued in Anderson, Hansen and Sargent (2003) – henceforth AHS – the agents believe the approximating model is only a useful benchmark. However, they are concerned about the possibility that the approximating model is misspecified. In order to incorporate doubts about model specification, the agents conceive a class of models surrounding the approximating model, and make optimal decisions based on the range of possible models.

The distorted model is

$$\frac{dD_t^i}{D_t^i} = \left(g_t^i + \sigma \nu_t^i\right) dt + \sigma d\tilde{Z}_t, \tag{16}$$

where $d\tilde{Z}_t = dZ_t - \nu_t^i dt$. An endogenous perturbation ν_t^i is introduced to parameterize the change of measure from $\mathbb{P}$ to $\mathbb{Q}$. Under the distorted probability measure, the household problem becomes

$$V\left(\hat{g}_t^h, \Sigma_t^h, W_t^h; Y_t^h\right) = \sup_{\{C_t^h, \varepsilon_t^h\}} \inf_{\nu_t^h} \mathbb{E} \int_0^\infty e^{-\rho^h t} \left[\ln C_t^h + \frac{1}{2\theta^h}\left(\nu_t^h\right)^2\right] dt$$

subject to (8), (9), and

$$dW_t^h = \left[\varepsilon_t^h (\pi_{R,t} - k_t) + r_t W_t^h - C_t^h\right] dt + \sigma_{W,t}^h \left(\nu_t^h dt + dZ_t\right), \tag{17}$$

where $\sigma_{W,t}^h \equiv \sigma_{R,t}\varepsilon_t^h$. The household then solves the following HJB equation:

$$\rho^h V = \sup_{\{C_t^h, \varepsilon_t^h\}} \inf_{\nu_t^h} \left[\ln C_t^h + \mathcal{D}V + \nu_t^h \sigma_{W,t}^h V_w + \frac{1}{2\theta^h}\left(\nu_t^h\right)^2\right]$$

where

$$\mathcal{D}V(\hat{g}_t^h, \Sigma_t^h, W_t^h; Y_t^h) = V_w\left[\varepsilon_t^h(\pi_{R,t} - k_t) + r_t W_t^h - C_t^h\right] + \frac{1}{2}V_{ww}(\varepsilon_t^h)^2\sigma_{R,t}^2$$
$$+ V_{wg}\frac{\sigma_{R,t}}{\sigma}\varepsilon_t^h\Sigma_t^h + V_g\rho_g\left(\bar{g} - \hat{g}_t^h\right) + \frac{1}{2}V_{gg}\left[\frac{\left(\Sigma_t^h\right)^2}{\sigma^2} + 2\kappa^h\Sigma_t^h\right]$$
$$+ V_\Sigma\left[\sigma_g^2 - 2\rho_g\Sigma_t^h - \frac{\left(\Sigma_t^h\right)^2}{\sigma^2} - 2\kappa^h\Sigma_t^h\right] + \mu_{Y,t}^h.$$

Solving first the infimization part yields $\nu_t^{h*} = -\theta^h V_w \sigma_{W,t}^h$. Substituting for ν_t^h in the HJB equation gives:

$$\rho^h V = \sup_{\{C_t^h, \varepsilon_t^h\}}\left[\ln C_t^h + \mathcal{D}V - \frac{\theta^h}{2}(\sigma_{W,t}^h V_w)^2\right]. \tag{18}$$

The following proposition summarizes the main results from the above model with:

Proposition 1 *Under robustness, the household's optimal consumption rule is:*

$$C_t^{h*} = \rho^h W_t^h, \tag{19}$$

and the optimal risk exposure is:

$$\varepsilon_t^{h*} = \frac{\pi_{R,t} - k_t}{\gamma^h \sigma_{R,t}^2} W_t^h, \tag{20}$$

and the distortion of the household is:

$$\nu_t^{h*} = -\frac{\theta^h}{\rho^h}\frac{\varepsilon_t^h \sigma_{R,t}}{W_t^h}, \tag{21}$$

where $\gamma^h = 1 + \theta^h/\rho^h$ is the effective coefficient of risk aversion for the household and θ^h governs the degree of robustness. Household's value function takes the addictively separable form

$$V\left(\hat{g}_t^h, \Sigma_t^h, W_t^h; Y_t^h\right) = \frac{1}{\rho^h}\ln W_t^h + F^h\left(\hat{g}_t^h, \Sigma_t^h\right) + Y_t^h,$$

where $F^h\left(\hat{g}_t^h, \Sigma_t^h\right)$ and Y_t^h is a function of aggregate state x_t and solves the following PDE system:

$$\rho^i F^i = F_g^i\rho_g\left(\bar{g} - \hat{g}_t^i\right) + \frac{1}{2}F_{gg}^i\left(\frac{\Sigma_t^{i2}}{\sigma^2} + 2\kappa^i\Sigma_t^i\right) + F_\Sigma^i\left(\sigma_g^2 - 2\rho_g\Sigma_t^i - \frac{\Sigma_t^{i2}}{\sigma^2} - 2\kappa^i\Sigma_t^i\right),\ i = h \tag{22}$$

$$\ln\rho^h - 1 + Y_t^{h\prime}\mu_{x,t}x_t + \frac{1}{2}Y_t^{h\prime\prime}\sigma_{x,t}^2 x_t^2 = \rho^h Y_t^h - \frac{(\pi_{R,t} - k_t)^2}{2\rho^h\gamma^h\sigma_{R,t}^2} - \frac{r_t}{\rho^h}. \tag{23}$$

Proof. See Appendix 5.1 for the derivations. ■

We now turn to the specialist's problem. The specialist problem can be written as:

$$J\left(\hat{g}_t, \Sigma_t, W_t; Y_t\right) = \sup_{\{C_t, \varepsilon_t\}}\inf_{\nu_t}\mathbb{E}\int_0^\infty e^{-\rho t}\left[\ln C_t + \frac{1}{2\theta}\left(\nu_t\right)^2\right]dt$$

subject to (8), (9), and

$$dW_t = [\varepsilon_t \pi_{R,t} + (q_t + r_t)W_t - C_t]\, dt + \sigma_{W,t} (\nu_t dt + dZ_t) \tag{24}$$

where $\sigma_{W,t} = \sigma_{R,t}\varepsilon_t$. The specialist solves the following HJB equation:

$$\rho J = \sup_{\{C_t,\varepsilon_t\}} \inf_{\nu_t} \left[\ln C_t + \mathcal{D}J + \nu_t \sigma_{W,t} J_w + \frac{1}{2\theta}\nu_t^2 \right],$$

where

$$\mathcal{D}J(\hat{g}_t^s, \Sigma_t^s, W_t; Y_t) = J_w [\varepsilon_t \pi_{R,t} + (q_t + r_t)W_t - C_t] + \frac{1}{2} J_{ww} {\varepsilon_t}^2 \sigma_{R,t}^2 + J_{wg} \frac{\sigma_{R,t}}{\sigma} \varepsilon_t \Sigma_t^s + J_g \rho_g (\bar{g} - \hat{g}_t^s)$$
$$+ \frac{1}{2} J_{gg} \left(\frac{\Sigma_t^{s2}}{\sigma^2} + 2\kappa \Sigma_t^s \right) + J_\Sigma \left(\sigma_g^2 - 2\rho_g \Sigma_t^s - \frac{\Sigma_t^{s2}}{\sigma^2} - 2\kappa \Sigma_t^s \right) + \mu_{Y,t}$$

Solving first the infimization part yields $\nu_t^* = -\theta J_w \sigma_{W,t}$. Substituting for ν_t in the HJB equation gives:

$$\rho J = \sup_{\{C_t,\varepsilon_t\}} \left[\ln C_t + \mathcal{D}J - \frac{\theta}{2} (\sigma_{W,t} J_w)^2 \right] \tag{25}$$

The following proposition summarizes the main results:

Proposition 2 *Under robustness, the specialist's optimal consumption rule is:*

$$C_t^* = \rho W_t, \tag{26}$$

the optimal risk exposure is:

$$\varepsilon_t^* = \frac{\pi_{R,t}}{\gamma \sigma_{R,t}^2} W_t, \tag{27}$$

and the distortion of the specialist is:

$$\nu_t^* = -\frac{\theta}{\rho} \frac{\varepsilon_t \sigma_{R,t}}{W_t}, \tag{28}$$

where $\gamma = 1 + \theta/\rho$ *is the effective risk aversion for the specialist and* θ *governs the degree of robustness. The specialist's value function takes the addictively separable form*

$$J(\hat{g}_t^s, \Sigma_t^s, W_t; Y_t) = \frac{1}{\rho} \ln W_t + F(\hat{g}_t, \Sigma_t^s) + Y_t,$$

where $F(\hat{g}_t^s, \Sigma_t^s)$ *and* Y_t *is a function of aggregate state* x_t *and solve the PDE system (22) (where* $i = s$*) and*

$$\ln \rho - 1 + Y_t' \mu_{x,t} x_t + \frac{1}{2} Y_t'' \sigma_{x,t}^2 x_t^2 = \rho Y_t - \frac{q_t + r_t}{\rho} - \frac{\pi_{R,t}^2}{2\rho\gamma\sigma_{R,t}^2}. \tag{29}$$

Proof. See Appendix 5.1 for the derivations. ■

2.4 Steady State Solution

In the steady state in which $d\Sigma_t^i = 0$, we have:

$$\Sigma^{i2} + 2\sigma^2\left(\kappa^i + \rho_g\right)\Sigma^i - \left(\sigma_g\sigma\right)^2 = 0,$$

which implies

$$\Sigma^i = \sigma^2\left[-\left(\kappa^i + \rho_g\right) + \sqrt{\left(\kappa^i + \rho_g\right)^2 + \left(\sigma_g/\sigma\right)^2}\right]. \tag{30}$$

It is straightforward to show that

$$\frac{d\Sigma^i}{d\kappa^i} = \sigma^2\left[-1 + \frac{\kappa^i + \rho_g}{\kappa^i + \rho_g + \left(\Sigma^i/\sigma^2\right)}\right] < 0, \tag{31}$$

that is, a higher channel capacity reduces the steady state conditional variance. In the steady state, (22) reduces to:

$$\rho^i F^i = F_g^i \rho_g\left(\bar{g} - \hat{g}\right) + \frac{1}{2}F_{gg}^i\hat{\sigma}^{i2},$$

where $\hat{\sigma}^{i2} = \sigma_g^2 - 2\rho_g\Sigma^i$ and $\hat{g}$ is the steady state expected value of posterior mean dividend growth. Following the method proposed in Chen and Kohn (2011), the solution to the above PDE is of the form:

$$F^i\left(\hat{g}\right) = C_1 H_{-v^i}\left(\frac{\bar{g} - \hat{g}}{\hat{\sigma}^i/\sqrt{2\rho_g}}\right) + C_2 H_{-v^i}\left(\frac{\hat{g} - \bar{g}}{\hat{\sigma}^i/\sqrt{2\rho_g}}\right),$$

where $v^i = \rho^i/\rho_g$ and H_{-v^i} has the explicit form (see Appendix 5.2). We show that: (1) H is decreasing in ρ^i and (2) H is increasing in σ^i. Since $\rho^h > \rho^s$ and $\kappa^h < \kappa^s$, we have $\Sigma^h > \Sigma^s$ and $\hat{\sigma}^h < \hat{\sigma}^s$, which imply that

$$F^h\left(\hat{g}\right) < F^s\left(\hat{g}\right).$$

That is, the specialist with higher channel capacity has higher steady state welfare than the household. We then have the following participation constraint for the household:

$$\bar{K} \equiv F^s\left(\hat{g}\right) - F^h\left(\hat{g}\right) > 0.$$

That is, whenever intermediary charges a fee lower than $\bar{K}$, the household would like to use the higher information channel capacity.

In the next section we will show that the equilibrium exposure in the constrained region is $\varepsilon_t^* = \frac{1}{1+m}P_t$. Hence, the steady state exposure price is

$$\bar{k} = (1+m)\frac{\bar{K}}{\bar{P}}. \tag{32}$$

From equation (54), the equilibrium per exposure price is a decreasing function of x_t: $x_t \in \left[\frac{1}{m\rho^h + \rho}, x_c\right]$. The maximum exposure price the household needs to pay is $k_{max} = \sigma^2\left(\gamma - \gamma^h\right)$.

In order to motivate the household to participate in the intermediary market, we assume the following participation constraint holds:

Proposition 3 *The household's participation constraint satisfies*

$$\bar{k} \geq k_{max}. \tag{33}$$

Ex post the household will purchase channel capacity from the specialist with higher capacity and delegate the portfolio decisions to the intermediary. The specialist becomes the only one who learn from the unobservable dividend growth. Afterwards, we drop the superscript in the filtering problem and denote the posterior mean and variance as $\hat{g}_t$ and Σ_t.

3 Theoretical Implications

3.1 Constrained and Unconstrained Region

The specialist's exposure supply is a step function:

$$\begin{cases} \frac{1-\beta_t^*}{\beta_t^*}\varepsilon_t^* \in [0, m\varepsilon_t^*], \text{ for any } \beta_t^* \in [\frac{1}{1+m}, 1] & \text{if } k_t = 0, \\ m\varepsilon_t^* \text{ with } \beta_t^* = \frac{1}{1+m} & \text{if } 0 < k_t < k_{max}. \end{cases},$$

with $\varepsilon_t^* = \frac{\pi_{R,t}}{\gamma\sigma_{R,t}^2}W_t$, and k_t denotes the per-unit exposure price, which is the only determinant for optimal contract β_t^*. In contrast, the household's exposure demand is $\varepsilon_t^h = \frac{\pi_{R,t}-k_t}{\gamma^h\sigma_{R,t}^2}W_t^h$, which follows the risk sharing constraint $\varepsilon_t^h \leq m\varepsilon_t^*$. From Figures 2 and 3, it is clear that both the exposure supply and demand functions are influenced by the robust parameters θ and θ^h.

3.2 Market Equilibrium

Here we provide a detailed definition of market equilibrium in our model economy:

Definition 4 *An equilibrium for the economy is a set of progressively, measurable price processes $\{P_t, r_t, R_t\}$ and $\{k_t\}$, households' decisions $\{C_t^{h*}, \varepsilon_t^{h*}\}$, and specialists' decisions $\{C_t^*, \varepsilon_t^*, \beta_t^*\}$ such that:*

1. *Given the processes, decisions optimally solve (10) and (11).*

2. *The intermediation market reaches equilibrium with risk exposure clearing condition,*

$$\varepsilon_t^{h*} = \frac{1-\beta_t^*}{\beta_t^*}\varepsilon_t^*. \tag{34}$$

3. *The stock market clears:*

$$\varepsilon_t^* + \varepsilon_t^{h*} = P_t. \tag{35}$$

4. The goods market clears:

$$C_t^* + C_t^{h*} = D_t. \tag{36}$$

5. Transversality conditions satisfy:

$$\lim_{t\to\infty} \mathbb{E}\left[\exp\left(-\rho^h t\right) V(W_t^h, t)\right] = 0 \textit{ and } \lim_{t\to\infty} \mathbb{E}\left[\exp\left(-\rho t\right) J(W_t, t)\right] = 0. \tag{37}$$

In the constrained region (see Figure 2), the exposure supply is less than demand, $k_t \geq 0$. The incentive-compatibility constraint is binding ($\beta_t^* = \frac{1}{1+m}$) and the equity capital constraint is binding (i.e., $\varepsilon_t^h = m\varepsilon_t$), such that:

$$\frac{\pi_{R,t} - k_t}{\gamma^h \sigma_{R,t}^2} W_t^h = m \frac{\pi_{R,t}}{\gamma \sigma_{R,t}^2} W_t \text{ or } W_t^h \geq \tilde{m} W_t, \tag{38}$$

where $\tilde{m} \equiv \frac{\gamma^h}{\gamma} m$. In equilibrium, the specialist earns a rent for scarce intermediary service. When k_t increases, ε_t^{h*} decreases, hence exposure demand drops. Households would not put all their wealth into the intermediary, $T_t^h = \tilde{m} W_t \leq W_t^h$, thus induces the financial constraint for the intermediation.

In the unconstrained region (see Figure 3), the exposure supply exceeds the demand. There exists an abundance of intermediary supply so that specialists must set the intermediation fee to zero to attract all the exposure demand from the household. In this case, the per-unit exposure price k_t is zero. The incentive-compatibility constraint is slack ($\beta_t^* > \frac{1}{1+m}$), as well as the risk-sharing constraint is slack (i.e., $\varepsilon_t^h|_{k_t=0} < m\varepsilon_t$), such that

$$\frac{\pi_{R,t}}{\gamma^h \sigma_{R,t}^2} W_t^h < m \frac{\pi_{R,t}}{\gamma \sigma_{R,t}^2} W_t,$$

where we assume risk premium $\pi_{R,t}$ is positive. The risk-sharing constraint is translated into the equity capital constraint:

$$W_t^h < \tilde{m} W_t.$$

Intermediary earns higher exposure, so that households put all the wealth into the intermediation, $T_t^h = W_t^h$.

Robustness concerns change the binding conditions for the economy through the effective financial constraint $\tilde{m}$. When $\gamma^h = \gamma$ ($\theta^h/\theta = \rho^h/\rho$), ambiguity parameters don't change the financial constraint, i.e., $\tilde{m} = m$. However, as in HK12 and HK13, assuming the specialist is more patient than the household, $\rho^h > \rho$, the existence of ambiguity causes the effective financial constraint to be scaled by relative ambiguity aversion. Thus, specialists become more constrained even if they face the same level of ambiguity as the households, i.e., when $\theta^h = \theta$, $\tilde{m} < m$. Later we will see in addition to the wealth distribution among households and specialists, robustness parameters influence the equity capital binding conditions as well as the conditions for whether the economy is in the constrained region or not. This is similar to the role of the financial constraint. We incorporate ambiguity into the financial constraint to make it "endogenous" by the agents' ambiguity. The effective financial constraint can also be treated as an "adjusted" financial constraint with the adjustment of $\frac{\gamma^h}{\gamma} = \frac{1+\theta^h/\rho^h}{1+\theta/\rho}$, which is the relative ratio of effective ambiguity aversion between the household and specialist. Using (**??**), we have

$$\frac{d\tilde{m}}{d\theta} = -\frac{\gamma^h}{\rho\gamma^2}m < 0 \text{ and } \frac{d\tilde{m}}{d\theta^h} = \frac{1}{\rho^h\gamma}m > 0.$$

From these results, we can see that θ and θ^h play opposite roles in determining $\tilde{m}$ which captures the inverse of agency friction.

3.3 Asset Pricing Implications

The Equity Capital Constraint and the Price-Dividend Ratio. Since bonds are in zero net supply, the asset market clears when aggregate wealth equals the market value of the risky asset,

$$W_t^h + W_t = P_t. \tag{39}$$

In equilibrium, from the goods market clearing condition (36) and the optimal consumption rules of households and specialists, we have

$$\rho W_t + \rho^h W_t^h = D_t. \tag{40}$$

The equilibrium price to dividend ratio can thus be written as:

$$\frac{P_t}{D_t} = \frac{1}{\rho^h} + (1 - \frac{\rho}{\rho^h})x_t = \frac{1 + \Delta\rho x_t}{\rho^h}, \tag{41}$$

where $\Delta\rho \equiv \rho^h - \rho$. Notice that robustness concerns do not have a first order effect on the price to dividend ratio. Robustness only indirectly influences it through a wealth effect. When the risk sharing constraint just starts to bind, the threshold level of the state x^c can be written as:

$$\varepsilon_t^h = m\varepsilon_t,$$

which means that $\frac{P_t - W_t}{\gamma^h} = \frac{mW_t}{\gamma}$. Together with the equilibrium price/dividend ratio above yields:

$$x^c = \frac{1}{\tilde{m}\rho^h + \rho}. \tag{42}$$

When $x_t \leq x^c$, the economy is within the constrained region; otherwise, when $x_t > x^c$, the economy is unconstrained. Agents are ambiguous, thus both the robust concerns from households and specialists influence the critical level of x^c through the effective financial constraint $\tilde{m}$.

Specialist's Portfolio Share. The specialist makes a portfolio choice to invest a share α_t of the total equity $T_t^I = W_t + T_t^h$ into the risky asset and the rest into the riskless bond. Thus, the total exposure is:

$$\varepsilon_t^I = \alpha_t T_t^I$$

which yields the following implementation constraint:

$$\varepsilon_t^* + \varepsilon_t^{h*} = \alpha_t (W_t + T_t^h). \tag{43}$$

This implementation constraint requires the specialist to choose α_t to reach the optimal risk exposure ε_t^*. Households obtain the desired exposure ε_t^{h*} by choosing how much wealth T_t^h to contribute to the intermediation.

Proposition 5 *In unconstrained region, the share of the return is*

$$\beta_t^U = \frac{1}{1 + \frac{\gamma}{\gamma^h}\left(\frac{1}{\rho^h x_t} - \frac{\rho}{\rho^h}\right)}. \tag{44}$$

In constrast, in constrained region,

$$\beta_t = \frac{1}{1+m}. \tag{45}$$

Proof: In the unconstrained case, per-unit exposure price is zero. Recall that the share of return contract $\beta_t \equiv \varepsilon_t^* / \varepsilon_t^I$. Since the robust concern distorts the specialist's desired risk exposure ε_t^*, the choice of share contract turns into

$$\beta_t^U = \frac{W_t}{W_t + \frac{\gamma}{\gamma^h} W_t^h} \text{ and } k_t = 0.$$

Now the specialist and household no longer hold the equity claims according to their wealth contributions as the benchmark case, but with a distortion term $\frac{\gamma}{\gamma^h}$ which equals the inverse of distortion on the financial constraint. Note that although agency friction m doesn't enter β_t^U in unconstrained region, both robustness parameters distort the contract share alternatively. Replacing W_t^h with asset market clearing condition (39) yields:

$$\beta_t^U = \frac{1}{1 + \frac{\gamma}{\gamma^h}\left(\frac{1}{\rho^h x_t} - \frac{\rho}{\rho^h}\right)}.$$

By the imposed assumption that $0 < \beta_t^U \leq 1$, x_t should be limited within $(0, 1/\rho]$. Later we will show that in order for the risk-free rate to be valid whenever robustness exists, $x_t \neq 1/\rho$. From now on, we assume

$$x_t \in \begin{cases} (0, 1/\rho] & \text{for } \theta = \theta^h = 0 \\ (0, 1/\rho) & \text{others.} \end{cases}$$

In the constrained region, the share of return is determined by the incentive constraint of specialist. In order to prevent the specialist from shirking, households need to pay a positive intermediation fee and exposure price to the intermediary, thus

$$\beta_t = \frac{1}{1+m} \text{ and } k_t > 0.$$

Proposition 6 *In the unconstrained region, the desired risk exposure and optimal portfolio choice are*

$$\varepsilon_t^{U*} = \frac{1}{1 + \frac{\gamma}{\gamma^h}\left(\frac{1}{\rho^h x_t} - \frac{\rho}{\rho^h}\right)} P_t \text{ and } \alpha_t^{U*} = 1,$$

respectively. In the constrained region,

$$\varepsilon_t^* = \frac{1}{1+m} P_t \text{ and } \alpha_t^* = \frac{\frac{1}{x_t} + \rho^h - \rho}{(1+\tilde{m})\rho^h}. \tag{46}$$

Proof. In the unconstrained region, $T_t^h = W_t^h$, both households and specialists put all their wealth into the intermediation, such that the total risk exposure equals $\varepsilon_t^I = \alpha_t(W_t + W_t^h)$. The equilibrium conditions (35) and (39) yield $\alpha_t^{U*} = 1$. Given that $\varepsilon_t^* + \varepsilon_t^{h*} = W_t + W_t^h$, the risk exposure for the specialist can be derived as follows:

$$\varepsilon_t^* + \frac{1-\beta_t^U}{\beta_t^U}\varepsilon_t^* = P_t \Longrightarrow \varepsilon_t^{U*} = \frac{1}{1 + \frac{\gamma}{\gamma^h}\left(\frac{1}{\rho^h x_t} - \frac{\rho}{\rho^h}\right)} P_t.$$

In the constrained region, the specialist holds $\beta_t = \frac{1}{1+m}$ share of risk. Hence, the specialist's risk exposure

$$\varepsilon_t^* = \beta_t \varepsilon_t^I = \frac{1}{1+m} P_t.$$

Furthermore, the specialist's portfolio share is

$$\alpha_t = \frac{\varepsilon_t^I}{W_t + T_t^h} = \frac{P_t}{(1+\tilde{m})W_t} \Longrightarrow \alpha_t^* = \frac{\frac{1}{x_t} + \rho^h - \rho}{(1+\tilde{m})\rho^h}.$$

From Figure 4, in the unconstrained region, $\alpha_t = 1$ such that the specialist invests all of the intermediary's equity capital into the risky asset. Once the constraint is binding. $\alpha_t > 1$ means the specialist holds above 100% of the total equity and borrows $(\alpha_t - 1)(W_t + T_t^h)$ riskless bonds.

Risky Asset Volatility.

Proposition 7 *In the unconstrained region,*

$$\sigma_{R,t}^U = \sigma\left(\frac{1}{1+\Delta\rho x_t}\right)\frac{\left(\rho^h\gamma^h - \rho\gamma\right)x_t + \gamma}{\rho\left(\gamma^h - \gamma\right)x_t + \gamma}. \tag{47}$$

In the constrained region,

$$\sigma_{R,t} = \sigma\left(\frac{\rho^h}{1+\Delta\rho x_t}\right)\left(\frac{1+m}{m\rho^h + \rho}\right). \tag{48}$$

Proof. The return volatility can be derived from matching the diffusion terms of equation (1), (2), and (41) that

$$\sigma_{R,t} = \frac{\sigma D_t}{\rho^h P_t - (\rho^h - \rho)\varepsilon_t^*} = \left(\frac{1}{P_t/D_t}\right)\frac{\sigma}{\rho^h - (\rho^h - \rho)\beta_t}. \tag{49}$$

Using Proposition 6 and 5, we have

$$\sigma_{R,t}^U = \frac{\sigma}{\rho^h}\left(\frac{1}{P_t/D_t}\right)\frac{\rho\left(\gamma^h - \gamma\right)x_t + \gamma}{\left(\rho^h\gamma^h - \rho\gamma\right)x_t + \gamma} \text{ and } \sigma_{R,t} = \left(\frac{1}{P_t/D_t}\right)\left(\frac{\sigma}{\rho^h - \frac{\rho^h - \rho}{1+m}}\right).$$

From equation (**??**) and (**??**), price/dividend ratio increases when x_t increases, thus the return volatilities decrease. In the constrained region, as x_t drops, the constraint tightens, thus return volatility rises only through price/dividend ratio. However, in the unconstrained region, decreasing in x_t not only increases $\sigma_{R,t}^U$ through price/dividend ratio from the first term in parentheses of equation (**??**), but also decreases $\sigma_{R,t}^U$ through the second term. Thus, the effect of x_t to $\sigma_{R,t}^U$ is ambiguous in the unconstrained case. When there is no ambiguity, i.e. $\theta = \theta^h = 0$, $\sigma_{R,t}^U = \sigma$ which is independent of x_t.

Lemma 8

$$\frac{d\sigma_{R,t}^U}{d\theta} \leq 0, \ \frac{d\sigma_{R,t}^U}{d\theta^h} \geq 0, \ \frac{d\sigma_{R,t}^U}{d\bar{\theta}} \leq 0; \ \textit{and} \ \frac{d\sigma_{R,t}}{d\theta} = \frac{d\sigma_{R,t}}{d\theta^h} = \frac{d\sigma_{R,t}}{d\bar{\theta}} = 0.$$

where $\bar{\theta}$ denotes homogeneous ambiguity when $\theta = \theta^h$. $\Delta\gamma \equiv \gamma^h - \gamma$ denotes the dispersion in effective ambiguity aversion.

Lemma 8 shows the opposite influence from two agents' ambiguities. From equation (49), the risky asset volatility come from two parts: price to dividend ratio and risk share contract. Price to dividend ratio is not a function of ambiguity under given wealth. In Proposition 5, heterogeneous agents' ambiguities have first order effect on β_t^U in the unconstrained region but no effect in constrained region, thus first order influence the risky asset volatility.

Risk Premium and Financial Constraint. The risk premium could be solved through optimal exposure supply by the specialist (27) and we have the following results.

Proposition 9 *In the unconstrained region,*

$$\pi_{R,t}^U = \frac{\sigma^2\gamma\gamma^h}{(1 + \Delta\rho x_t)}\frac{\left[\left(\rho^h\gamma^h - \rho\gamma\right)x_t + \gamma\right]}{\left[\rho\left(\gamma^h - \gamma\right)x_t + \gamma\right]^2}. \tag{50}$$

In the constrained region,

$$\pi_{R,t} = \frac{\sigma^2\rho^h\gamma}{x_t\left(1 + \Delta\rho x_t\right)}\frac{1+m}{\left(m\rho^h + \rho\right)^2}. \tag{51}$$

Proof. See Appendix 5.3.

Lemma 10

$$\frac{d\pi^U_{R,t}}{d\theta} > 0, \ \frac{d\pi^U_{R,t}}{d\theta^h} \geq 0, \ \frac{d\pi^U_{R,t}}{d\bar{\theta}} \geq 0; \ \textit{and} \ \frac{d\pi_{R,t}}{d\theta} > 0, \ \frac{d\pi_{R,t}}{d\theta^h} = 0, \ \frac{d\pi_{R,t}}{d\bar{\theta}} > 0.$$

Proof. See Appendix 5.4. It is interesting to notice that, θ positively changes the risk premium both in the unconstrained and constrained region, while θ^h also has a positive impact but *only* in the unconstrained region, as shown in Figure 5.8.

Market Price of Risk and Uncertainty. The market price of risk is defined as the Sharpe ratio. Using Proposition 7 and 9 directly gets the following result.

Proposition 11 *In the unconstrained region, the market price of risk is*

$$\frac{\pi^U_{R,t}}{\sigma^U_{R,t}} = \frac{\sigma\gamma\gamma^h}{\rho\left(\gamma^h - \gamma\right)x_t + \gamma}. \tag{52}$$

In the constrained region,

$$\frac{\pi_{R,t}}{\sigma_{R,t}} = \sigma\left(\frac{\gamma}{m\rho^h + \rho}\right)\frac{1}{x_t}. \tag{53}$$

In the constrained region, only the specialist ambiguity degree has first order effect on the Sharpe ratio. This is consistent with the argument in intermediary asset pricing that marginal investors rather than households truly dominate the asset market.

Lemma 12

$$\frac{d\left(\pi^U_{R,t}/\sigma^U_{R,t}\right)}{d\theta} > 0, \ \frac{d\left(\pi^U_{R,t}/\sigma^U_{R,t}\right)}{d\theta^h} \geq 0, \ \frac{d\left(\pi^U_{R,t}/\sigma^U_{R,t}\right)}{d\bar{\theta}} > 0;$$
$$\textit{and} \ \frac{d\left(\pi_{R,t}/\sigma_{R,t}\right)}{d\theta} > 0, \ \frac{d\left(\pi_{R,t}/\sigma_{R,t}\right)}{d\theta^h} = 0, \ \frac{d\left(\pi_{R,t}/\sigma_{R,t}\right)}{d\bar{\theta}} > 0.$$

Proof. See Appendix 5.4.

Exposure Price and Intermediation Fee.

Proposition 13 *In the unconstrained region, the per-unit exposure price* $k^U_t = 0$. *In the constrained region,*

$$k_t = \frac{\sigma^2(1+m)}{\left(m\rho^h + \rho\right)^2}\left(\gamma - \frac{\rho^h\gamma^h m x_t}{1 - \rho x_t}\right)\frac{\rho^h}{\left(1 + \Delta\rho x_t\right)x_t} \tag{54}$$

Proof. See Appendix **??**.

Lemma 14

$$\frac{dk_t}{d\theta} > 0, \ \frac{dk_t}{d\theta^h} < 0, \ \frac{dk_t}{d\bar{\theta}} > 0.$$

Proof. See Appendix 5.4.

The Interest Rate.

Proposition 15 *In the unconstrained region, the interest rate is*

$$r_t^U = \rho^h + \hat{g}_t - \rho\Delta\rho x_t + \sigma^2 \frac{\gamma\gamma^h - \left(\gamma^h + \gamma\right)\left[\rho x_t\left(\gamma^h - \gamma\right) + \gamma\right]}{\left[\rho\left(\gamma^h - \gamma\right)x_t + \gamma\right]^2}. \tag{55}$$

In the constrained region,

$$r_t = \rho^h + \hat{g}_t - \rho\Delta\rho x_t - \sigma^2 \frac{\left(1 - \rho x_t\right)\left[\rho\left(1 + \gamma m\right) + \rho^h m^2 \gamma^h\right] + \rho^h m^2\left(\rho^h x_t - \gamma^h\right)}{\left(1 - \rho x_t\right)\left(\rho + m\rho^h\right)^2 x_t}. \tag{56}$$

Proof. See Appendix 5.4.

Lemma 16

$$\begin{cases} \frac{dr_t^U}{d\theta} < 0 & \text{if } \gamma^h < \gamma \\ \frac{dr_t^U}{d\theta} \geq 0 & \text{if } \gamma^h \geq \gamma, \end{cases} \quad \begin{cases} \frac{dr_t^U}{d\theta^h} > 0 & \text{if } \gamma^h < \gamma \\ \frac{dr_t^U}{d\theta^h} \leq 0 & \text{if } \gamma^h \geq \gamma, \end{cases} \quad \frac{dr_t^U}{d\bar{\theta}} \leq 0.$$

$$\text{and } \frac{dr_t}{d\theta} < 0, \ \frac{dr_t}{d\theta^h} > 0, \ \frac{dr_t}{d\bar{\theta}} < 0.$$

Proof: See Appendix 5.4.

3.4 The Stationary Wealth Distribution

Proposition 17 *The specialist scaled wealth process follows,*

$$\frac{dx_t}{x_t} = \mu_{x,t}dt + \sigma_{x,t}dZ_t.$$

In unconstrained region,

$$\mu_{x,t}^U = \Delta\rho\left(1 - \rho x_t\right) + \sigma^2 + \sigma^2 \frac{A_0^U x_t^2 + A_1^U x_t + A_2^U}{\left(1 - \rho x_t\right)\left[\rho\left(\gamma^h - \gamma\right)x_t + \gamma\right]^2}, \tag{57}$$

$$\sigma_{x,t}^U = \sigma\left[\frac{\gamma^h}{\rho\left(\gamma^h - \gamma\right)x_t + \gamma} - 1\right], \tag{58}$$

where $A_0^U = \rho^2\left(\gamma^h - \gamma\right)\left(2\gamma^h + \gamma\right)$, $A_1^U = \rho\left(2\gamma^2 - 3\gamma^{h2} + 2\gamma\gamma^h\right)$, *and* $A_2^U = \gamma^{h2} - \gamma^2 - \gamma\gamma^h$. *In constrained region,*

$$\mu_{x,t} = \Delta\rho\left(1 - \rho x_t\right) + \sigma^2 + \sigma^2 \frac{A_0 x_t^2 + A_1 x_t + A_2}{\left(m\rho^h + \rho\right)^2\left(1 - \rho x_t\right)x_t^2}, \tag{59}$$

$$\sigma_{x,t} = \sigma\left[\frac{1}{\left(m\rho^h + \rho\right)x_t} - 1\right], \tag{60}$$

where $A_0 = \rho^h\left(\rho\gamma^h - \rho^h\right)m^2 + \rho\left(\rho\gamma + \rho^h\right)m + 2\rho^2$, $A_1 = -\rho^h\gamma^h m^2 - \left(\rho^h + 2\rho\gamma\right)m - 3\rho$, *and* $A_2 = \gamma m + 1$

Proof. See Appendix 5.5.

Proposition 18 *The theoretical density of specialist scaled wealth satisfies*

$$f(x) = \frac{C_1}{\sigma_c^2(x)} \exp\left(\int_0^{x^c} \frac{2\mu_c(s)}{\sigma_c^2(s)} ds\right) + \frac{C_2}{\sigma_u^2(x)} \exp\left(\int_{x^c}^{\frac{1}{\rho}} \frac{2\mu_u(s)}{\sigma_u^2(s)} ds\right), \tag{61}$$

where C_1 and C_2 satisfy:

1. $\int f(x)dx = 1$;
2. *continuous at* x^c.

See Appendix 5.5 for the derivation of the pdf. Define scaled household wealth $x_t^h \equiv \frac{W_t^h}{D_t}$,

$$x_t^h = \frac{P_t - W_t}{D_t} = \frac{1 - \rho x_t}{\rho^h}.$$

The wealth evolution for households is

$$\frac{dW_t^h}{W_t^h} - \frac{dW_t}{W_t} = -\frac{1}{1 - \rho x_t}\left(\frac{dx_t}{x_t} + \frac{\sigma}{\gamma}\frac{\pi_{R,t}}{\sigma_{R,t}} dt - \sigma^2 dt\right).$$

In our model, the effective ambiguity aversion depends on the agent's time preference. Specialists with a low rate of time preference are endogenously more ambiguity averse, since they care more about the future. As a result, their pessimistic drift distortions are greater. This is important because it allows households to survive in the long-run, despite their greater impatience. Our model produces a stationary wealth distribution as in Figure 10, with 1.64% probability the constraint binds annually theoretically.[7]

4 Quantitative Results

4.1 Calibrating Ambiguity Parameter Using Detection-error Probabilities

Obtain the relative entropies $|\nu_t^h|$ and $|\nu_t|$ from household and specialist optimal robust problem, we finally get

$$g^h(x_t) \equiv |\nu_t^h| = \sigma\gamma\left(\gamma^h - 1\right)\left[\frac{m\rho^h}{\gamma(m\rho^h + \rho)(1 - \rho x_t)}\mathbf{1}_{x_t \in (x_{min}, x^c]} + \frac{1}{\rho(\gamma^h - \gamma)x_t + \gamma}\mathbf{1}_{x_t \in (x^c, \frac{1}{\rho}]}\right]$$

$$g(x_t) \equiv |\nu_t| = \sigma\gamma^h(\gamma - 1)\left[\frac{1}{\gamma^h(m\rho^h + \rho)x_t}\mathbf{1}_{x_t \in (x_{min}, x^c]} + \frac{1}{\rho(\gamma^h - \gamma)x_t + \gamma}\mathbf{1}_{x_t \in (x^c, \frac{1}{\rho}]}\right] \tag{62}$$

where $x_{min} = \frac{1}{m\rho^h + \rho}$ and $\mathbf{1}$ denotes the indicator function. See Appendix 5.7 for the proof. Follow the method of Maenhout (2006), define Radon-Nikodym derivative $\Xi_{1,t}^h \equiv \frac{d\mathbb{Q}^h}{d\mathbb{P}}$ $\left(\Xi_{1,t} \equiv \frac{d\mathbb{Q}}{d\mathbb{P}}\right)$ as

[7] In HK12 case, $\frac{\pi_{R,t}}{\sigma_{R,t}} = \sigma^2$, $\gamma = 1$, then $\frac{dW_t^h}{W_t^h} - \frac{dW_t}{W_t} = -\Delta\rho \leq 0$. Households will eventually become extinct and only specialists will survive in the economy.

households (specialists) distorted model $\mathbb{Q}^h$ ($\mathbb{Q}$) with respect to approximating model $\mathbb{P}$. Then log likelihoods for two agents are

$$\xi_{1,t}^h \equiv \log \Xi_{1,t}^h = -\int_0^t g^h(x_s)\, dZ_s - \frac{1}{2}\int_0^t \left\| g^h(x_s) \right\|^2 ds \tag{63}$$

$$\xi_{1,t} \equiv \log \Xi_{1,t} = -\int_0^t g(x_s)\, dZ_s - \frac{1}{2}\int_0^t \left\| g(x_s) \right\|^2 ds. \tag{64}$$

The log of Radon-Nikodym derivative $\Xi_{2,t}^h \equiv \frac{d\mathbb{P}}{d\mathbb{Q}^h}$ $\left(\Xi_{2,t} \equiv \frac{d\mathbb{P}}{d\mathbb{Q}}\right)$ of the approximating model $\mathbb{P}$ with respect to households (specialists) distorted model $\mathbb{Q}^h$ ($\mathbb{Q}$) is

$$\xi_{2,t}^h \equiv \log \Xi_{2,t}^h = \int_0^t g^h(x_s)\, dZ_s + \frac{1}{2}\int_0^t \left\| g^h(x_s) \right\|^2 ds \tag{65}$$

$$\xi_{2,t} \equiv \log \Xi_{2,t} = \int_0^t g(x_s)\, dZ_s + \frac{1}{2}\int_0^t \left\| g(x_s) \right\|^2 ds. \tag{66}$$

When approximating model $\mathbb{P}$ generates the data, q_P measures the probability of the likelihood ratio of making detection errors in selecting model $\mathbb{Q}$. Define

$$q_P^h = \Pr\left(\xi_{1,t}^h > 0 | \mathbb{P}, \mathcal{F}_0\right) \tag{67}$$

$$q_P = \Pr\left(\xi_{1,t} > 0 | \mathbb{P}, \mathcal{F}_0\right). \tag{68}$$

Similarly, when model $\mathbb{Q}$ generates the data,

$$q_Q^h = \Pr\left(\xi_{2,t}^h > 0 | \mathbb{Q}, \mathcal{F}_0\right) \tag{69}$$

$$q_Q = \Pr\left(\xi_{2,t} > 0 | \mathbb{Q}, \mathcal{F}_0\right). \tag{70}$$

Given the equal weight of prior probabilities over model $\mathbb{P}$ and $\mathbb{Q}$, the conditional probability of the detection error for two agents over sample length N are

$$p^h\left(\theta^h; N\right) = \frac{1}{2} q_P^h + \frac{1}{2} q_Q^h \tag{71}$$

$$p(\theta; N) = \frac{1}{2} q_P + \frac{1}{2} q_Q. \tag{72}$$

4.2 Measurements from Simulation

In this section we evaluate the model's ability to match the empirical moments. We calibrate the model at an annual frequency and evaluate its ability to replicate key moments of asset returns and the probability and severity of financial crisis. We simulate 5000 years and 5000 sample paths with quarterly frequency based on calibrated values in Table 1. To match the data from 1970-2017, we report 47 years simulated results in stationary distribution. Numerically we prove that it is long enough to ensure the stationary distribution of state variable x_t. Also, x_t is independent of the initial values.

We use equity premium data from Kenneth French's website. The real risk-free rate is from Robert Shiller's website. The data moments and main simulated results are summarized in Table 2. We did two calibration experiments based on parameter values in Table 1. The benchmark model has the ambiguity aversion parameter values close to zero.[8] This model represents the rational expectation case where two agents have no preference for robustness. The adjacent column documents our main calibrated results when two agents have the same ambiguity aversion $\theta = \theta^h = 0.3$.

We have the following observations. First, without ambiguity, the benchmark model cannot explain the equity premium puzzle. We can get 6.22% equity premium and 55.92% Sharpe ratio in the model compared to 6.93% and 42.86% in the data with calibrated ambiguity parameter. Second, we produce a lower interest rate of 1.31% compared to the benchmark model, even though still higher than the data. That is mainly because logarithm utility excludes the precautionary saving motive. Introducing recursive preference or CARA utility would modify our results. Third, benchmark model does not have any probabilities for the financial constraint to be binding. The model performs well when we consider both agents have ambiguity aversion. Figure5.8 plots the simulated time path for the probability of constraint binding. We see an average 3.09% annually the constraint binds. Fourth, the benchmark model cannot predict a severe financial crisis. With ambiguity, there are 33.76% probability the Sharpe ratio will exceed 72.70% (30% of the mean). The model also predicts a severer crisis, with 4.20% probability the Sharpe ratio exceeds 78.29% (40% of the mean).

Our main story can be summarized in the simulated sample paths in Figure 12 and 13. The red rectangular areas indicate the time when the constraint binds. While specialists have the same ambiguity attitude as households ($\theta = \theta^h$), they are more patient than households ($\rho < \rho^h$, which in our model makes them more effectively ambiguity averse $\gamma > \gamma^h$. With a decrease in specialist scaled wealth, the specialist becomes relatively more pessimistic. This tightens the specialist's capital constraint, making crisis episodes more likely. The constraint binds when households want to invest in the risky asset, but specialists do not. In the graph we see a rising difference in two agents' pessimism. This further reduces the intermediary's net worth and increases the probability of severe crisis. As a result, the risk premium and Sharpe ratio rise and interest rate decreases correspondingly.

[8] In order to ensure non-degenerate distribution, we set ambiguity aversion as 0.0001 instead of 0.

5 Conclusion

This paper has examined the implications of ambiguity and information processing constraints for equilibrium asset prices in an otherwise standard He-Krishnamurthy model of intermediary asset pricing. We found that households with less information-processing capacity optimally choose to delegate their investment decisions to specialists. We also showed that when the fundamental volatility increases, the drift distortions and the amount of model uncertainty increase because specialists become relatively pessimistic, which tightens the capital constraint and accelerates the onset of a financial crisis.

References

[1] Adrian, T., Shin, H.S., 2014. Procyclical Leverage and Value-at-Risk. *The Review of Financial Studies* 27, 373-403

[2] Ahn, D., Choi, S., Gale, D., Kariv, S., 2014. Estimating ambiguity aversion in a portfolio choice experiment. *Quantitative Economics* 5, 195-223

[3] Anderson, E.W., Hansen, L.P., Sargent, T.J., 2003. A Quartet of Semigroups for Model Specification, Robustness, Prices of risk, and Model Detection. *Journal of the European Economic Association* 1, 68-123

[4] Antoniou, C., Harris, R.D.F., Zhang, R., 2015. Ambiguity aversion and stock market participation: An empirical analysis. *Journal of Banking and Finance* 58, 57-70

[5] Backus, D., Ferriere, A., Zin, S., 2015. Risk and ambiguity in models of business cycles. *Journal of Monetary Economics* 69, 42-63

[6] Barberis, N., Greenwood, R., Jin, L., Shleifer, A., 2015. X-CAPM: An extrapolative capital asset pricing model. *Journal of Financial Economics* 115, 1-24

[7] Benhabib, J., Bisin, A., Zhu, S., 2011. The distribution of wealth and fiscal policy in economies with finitely lived agents. *Econometrica* 79, 123-157

[8] Benhabib, J., Bisin, A., Zhu, S., 2015. The wealth distribution in Bewley economies with capital income risk. *Journal of Economic Theory* 159, 489-515

[9] Bhandari, A., Borovička, J., & Ho, P. (2016). Identifying Ambiguity Shocks in Business Cycle Models Using Survey Data (No. w22225). *National Bureau of Economic Research.*

[10] Bloom, N., 2014. Fluctuations in Uncertainty. *Journal of Economic Perspectives* 28, 153-76

[11] Borovička, J., 2016. Survival and long-run dynamics with heterogeneous beliefs under recursive preferences. *Job Market Paper*

[12] Bossaerts, P., Ghirardato, P., Guarnaschelli, S., Zame, W.R., 2010. Ambiguity in asset markets : theory and experiment. *Review of Financial Studies* 23, 1325-1359

[13] Boyarchenko, N., 2012. Ambiguity shifts and the 2007–2008 financial crisis. *Journal of Monetary Economics* 59, 493-507

[14] Brenner, M., Izhakian, Y.Y., 2018. Asset pricing and ambiguity: Empirical evidence. *Journal of Financial Economics, forthcoming*

[15] Brunnermeier, M.K., Pedersen, L.H., 2009. Market Liquidity and Funding Liquidity. *The Review of Financial Studies* 22, 2201-2238

[16] Brunnermeier, M.K., Sannikov, Y., 2016. Macro, Money, and Finance: A Continuous-Time Approach. *Handbook of Macroeconomics*, pp. 1497-1545.

[17] Caballero, R.J., Krishnamurthy, A., 2008. Collective Risk Management in a Flight to Quality Episode. *Journal of Finance* 63, 2195-2230

[18] Caballero, R.J., Simsek, A., 2013. Fire Sales in a Model of Complexity. *Journal of Finance* 68, 2549-2587

[19] Cao, H.H., Wang, T., Zhang, H.H., 2005. Model Uncertainty, Limited Market Participation, and Asset Prices. *The Review of Financial Studies* 18, 1219-1251

[20] Chao, T., 2017. Ambiguity and Investment Decisions: An Empirical Analysis on Mutual Fund Investor Behaviour. *Academic Journal of Economic Studies* 3, 38-46

[21] Chen, Z., Epstein, L., 2002. Ambiguity, Risk, and Asset Returns in Continuous Time. *Econometrica* 70, 1403-1443

[22] Cochrane, J.H., 2017. Macro-Finance *. *Review of Finance* 21, 945-985

[23] Dimmock, S.G., Kouwenberg, R., Mitchell, O.S., Peijnenburg, K., 2016. Ambiguity aversion and household portfolio choice puzzles: Empirical evidence. *Journal of Financial Economics* 119, 559-577

[24] Driouchi, T., Trigeorgis, L., So, R.H.Y., 2015. Option implied ambiguity and its information content: Evidence from the subprime crisis. *Annals of Operations Research*

[25] Easley, D., O'Hara, M., 2009. Ambiguity and Nonparticipation: The Role of Regulation. *The Review of Financial Studies* 22, 1817-1843

[26] Ellsberg, D., 1961. Risk, Ambiguity, and the Savage Axioms. *Quarterly Journal of Economics* 75, 643-669

[27] Epstein, L.G., Wang, T., 1994. Intertemporal Asset Pricing under Knightian Uncertainty. *Econometrica* 62, 283-322

[28] Epstein, L.G., Schneider, M., 2007. Learning under ambiguity. *Review of Economic Studies* 74, 1275-1303

[29] Eisfeldt, A.L., Lustig, H., Zhang, L., 2017. Complex Asset Markets. *NBER working paper*

[30] Gabaix, X., 2009. Power Laws in Economics and Finance. *Annual Review of Economics* 1, 255-294

[31] Gilboa, I., Schmeidler, D., 1989. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics* 18, 141-153

[32] Golman, R., Hagmann, D., Loewenstein, G., 2017. Information Avoidance. *Journal of Economic Literature* 55, 96-135

[33] Hansen, L.P., Sargent, T.J., 2001. Robust Control and Model Uncertainty. *American Economic Review* 91, 60-66

[34] Hansen, L.P., Sargent, T.J., 2005. Robust Estimation and Control under Commitment. *Journal of Economic Theory* 124, 258-301

[35] Hansen, L.P., Sargent, T.J., 2007. Recursive Robust Estimation and Control without Commitment. *Journal of Economic Theory* 136, 1-27

[36] Hansen, L.P., 2008. Robustness. Princeton University Press, Princeton Oxford.

[37] Hansen, L.P., Sargent, T.J., 2010. Fragile Beliefs and the Price of Uncertainty. *Quantitative Economics* 1, 129-162

[38] Hansen, L.P., Sargent, T.J., 2011. Robustness and Ambiguity in Continuous Time. *Journal of Economic Theory* 146, 1195-1223

[39] Hansen, L.P., Sargent, T.J., Tallarini, T.D., 1999. Robust Permanent Income and Pricing. *Review of Economic Studies* 66, 873-907

[40] He, Z., Krishnamurthy, A., 2012. A Model of Capital and Crises. *Review of Economic Studies* 79, 735-777

[41] He, Z., Krishnamurthy, A., 2013. Intermediary Asset Pricing. *American Economic Review* 103, 732-770

[42] He, Z., & Krishnamurthy, A., 2018. Intermediary Asset Pricing and the Financial Crisis. *Annual Review of Financial Economics*, 10(1), 173-197.

[43] He, Z., Kelly, B., Manela, A., 2017. Intermediary asset pricing: New evidence from many asset classes. *Journal of Financial Economics* 126, 1-35

[44] Ju, N., Miao, J., 2012. Ambiguity, Learning, and Asset Returns. *Econometrica* 80, 559-591

[45] Kacperczyk, M., Nosal, J., Stevens, L., 2018. Investor sophistication and capital income inequality. Journal of Monetary Economics

[46] Kasa, K., 2006. Robustness and information processing. *Review of Economic Dynamics* 9, 1-33

[47] Kasa, K., Lei, X., 2018. Risk, uncertainty, and the dynamics of inequality. *Journal of Monetary Economics* 94, 60-78

[48] Kocher, M.G., Lahno, A.M., Trautmann, S.T., 2018. Ambiguity aversion is not universal. *European Economic Review* 101, 268-283

[49] Li, K., 2016. Asset Pricing with a Financial Sector. *Working paper, HKUST.*

[50] Luo, Y., 2016. Robustly Strategic Consumption–Portfolio Rules with Informational Frictions. *Management Science* 63, 4158-4174

[51] Luo, Y., Young, E., 2016. Induced uncertainty, market price of risk, and the dynamics of consumption and wealth. *Journal of Economic Theory* 163, 1

[52] Maenhout, P.J., 2004. Robust Portfolio Rules and Asset Pricing. *The Review of Financial Studies* 17, 951-983

[53] Maenhout, P.J., 2006. Robust Portfolio Rules and Detection-error Probabilities for a Mean-reverting Risk Premium. *Journal of Economic Theory* 128, 136-163

[54] Malmendier, U., Nagel, S., 2011. Depression Babies: Do Macroeconomic Experiences Affect Risk Taking? *Quarterly Journal of Economics* 126, 373

[55] Miao, J., Rivera, A., 2016. Robust Contracts in Continuous Time. *Econometrica* 84, 1405-1440

[56] Pagel, M., 2018. A News-Utility Theory for Inattention and Delegation in Portfolio Choice. *Econometrica* 86, 491-522

[57] Peng, L., 2005. Learning with Information Capacity Constraints. *Journal of Financial and Quantitative Analysis.* 40, 307-329

[58] Sims, C., 2003. Implications of rational inattention. *Journal of Monetary Economics* 50, 665-690

[59] Sims, C.A., 2006. Rational Inattention: Beyond the Linear-Quadratic Case. *The American Economic Review* 96, 158-163

[60] Tallarini, T.D., 2000. Risk-sensitive Real Business Cycles. *Journal of Monetary Economics* 45, 507-532

Appendix

5.1 Solving the Agents' Optimization Problems

Proof of Proposition 1. Optimal household consumption and portfolio rule under robustness are

$$C_t^h = \frac{1}{V_w} \tag{73}$$

and

$$\varepsilon_t^h = \frac{-V_w}{V_{ww} - \theta^h V_w^2} \frac{(\pi_{R,t} - k_t)}{\sigma_{R,t}^2}, \tag{74}$$

respectively. Guess value function takes the form

$$V\left(\hat{g}_t^h, \Sigma_t^h, W_t^h; Y_t^h\right) = \frac{1}{\rho^h} \ln W_t^h + F^h\left(\hat{g}_t^h, \Sigma_t^h\right) + Y_t^h \tag{75}$$

where Y_t^h is a function of aggregate state variable x_t. Now define Y_t^h andY_t as a function of x_t,

$$dY^h(x_t) = \mu_{Y,t}^h dt + \sigma_{Y,t}^h dZ_t$$

$$dY(x_t) = \mu_{Y,t} dt + \sigma_{Y,t} dZ_t.$$

Using Ito's formula,

$$\mu_{Y,t}^h = Y^{h\prime}(x_t)\mu_{x,t}x_t + \frac{1}{2}Y^{h\prime\prime}(x_t)\sigma_{x,t}^2 x_t^2 \tag{76}$$

$$\sigma_{Y,t}^h = Y^{h\prime}(x_t)\sigma_{x,t}x_t \tag{77}$$

$$\mu_{Y,t} = Y\prime(x_t)\mu_{x,t}x_t + \frac{1}{2}Y''(x_t)\sigma_{x,t}^2 x_t^2 \tag{78}$$

$$\sigma_{Y,t} = Y\prime(x_t)\sigma_{x,t}x_t. \tag{79}$$

Under this conjecture, $V_w = \frac{1}{\rho^h W_t^h}$ and $V_{ww} = -\frac{1}{\rho^h (W_t^h)^2}$. Substituting these conjectures into FOCs (73) and (74) gives the results.

5.2 Solving the Steady State Optimal Contract

Following Chen and Kohn (2011), the general solution for steady state F PDE has the form

$$F^i(\hat{g}) = C_1 H_{-v^i}\left(\frac{\bar{g} - \hat{g}}{\sigma^i/\sqrt{2\rho_g}}\right) + C_2 H_{-v^i}\left(\frac{\hat{g} - \bar{g}}{\sigma^i/\sqrt{2\rho_g}}\right)$$

where $v^i = \rho^i/\rho_g$. H_{-v^i} has the series expansion given in the Appendix of Chen and Kohn (2011),

$$H_{-v^i}(w) = \sqrt{\frac{\pi}{2^{v^i}}}\left\{\frac{1}{\Gamma\left(\frac{v^i+1}{2}\right)}\left(1 + \sum_{j=1}^{\infty}\frac{v^i\left(v^i+2\right)\cdots\left(v^i+2j-2\right)}{(2j)!}w^{2j}\right)\right.$$
$$\left. - \frac{\sqrt{2}w}{\Gamma\left(\frac{v^i}{2}\right)}\left(1 + \sum_{j=1}^{\infty}\frac{\left(v^i+1\right)\left(v^i+3\right)\cdots\left(v^i+2j-1\right)}{(2j+1)!}w^{2j}\right)\right\}.$$

5.3 Solving for the Key Moments of Asset Prices

The risk premium Given that

$$\pi_{R,t} = \frac{\gamma\sigma_{R,t}^2\varepsilon_t^*}{W_t} = \frac{\gamma\sigma_{R,t}^2\beta_t P_t}{W_t} = \frac{\gamma\sigma_{R,t}^2\beta_t(P_t/D_t)}{x_t},$$

we have

$$\pi_{R,t}^U = \frac{\gamma\sigma_{R,t}^{U2}\beta_t^U(P_t/D_t)}{x_t} = \frac{\sigma^2\gamma\gamma^h}{(1+\Delta\rho x_t)}\frac{\left[\left(\rho^h\gamma^h - \rho\gamma\right)x_t + \gamma\right]}{\left[\rho\left(\gamma^h-\gamma\right)x_t+\gamma\right]^2}$$

in the unconstrained region. By contrast, in the constrained region,

$$\pi_{R,t} = \frac{\gamma\sigma_{R,t}^2\beta_t(P_t/D_t)}{x_t} = \frac{\sigma^2\gamma\rho^h}{x_t\left(1+\Delta\rho x_t\right)}\frac{1+m}{\left(m\rho^h+\rho\right)^2}.$$

Solving the exposure price and intermediation fee. In the constrained region, $k_t \geq 0$. When household desired exposure demand (20) equals specialist exposure supply (27), i.e., $\varepsilon_t^{h*}(k_t) = m\varepsilon_t^*$, we have

$$\frac{\pi_{R,t}-k_t}{\gamma^h\sigma_{R,t}^2}W_t^h = m\frac{\pi_{R,t}}{\gamma\sigma_{R,t}^2}W_t,$$

which gives

$$\begin{aligned} k_t &= \left(1-\frac{\tilde{m}\rho^h x_t}{1-\rho x_t}\right)\pi_{R,t} = \left(1-\frac{\gamma^h}{\gamma}\frac{m\rho^h x_t}{1-\rho x_t}\right)\pi_{R,t} \\ &= \frac{\sigma^2(1+m)}{\left(m\rho^h+\rho\right)^2}\left(\gamma-\frac{\rho^h\gamma^h m x_t}{1-\rho x_t}\right)\frac{\rho^h}{\left(1+\Delta\rho x_t\right)x_t}. \end{aligned} \tag{80}$$

Solving the risk free rate. From household's Euler equation under distorted model,

$$r_t dt = \rho^h dt + \mathbb{E}_t\left[\frac{dC_t^{h*}}{C_t^{h*}}\right] - \mathrm{var}_t\left[\frac{dC_t^{h*}}{C_t^{h*}}\right].$$

$$\frac{dC_t^{h*}}{C_t^{h*}} = \frac{d\left(\rho^h W_t^h\right)}{\rho^h W_t^h} = \frac{d\left(P_t - W_t\right)}{P_t - W_t}.$$

$$dW_t = \left(\varepsilon_t\pi_{R,t} + (q_t+r_t)W_t - C_t\right)dt + \sigma_{R,t}\varepsilon_t\left(\sigma_{R,t}\varepsilon_t\nu_t dt + dZ_t\right)$$

$$\begin{aligned} \Rightarrow \frac{dW_t}{W_t} &= \left[\frac{1}{\gamma}\left(1-\frac{\gamma-1}{\gamma}\right)\frac{\pi_{R,t}^2}{\sigma_{R,t}^2} + q_t + r_t - \rho\right]dt + \frac{\pi_{R,t}}{\gamma\sigma_{R,t}}dZ_t \\ &= \phi_{W,t}dt + r_t dt + \frac{\pi_{R,t}}{\gamma\sigma_{R,t}}dZ_t \end{aligned} \tag{81}$$

where $\phi_{W,t} = \left(\frac{\pi_{R,t}}{\gamma\sigma_{R,t}}\right)^2 + q_t - \rho$. From equation (39) and (40), we have

$$P_t - W_t = \frac{D_t}{\rho^h} - \frac{\rho}{\rho^h}W_t \Rightarrow d\left(P_t - W_t\right) = \frac{(g_t dt + \sigma dZ_t)D_t - \rho dW_t}{\rho^h},$$

$$\frac{d\left(P_t-W_t\right)}{P_t-W_t}=\frac{(g_t dt+\sigma dZ_t)D_t-\rho dW_t}{D_t-\rho W_t}=\frac{(g_t dt+\sigma dZ_t)-\rho\frac{dW_t}{W_t}x_t}{1-\rho x_t}$$

$$\Rightarrow \mathbb{E}_t\left[\frac{d(P_t-W_t)}{P_t-W_t}\right]=\frac{\hat{g}_t-\rho x_t\left(\phi_{W,t}+r_t\right)}{1-\rho x_t}dt$$

$$\mathrm{var}_t\left[\frac{d(P_t-W_t)}{P_t-W_t}\right]=\left(\frac{\sigma-\frac{\rho x_t}{\gamma}\frac{\pi_{R,t}}{\sigma_{R,t}}}{1-\rho x_t}\right)^2 dt$$

$$r_t=\rho^h+\frac{\hat{g}_t-\rho x_t\left(\phi_{W,t}+r_t\right)}{1-\rho x_t}-\left(\frac{\sigma-\frac{\rho x_t}{\gamma}\frac{\pi_{R,t}}{\sigma_{R,t}}}{1-\rho x_t}\right)^2$$

$$\Rightarrow r_t=\rho^h+\hat{g}_t-\rho\left(\rho^h-\rho\right)x_t-\rho q_t x_t-\frac{\rho x_t\left(\left[\frac{\pi_{R,t}}{\gamma\sigma_{R,t}}\right]^2-2\sigma\frac{\pi_{R,t}}{\gamma\sigma_{R,t}}\right)+\sigma^2}{1-\rho x_t}.$$

Using the expressions for $\pi_{R,t}/\sigma_{R,t}$ and q_t in constrained and unconstrained regions by propositions 11 and 13,

$$r_t^U=\rho^h+\hat{g}_t-\rho\varDelta\rho x_t+\sigma^2\frac{\gamma\gamma^h-\left(\gamma^h+\gamma\right)\left(\rho x_t\left(\gamma^h-\gamma\right)+\gamma\right)}{\left[\rho\left(\gamma^h-\gamma\right)x_t+\gamma\right]^2}.$$

$$r_t=\rho^h+\hat{g}_t-\rho\varDelta\rho x_t-\sigma^2\frac{(1-\rho x_t)\left[\rho\left(1+\gamma m\right)+\rho^h m^2\gamma^h\right]+\rho^h m^2\left(\rho^h x_t-\gamma^h\right)}{\left(1-\rho x_t\right)\left(\rho+m\rho^h\right)^2 x_t}.$$

From equation (**??**),

$$\frac{dW_t^h}{W_t^h}=\frac{d(P_t-W_t)}{P_t-W_t}=\frac{(g_t dt+\sigma dZ_t)-\rho\frac{dW_t}{W_t}x_t}{1-\rho x_t}$$

$$\Leftrightarrow \frac{dW_t^h}{W_t^h}=\left(1-\frac{1}{1-\rho x_t}\right)\frac{dW_t}{W_t}+\frac{1}{1-\rho x_t}\frac{dD_t}{D_t}$$

$$\frac{dW_t^h}{W_t^h}-\frac{dW_t}{W_t}=-\frac{1}{1-\rho x_t}\left(\frac{dx_t}{x_t}+\frac{\sigma}{\gamma}\frac{\pi_{R,t}}{\sigma_{R,t}}dt-\sigma^2 dt\right).$$

5.4 Comparative Analysis

5.4.1 Risk premium.

$$\pi_{R,t}^U=\frac{\sigma^2(1+\theta/\rho)\left(1+\theta^h/\rho^h\right)}{\rho^2 x_t\left[1+(\rho^h-\rho)x_t\right]}\frac{\left[\left(\rho^h-\rho\right)+\left(\theta^h-\theta\right)+(1+\theta/\rho)\frac{1}{x_t}\right]}{\left[(1+\theta^h/\rho^h)-(1+\theta/\rho)+(1+\theta/\rho)\frac{1}{\rho x_t}\right]^2}$$

$$\Rightarrow \frac{d\pi_{R,t}^U}{d\theta}=\frac{2\sigma^2\left(\gamma^h\right)^2\left[\rho^h+\theta^h+\gamma\left(\frac{1}{x_t}-\rho\right)\right]}{\rho^3 x_t\left[1+(\rho^h-\rho)x_t\right]\left[\gamma^h-\gamma+\gamma\frac{1}{\rho x_t}\right]^3}.$$

If $\gamma^h>\gamma\geq 1$,

$$x_t>0>-\frac{\gamma}{\rho\left(\gamma^h-\gamma\right)}\Leftrightarrow \gamma^h-\gamma+\gamma\frac{1}{\rho x_t}>0.$$

If $\gamma > \gamma^h \geq 1$,

$$\frac{\gamma}{\gamma - \gamma^h} > 1 \Leftrightarrow \frac{\gamma}{\rho\left(\gamma - \gamma^h\right)} > \frac{1}{\rho} \geq x_t \Leftrightarrow \gamma^h - \gamma + \gamma \frac{1}{\rho x_t} > 0.$$

If $\gamma^h = \gamma \geq 1$,

$$\gamma^h - \gamma + \gamma \frac{1}{\rho x_t} = \gamma \frac{1}{\rho x_t} > 0.$$

$$\Rightarrow \frac{d\pi_{R,t}^U}{d\theta} = \frac{2\sigma^2 \left(\gamma^h\right)^2 \left[\rho^h + \theta^h + \gamma \left(\frac{1}{x_t} - \rho\right)\right]}{\rho^3 x_t \left[1 + \Delta\rho x_t\right] \left[\gamma^h - \gamma + \gamma \frac{1}{\rho x_t}\right]^3} > 0.$$

$$\pi_{R,t}^U = \frac{1}{P/D} \frac{\sigma^2 \left(1 + \theta/\rho\right) \beta_t^U}{x_t \left[\rho^h - \left(\rho^h - \rho\right) \beta_t^U\right]^2}$$

$$\Rightarrow \frac{d\pi_{R,t}^U}{d\theta^h} = \frac{1}{P/D} \frac{\sigma^2 \left(1 + \theta/\rho\right)}{x_t \left[\rho^h - \left(\rho^h - \rho\right) \beta_t^U\right]^3} \left[\frac{d\beta_t^U}{d\theta^h} \left(\rho^h - \left(\rho^h - \rho\right) \beta_t^U\right) + 2\left(\rho^h - \rho\right) \frac{d\beta_t^U}{d\theta^h} \beta_t^U\right]$$

$$= \frac{\rho^h}{\left[1 + \left(\rho^h - \rho\right) x_t\right] x_t} \frac{\sigma^2 \left(1 + \theta/\rho\right)^2 \left(\frac{1}{x_t} - \rho\right) \left(\rho^h + \left(\rho^h - \rho\right) \beta_t^U\right) \left(\beta_t^U\right)^2}{\left(\rho^h - \left(\rho^h - \rho\right) \beta_t^U\right)^3 \rho^h \left(1 + \theta^h/\rho^h\right)}.$$

$$0 \leq \beta_t^U \leq 1 \Rightarrow \rho \leq \rho^h - \left(\rho^h - \rho\right) \beta_t^U \leq \rho^h$$

$$\Rightarrow \frac{d\pi_{R,t}^U}{d\theta^h} = \frac{\sigma^2 \gamma^2 \left(\frac{1}{x_t} - \rho\right) \left(\rho^h + \Delta\rho\beta_t^U\right) \left(\beta_t^U\right)^2}{\gamma^h x_t \left(1 + \Delta\rho x_t\right) \left(\rho^h - \Delta\rho\beta_t^U\right)^3} \geq 0.$$

5.4.2 Sharpe ratio.

$$\frac{d\left(\pi_{R,t}^U/\sigma_{R,t}^U\right)}{d\theta} = \frac{\sigma \left(\rho^h + \theta^h\right)^2 x_t}{\left[\left(\rho\theta^h - \rho^h\theta\right) x_t + \rho^h \left(1 + \theta/\rho\right)\right]^2}.$$

$$\Rightarrow \frac{d\left(\pi_{R,t}^U/\sigma_{R,t}^U\right)}{d\theta} = \frac{\sigma\gamma^{h2} x_t}{\left[\rho\left(\gamma^h - \gamma\right) x_t + \gamma\right]^2} > 0.$$

$$\frac{d\left(\pi_{R,t}^U/\sigma_{R,t}^U\right)}{d\theta^h} = \frac{\sigma \left(1 + \theta/\rho\right) \rho^h \left(\theta + \rho\right) \left(\frac{1}{\rho} - x_t\right)}{\left[\left(\rho\theta^h - \rho^h\theta\right) x_t + \rho^h \left(1 + \theta/\rho\right)\right]^2}$$

$$\Rightarrow \frac{d\left(\pi_{R,t}^U/\sigma_{R,t}^U\right)}{d\theta^h} = \frac{\sigma\gamma^2 \left(1 - \rho x_t\right)}{\rho^h \left[\rho\left(\gamma^h - \gamma\right) x_t + \gamma\right]^2} \geq 0.$$

$$\frac{d\left(\pi_{R,t}^{U}/\sigma_{R,t}^{U}\right)}{d\bar{\theta}}=\frac{\sigma\left(1+\bar{\theta}/\rho\right)\left[\Delta\rho\left(\rho^{h}-\frac{\bar{\theta}^{2}}{\rho}\right)x_{t}+\rho^{h}\left(1+\bar{\theta}/\rho\right)^{2}\right]}{\left[-\Delta\rho\bar{\theta}x_{t}+\rho^{h}\left(1+\bar{\theta}/\rho\right)\right]^{2}}.$$

When $\Delta\rho\left(\rho^{h}-\frac{\bar{\theta}^{2}}{\rho}\right)x_{t}+\rho^{h}\left(1+\bar{\theta}/\rho\right)^{2}>0$, $\frac{d\left(\pi_{R,t}^{U}/\sigma_{R,t}^{U}\right)}{d\theta}>0$. Hence,

$$\Delta\rho\left(\rho^{h}-\frac{\bar{\theta}^{2}}{\rho}\right)x_{t}+\rho^{h}\left(1+\bar{\theta}/\rho\right)^{2}>0$$

$$\begin{array}{lll} & \rho^{h}-\frac{\bar{\theta}^{2}}{\rho}=0 & \Leftrightarrow\bar{\theta}=\sqrt{\rho\rho^{h}} \\ \Leftrightarrow\{ & \{\begin{array}{l}\rho^{h}-\frac{\bar{\theta}^{2}}{\rho}<0 \\ x_{t}<-\frac{\rho^{h}\left(1+\bar{\theta}/\rho\right)^{2}}{\Delta\rho\left(\rho^{h}-\frac{\bar{\theta}^{2}}{\rho}\right)}\end{array} & \begin{array}{l}\Leftrightarrow\bar{\theta}>\sqrt{\rho\rho^{h}} \\ =\phi\end{array}\end{array}$$

Take first derivative of ϕ with respect to $\bar{\theta}$,

$$\frac{d\phi}{d\bar{\theta}}=\frac{-2\rho^{h}\left(1+\bar{\theta}/\rho\right)}{\Delta\rho\left(\rho\rho^{h}-\bar{\theta}^{2}\right)^{2}}\left(\rho^{h}-\bar{\theta}-2\frac{\bar{\theta}^{2}}{\rho}\right).$$

$$\rho^{h}-\bar{\theta}-2\frac{\bar{\theta}^{2}}{\rho}=0\Leftrightarrow 2\bar{\theta}^{2}+\rho\bar{\theta}-\rho\rho^{h}=0$$

$$\Rightarrow\bar{\theta}=\frac{-\rho\pm\sqrt{\rho^{2}+8\rho\rho^{h}}}{4}.$$

Since $\bar{\theta}\geq 0$, the negative root is not valid. Now show $\bar{\theta}=\frac{-\rho+\sqrt{\rho^{2}+8\rho\rho^{h}}}{4}<\sqrt{\rho\rho^{h}}$,

$$\frac{\bar{\theta}}{\sqrt{\rho\rho^{h}}}=\frac{-\sqrt{\frac{\rho}{\rho^{h}}}+\sqrt{\frac{\rho}{\rho^{h}}+8}}{4}<\frac{3}{4}<1.$$

where we have used $0<\frac{\rho}{\rho^{h}}\leq 1$.

$$\Rightarrow\rho^{h}-\bar{\theta}-2\frac{\bar{\theta}^{2}}{\rho}<0\,\mathrm{for}\,\bar{\theta}\sqrt{\rho\rho^{h}}.$$

Thus,

$$\frac{d\phi}{d\theta}>0.$$

Now show that $\phi>1/\rho$,

$$-\frac{\rho^{h}\left(1+\bar{\theta}/\rho\right)^{2}}{\Delta\rho\left(\rho^{h}-\frac{\bar{\theta}^{2}}{\rho}\right)}>\frac{1}{\rho}\Leftrightarrow-\rho\rho^{h}\left(1+\frac{2\bar{\theta}}{\rho}+\frac{\bar{\theta}^{2}}{\rho}\right)>\rho^{h2}-\frac{\rho^{h}}{\rho}\bar{\theta}^{2}-\rho\rho^{h}+\bar{\theta}^{2}$$

$$\Leftrightarrow\left(\bar{\theta}+\rho^{h}\right)^{2}>0$$

$$\Rightarrow x_{t}<\frac{1}{\rho}<\phi.$$

Finally,

$$\Rightarrow\frac{d\left(\pi_{R,t}^{U}/\sigma_{R,t}^{U}\right)}{d\bar{\theta}}>0.$$

5.4.3 Exposure price.

$$\frac{dk_t}{d\bar{\theta}} = \frac{\sigma^2(1+m)}{(m\rho^h+\rho)^2}\frac{\rho^h(1-\rho x_t-\rho m x_t)}{[1+(\rho^h-\rho)x_t]\,x_t}.$$

Since

$$\begin{aligned}\bar{x}^c = \frac{1}{\tilde{m}\rho^h+\rho} = \frac{1}{\rho}\left(\frac{1}{1+\frac{\rho^h+\bar{\theta}}{\rho+\bar{\theta}}m}\right) \le \frac{1}{\rho}\left(\frac{1}{1+m}\right)\\ \Rightarrow x_t \le \bar{x}^c \le \frac{1}{\rho}\left(\frac{1}{1+m}\right)\\ \Leftrightarrow 1-\rho x_t(1+m) > 0.\end{aligned} \tag{82}$$

$$\frac{dk_t}{d\bar{\theta}} = \frac{\sigma^2(1+m)}{(m\rho^h+\rho)^2}\frac{\rho^h[1-\rho x_t(1+m)]}{[1+(\rho^h-\rho)x_t]\,x_t} > 0.$$

5.4.4 Interest rate.

Denote Sharpe ratio $\frac{\pi_{R,t}}{\sigma_{R,t}} = sp$,

$$\begin{aligned}r_t = \rho^h + \hat{g}_t - \rho\left(\rho^h-\rho\right)x_t - \rho q_t x_t - \frac{\rho x_t\left[\left(\frac{1}{\gamma}sp\right)^2 - 2\sigma\frac{1}{\gamma}sp\right]+\sigma^2}{1-\rho x_t}\\ \Rightarrow \frac{dr_t}{d\theta} = -\rho x_t\frac{dq_t}{d\theta} - \frac{2x_t}{(1-\rho x_t)\gamma^2}\left[\left(\frac{sp}{\gamma}-\sigma\right)\left(-sp+\frac{dsp}{d\theta}\rho\gamma\right)\right]\end{aligned} \tag{83}$$

$$\Leftrightarrow \frac{dr_t^U}{d\theta} = \frac{2\sigma^2 x_t\gamma^h(1-\rho x_t)}{[\rho(\gamma^h-\gamma)x_t+\gamma]^3}\left(\gamma^h-\gamma\right).$$

$$\Rightarrow \begin{cases}\frac{dr_t^U}{d\theta} < 0 & \text{if } \gamma^h < \gamma\\ \frac{dr_t^U}{d\theta} \ge 0 & \text{if } \gamma^h \ge \gamma.\end{cases}$$

$$\frac{dr_t}{d\theta^h} = -\rho x_t\frac{dq_t}{d\theta^h} - \frac{2\rho x_t}{(1-\rho x_t)\gamma}\left(\frac{sp}{\gamma}\frac{dsp}{d\theta^h} - \sigma\frac{dsp}{d\theta^h}\right) \tag{84}$$

$$\frac{dr_t^U}{d\theta^h} = -\frac{2\sigma^2\rho\gamma x_t(1-\rho x_t)\left(\gamma^h-\gamma\right)}{\rho^h[\rho(\gamma^h-\gamma)x_t+\gamma]^3}.$$

$$\Rightarrow \begin{cases}\frac{dr_t^U}{d\theta^h} > 0 & \text{if } \gamma^h < \gamma\\ \frac{dr_t^U}{d\theta^h} \le 0 & \text{if } \gamma^h \ge \gamma.\end{cases}$$

$$\frac{dr_t^U}{d\bar{\theta}} = \frac{2\sigma^2\bar{\theta}\Delta\rho x_t\left[\Delta\rho\left(\rho^h\bar{\theta}+\rho\rho^h\right)x_t - \Delta\rho\rho^h\left(1+\bar{\theta}/\rho\right)\right]}{\rho\left(1+\bar{\theta}/\rho\right)\left[-\Delta\rho\bar{\theta}x_t+\rho^h\left(1+\bar{\theta}/\rho\right)\right]^3}$$

$$\Rightarrow \frac{dr_t^U}{d\bar{\theta}} = -\frac{2\sigma^2\bar{\theta}\rho^h (\Delta\rho)^2 x_t (1-\rho x_t)}{\rho\left[-\Delta\rho\bar{\theta}x_t + \rho^h\left(1+\bar{\theta}/\rho\right)\right]^3} \leq 0.$$

In constrained case, from equations (83) and (84),

$$\frac{dr_t}{d\theta} = -\frac{\sigma^2 m}{(m\rho^h+\rho)^2 x_t} - \frac{2x_t}{(1-\rho x_t)\gamma^2}\left[\left(\frac{\sigma}{(m\rho^h+\rho)x_t} - \sigma\right)\left(-\frac{\sigma\gamma}{(m\rho^h+\rho)x_t} + \frac{\sigma(\rho+\theta)}{\rho(m\rho^h+\rho)x_t}\right)\right]$$

$$\frac{dr_t}{d\theta} = -\frac{\sigma^2 m}{(m\rho^h+\rho)^2 x_t} < 0.$$

$$\frac{dr_t}{d\theta^h} = \frac{\rho\sigma^2 m^2}{(m\rho^h+\rho)^2(1-\rho x_t)} > 0.$$

$$\frac{dr_t}{d\bar{\theta}} = -\frac{\sigma^2 m\left[1-\rho x_t(1+m)\right]}{(m\rho^h+\rho)^2(1-\rho x_t)x_t} < 0$$

where we used the condition (82).

5.5 Solving the Stochastic Process of Aggregate State

In order to derive the unconditional mean and variance of risk premium and interest rate, we need to know the distribution of the state variable x_t. Using Ito's formula,

$$dx_t = d\left(\frac{W_t}{D_t}\right) = \frac{dW_t}{D_t} - \frac{W_t dD_t}{D_t^2} - \frac{dW_t dD_t}{D_t^2} + \frac{W_t}{D_t^3}[dD_t, dD_t]$$
$$\Rightarrow \frac{dx_t}{x_t} = \frac{dW_t}{W_t} - \frac{dD_t}{D_t} - \frac{dD_t}{D_t}\frac{dW_t}{W_t} + \left(\frac{dD_t}{D_t}\right)^2.$$

Plug in the assumed dividend process (1) and derived specialist wealth process (81),

$$\frac{dx_t}{x_t} = \frac{dW_t}{W_t} - \frac{dD_t}{D_t} - \frac{\sigma}{\gamma}\frac{\pi_{R,t}}{\sigma_{R,t}}dt + \sigma^2 dt$$

$$\frac{dx_t}{x_t} = \left(\sigma^2 - \hat{g}_t - \rho + q_t + r_t + \frac{1}{\gamma^2}\frac{\pi_{R,t}^2}{\sigma_{R,t}^2} - \frac{\sigma}{\gamma}\frac{\pi_{R,t}}{\sigma_{R,t}}\right)dt + \left(\frac{\pi_{R,t}}{\gamma\sigma_{R,t}} - \sigma\right)dZ_t.$$

Denote $E = \frac{\pi_{R,t}}{\gamma\sigma_{R,t}}$, drift and diffusion of aggregate state process as $\mu_{x,t}$ and $\sigma_{x,t}$, respectively.

$$\begin{aligned}\mu_{x,t} &= \sigma^2 - \hat{g}_t - \rho + q_t + r_t + E^2 - \sigma E \\ &= \sigma^2 + \rho^h - \rho - \rho\left(\rho^h - \rho\right)x_t + (1-\rho x_t)q_t + \frac{E^2(1-2\rho x_t) + (3\rho x_t - 1)\sigma E - \sigma^2}{1-\rho x_t} \\ \sigma_{x,t} &= E - \sigma\end{aligned}$$

5.6 Stationary Specialist Wealth Distribution

We can solve the stationary specialist wealth distribution explicitly. Let $G = 1 - \frac{\gamma^h}{\gamma} \in [0,1]$, $E = \frac{\sigma \frac{\gamma^h}{\gamma}}{1-G\rho x}$,

$$\mu_u(x) = \sigma^2 + \Delta\rho(1-\rho x) - \frac{\rho x\left(E^2 - 2\sigma E\right) + \sigma^2}{1-\rho x} + E^2 - \sigma E$$

$$= \sigma^2 + \Delta\rho(1-\rho x) + \sigma^2 \frac{\left(\frac{\gamma^h}{\gamma}\right)^2(1-2\rho x) + \left(\frac{\gamma^h}{\gamma}\right)(1-\rho G x)(3\rho x - 1) - (1-\rho G x)^2}{(1-\rho x)(1-\rho G x)^2}$$

$$\sigma_u^2(x) = \sigma^2 G^2 \left(\frac{1-\rho x}{1-\rho G x}\right)^2$$

$$\int \frac{2\mu_u(x)}{\sigma_u^2(x)} = \frac{2}{G^2}\Bigg[\int \left(\frac{1-\rho G s}{1-\rho s}\right)^2 ds + \frac{\Delta\rho}{\sigma^2}\int \frac{(1-\rho G s)^2}{1-\rho s} ds + \left(\frac{\gamma^h}{\gamma}\right)^2 \int \frac{1-2\rho s}{(1-\rho s)^3} ds$$
$$+ \frac{\gamma^h}{\gamma}\int \frac{(1-\rho G s)(3\rho s - 1)}{(1-\rho s)^3} ds - \int \frac{(1-\rho G s)^2}{(1-\rho s)^3} ds\Bigg]$$
$$= \frac{2}{G^2}\Bigg[\frac{\rho G^2 x(\rho x - 2) + 2(G-1)G(\rho x - 1)\, log(\rho x - 1) + 2G - 1}{\rho(\rho x - 1)}$$
$$- \frac{\Delta\rho}{\sigma^2}\frac{\rho G x[\rho(Gx - 4) + 2G] + 2(\rho - G)^2 \, log(1-\rho x)}{2\rho^3}$$
$$+ \left(\frac{\gamma^h}{\gamma}\right)^2 \frac{3 - 4\rho x}{2\rho(\rho x - 1)^2}$$
$$+ \frac{\gamma^h}{\gamma}\frac{G(4 - 5\rho x) + 3G(\rho x - 1)^2 \, log(\rho x - 1) + 3\rho x - 2}{\rho(\rho x - 1)^2}$$
$$+ \frac{2G^2 log(\rho x - 1) - \frac{(G-1)[G(4\rho x - 3) - 1]}{(\rho x - 1)^2}}{2\rho}\Bigg]$$

Let $H = m\rho^h + \rho$

$$\mu_c(x) = \sigma^2 + \Delta\rho(1-\rho x) + \sigma^2 \frac{A_0 x^2 + A_1 x + A_2}{(1-\rho x)(Hx)^2}$$

$$\sigma_c^2(x) = \sigma^2\left(\frac{1-Hx}{Hx}\right)^2$$

$$\frac{2\mu_c(x)}{\sigma_c^2(x)} = 2\left(\frac{Hx}{1-Hx}\right)^2 + 2\frac{\Delta\rho}{\sigma^2}(1-\rho x)\left(\frac{Hx}{1-Hx}\right)^2$$
$$+ \frac{2A_0 x^2}{(1-\rho x)(1-Hx)^2} + \frac{2A_1 x}{(1-\rho x)(1-Hx)^2} + \frac{2A_2}{(1-\rho x)(1-Hx)^2}$$

$$
\begin{aligned}
\int \frac{2\mu_c(x)}{\sigma_c^2(x)} =& \frac{2\left[Hx + \frac{1}{1-Hx} + 2log\left(1-Hx\right)\right]}{H} \\
&- \frac{2\Delta\rho}{\sigma^2}\frac{H^2\rho x^2 - 2Hx\left(H-2\rho\right) + \frac{2(H-\rho)}{Hx-1} + 2\left(3\rho - 2H\right)log\left(1-Hx\right)}{H^2} \\
&- 2A_0\frac{\frac{H-\rho}{H^2(Hx-1)} + \frac{(\rho-2H)log(1-Hx)}{H^2} + \frac{log(1-\rho x)}{\rho}}{\left(H-\rho\right)^2} \\
&- 2A_1\frac{\frac{H-\rho}{H(Hx-1)} - log\left(1-Hx\right) + log\left(1-\rho x\right)}{\left(H-\rho\right)^2} \\
&+ 2A_2\frac{\rho\left(Hx-1\right)log\left(1-Hx\right) + \left(\rho - H\rho x\right)log\left(1-\rho x\right) - H + \rho}{\left(H-\rho\right)^2\left(Hx-1\right)}
\end{aligned}
$$

5.7 Detection Error Probabilities

Obtain the relative entropies from household and specialist optimal robust problem,

$$
g^h\left(x_t\right) = |\nu_t^h| = \frac{\theta^h\sigma_{R,t}\varepsilon_t^h}{\rho^h W_t^h} = \frac{\theta^h}{\rho^h\gamma^h}\frac{\pi_{R,t} - k_t}{\sigma_{R,t}} = \frac{\gamma^h - 1}{\gamma^h}\left(\frac{\pi_{R,t}}{\sigma_{R,t}} - \frac{k_t}{\sigma_{R,t}}\right).
$$

In constraint case, from equation (80),

$$
\frac{\pi_{R,t}}{\sigma_{R,t}} - \frac{k_t}{\sigma_{R,t}} = \frac{\tilde{m}\rho^h\sigma\gamma}{\left(1-\rho x_t\right)\left(m\rho^h + \rho\right)}.
$$

In unconstrained case, $k_t = 0$,

$$
\frac{\pi_{R,t}}{\sigma_{R,t}} - \frac{k_t}{\sigma_{R,t}} = \frac{\sigma\gamma\gamma^h}{\rho\left(\gamma^h - \gamma\right)x_t + \gamma}.
$$

$$
\begin{aligned}
\Rightarrow g^h\left(x_t\right) =& \frac{\gamma^h - 1}{\gamma^h}\left[\frac{\gamma^h m\rho^h\sigma}{\left(1-\rho x_t\right)\left(m\rho^h + \rho\right)}\mathbf{1}_{x_t\in(x_{min},x^c]} + \frac{\sigma\gamma\gamma^h}{\rho\left(\gamma^h - \gamma\right)x_t + \gamma}\mathbf{1}_{x_t\in(x^c,\frac{1}{\rho}]}\right]. \\
=& \sigma\gamma\left(\gamma^h - 1\right)\left[\frac{m\rho^h}{\gamma\left(m\rho^h + \rho\right)\left(1-\rho x_t\right)}\mathbf{1}_{x_t\in(x_{min},x^c]} + \frac{1}{\rho\left(\gamma^h - \gamma\right)x_t + \gamma}\mathbf{1}_{x_t\in(x^c,\frac{1}{\rho}]}\right]
\end{aligned}
$$

$$
g\left(x_t\right) = |\nu_t| = -\frac{\theta\sigma_{R,t}\varepsilon_t}{\rho W_t} = -\frac{\theta}{\rho\gamma}\frac{\pi_{R,t}}{\sigma_{R,t}} = \frac{\gamma - 1}{\gamma}\frac{\pi_{R,t}}{\sigma_{R,t}}
$$

$$
\Rightarrow g\left(x_t\right) = \sigma\gamma^h\left(\gamma - 1\right)\left[\frac{1}{\gamma^h\left(m\rho^h + \rho\right)x_t}\mathbf{1}_{x_t\in(x_{min},x^c]} + \frac{1}{\rho\left(\gamma^h - \gamma\right)x_t + \gamma}\mathbf{1}_{x_t\in(x^c,\frac{1}{\rho}]}\right]
$$

where $\mathbf{1}$ denotes the indicator function.

5.8 Figures and Tables

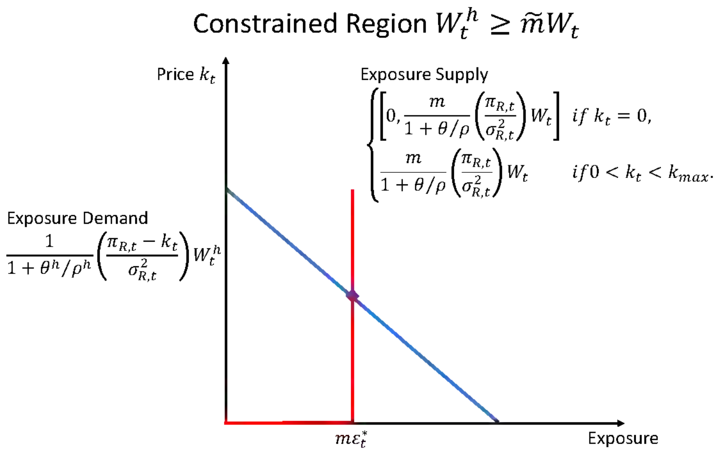

Figure 2: Constrained Region in Equilibrium

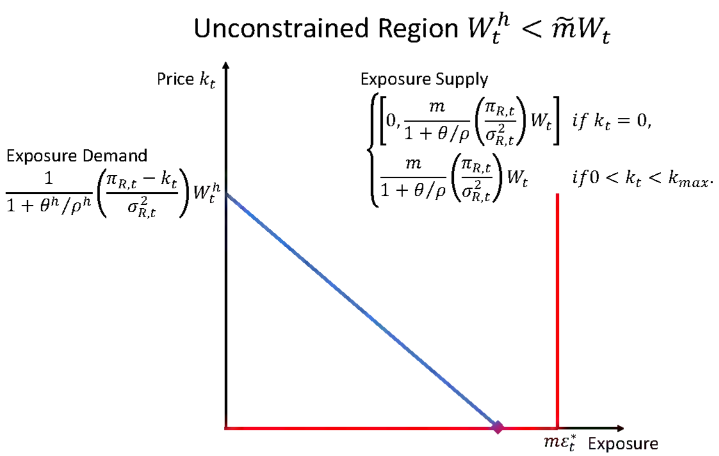

Figure 3: Unconstrained Region in Equilibrium

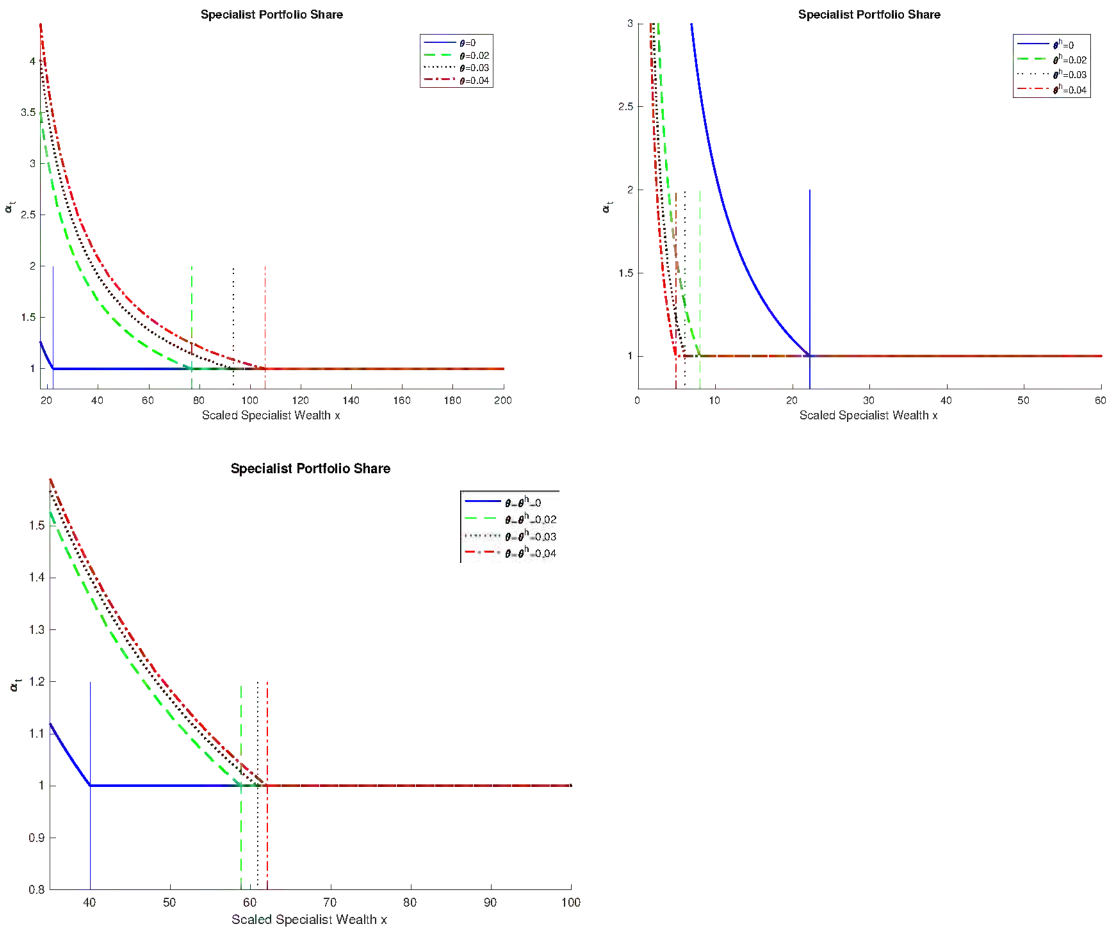

Figure 4: Specialist Portfolio Share

The specialist's portfolio share for risky asset α_t is graphed against the specialist scaled wealth x_t for different ambiguity parameters (θ and θ^h) varying from 0.02 to 0.04. The threshold value x^c (vertical line) separates the constrained (left) and unconstrained (right) region. The solid blue line shuts down the ambiguity. The left panel plots the portfolio share policy function with a fixed $\theta^h = 0.03$ but different values of θ. The middle panel fixes $\theta = 0.03$ with different values of θ^h. The right panel plots the homogeneous ambiguity from two agents ($\theta = \theta^h = \bar{\theta} = 0.3$).

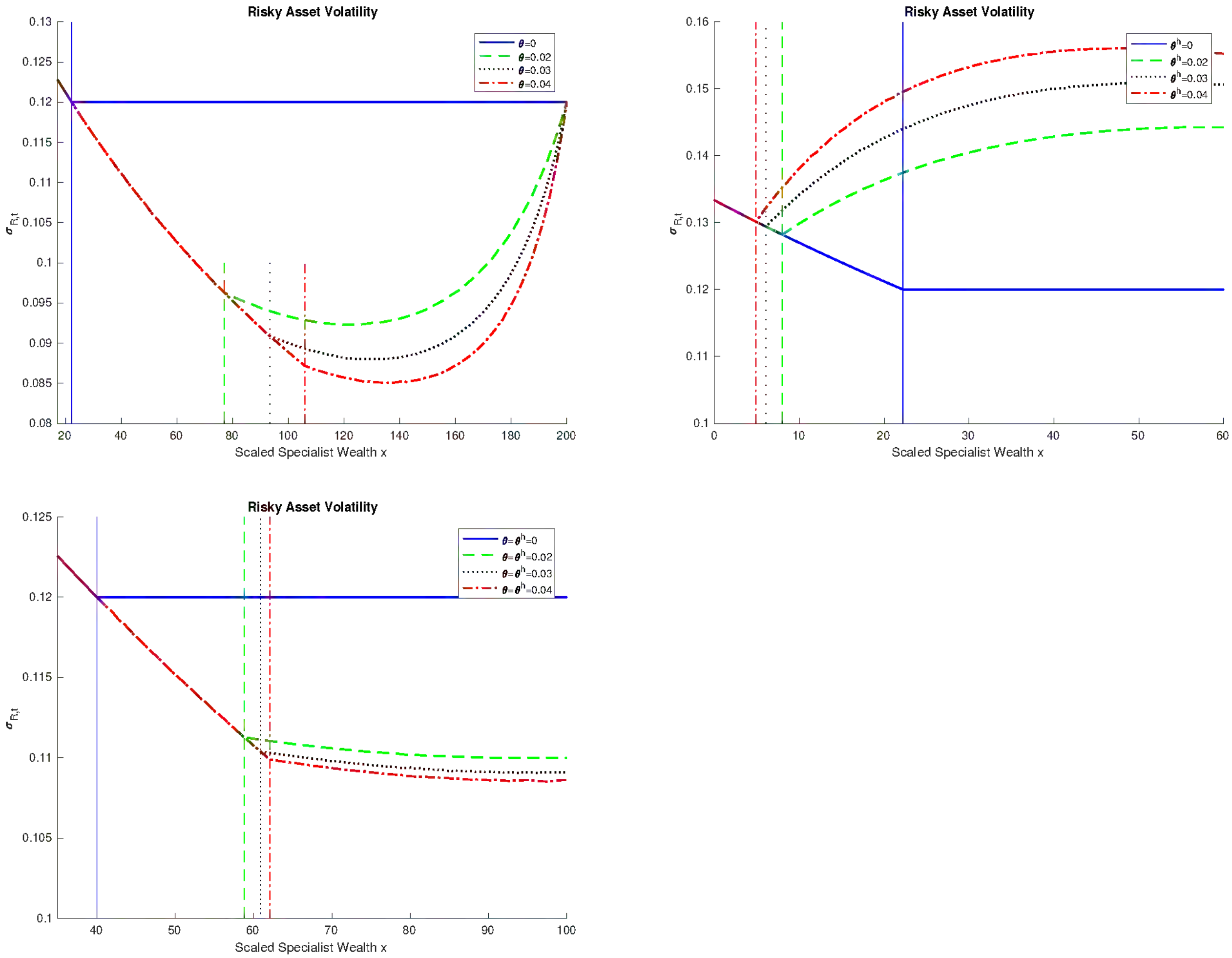

Figure 5: Risky Asset Volatility

The risky asset volatility $\sigma_{R,t}$ is graphed against the specialist scaled wealth x_t for different ambiguity parameters (θ and θ^h) varying from 0.02 to 0.04. The threshold value x^c (vertical line) separates the constrained (left) and unconstrained (right) region. The solid blue line shuts down the ambiguity. The left panel plots the portfolio share policy function with a fixed $\theta^h = 0.03$ but different values of θ. The middle panel fixes $\theta = 0.03$ with different values of θ^h. The right panel plots the homogeneous ambiguity from two agents ($\theta = \theta^h = \bar{\theta} = 0.3$).

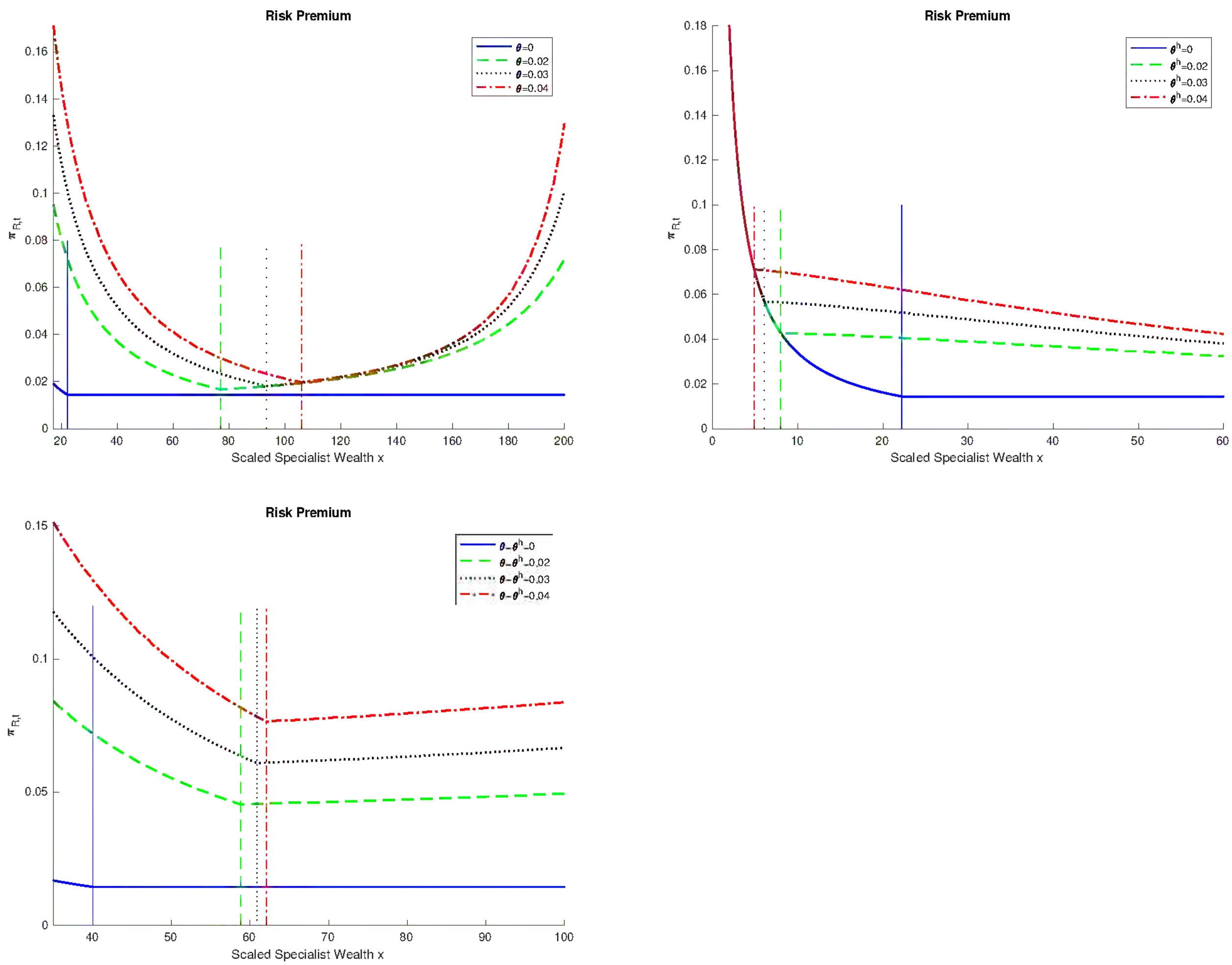

Figure 6: Risk Premium

The risk premium $\pi_{R,t}$ is graphed against the specialist scaled wealth x_t for different ambiguity parameters (θ and θ^h) varying from 0.02 to 0.04. The threshold value x^c (vertical line) separates the constrained (left) and unconstrained (right) region. The solid blue line shuts down the ambiguity. The left panel plots the portfolio share policy function with a fixed $\theta^h = 0.03$ but different values of θ. The middle panel fixes $\theta = 0.03$ with different values of θ^h. The right panel plots the homogeneous ambiguity from two agents ($\theta = \theta^h = \bar{\theta} = 0.3$).

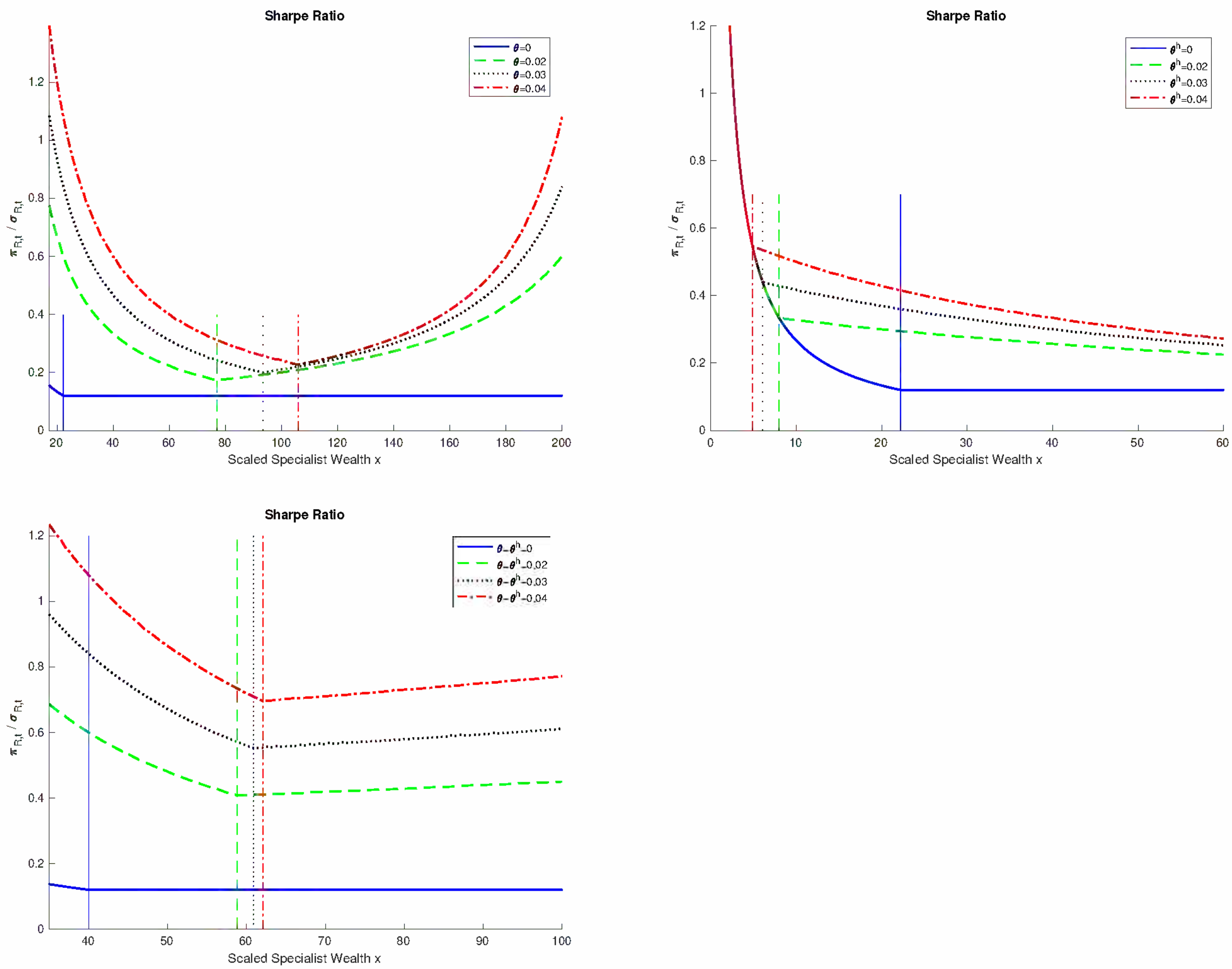

Figure 7: Sharpe Ratio

The Sharpe ratio $\pi_{R,t}/\sigma_{R,t}$ is graphed against the specialist scaled wealth x_t for different ambiguity parameters (θ and θ^h) varying from 0.02 to 0.04. The threshold value x^c (vertical line) separates the constrained (left) and unconstrained (right) region. The solid blue line shuts down the ambiguity. The left panel plots the portfolio share policy function with a fixed $\theta^h = 0.03$ but different values of θ. The middle panel fixes $\theta = 0.03$ with different values of θ^h. The right panel plots the homogeneous ambiguity from two agents ($\theta = \theta^h = \bar{\theta} = 0.3$).

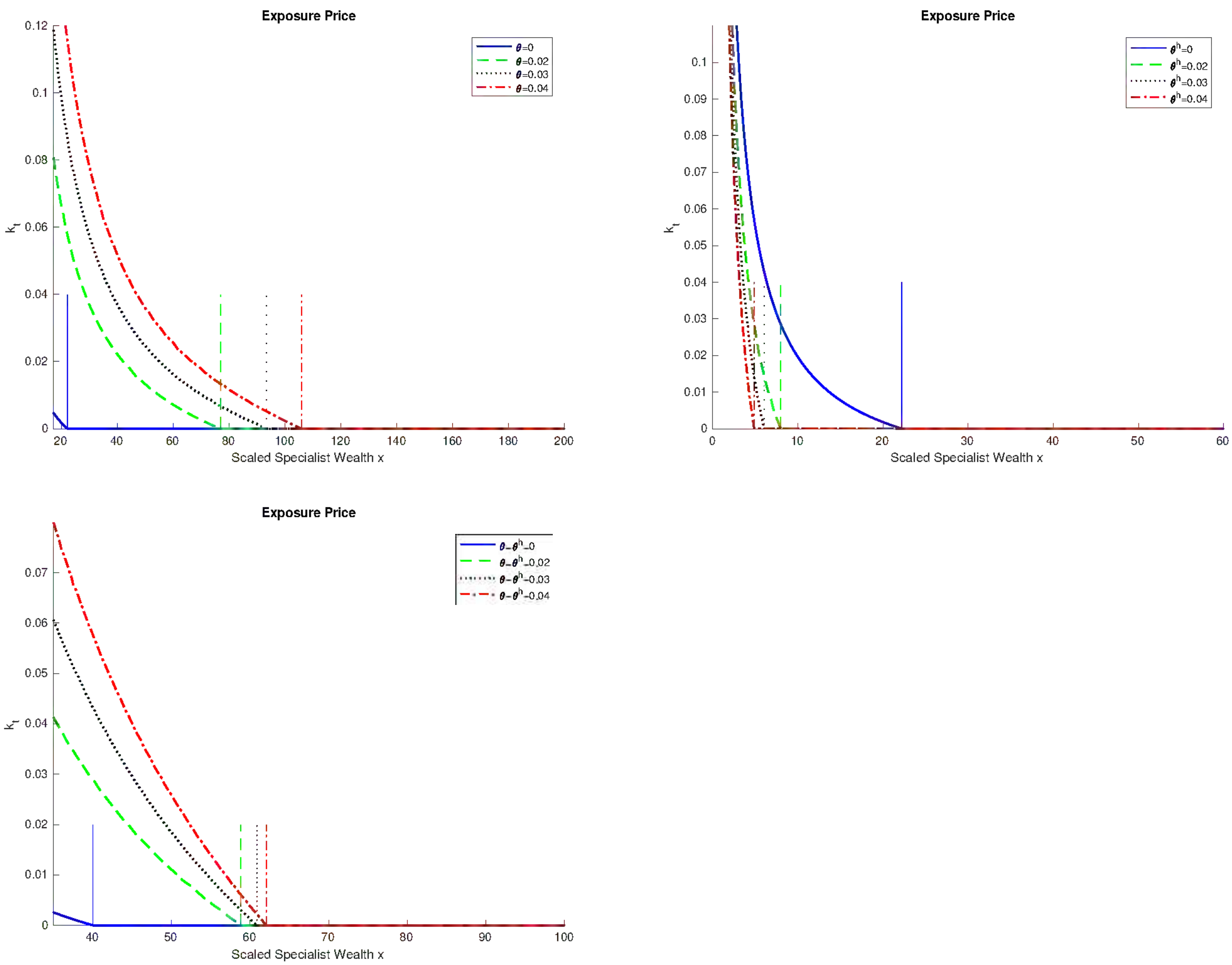

Figure 8: Intermediation Exposure Price

The exposure price k_t is graphed against the specialist scaled wealth x_t for different ambiguity parameters (θ and θ^h) varying from 0.3 to 0.5. The threshold value x^c (vertical line) separates the constrained (left) and unconstrained (right) region. The solid blue line shuts down the ambiguity. The left panel plots the portfolio share policy function with a fixed $\theta^h = 0.04$ but different values of θ. The middle panel fixes $\theta = 0.04$ with different values of θ^h. The right panel plots the homogeneous ambiguity from two agents ($\theta = \theta^h = \bar{\theta} = 0.3$).

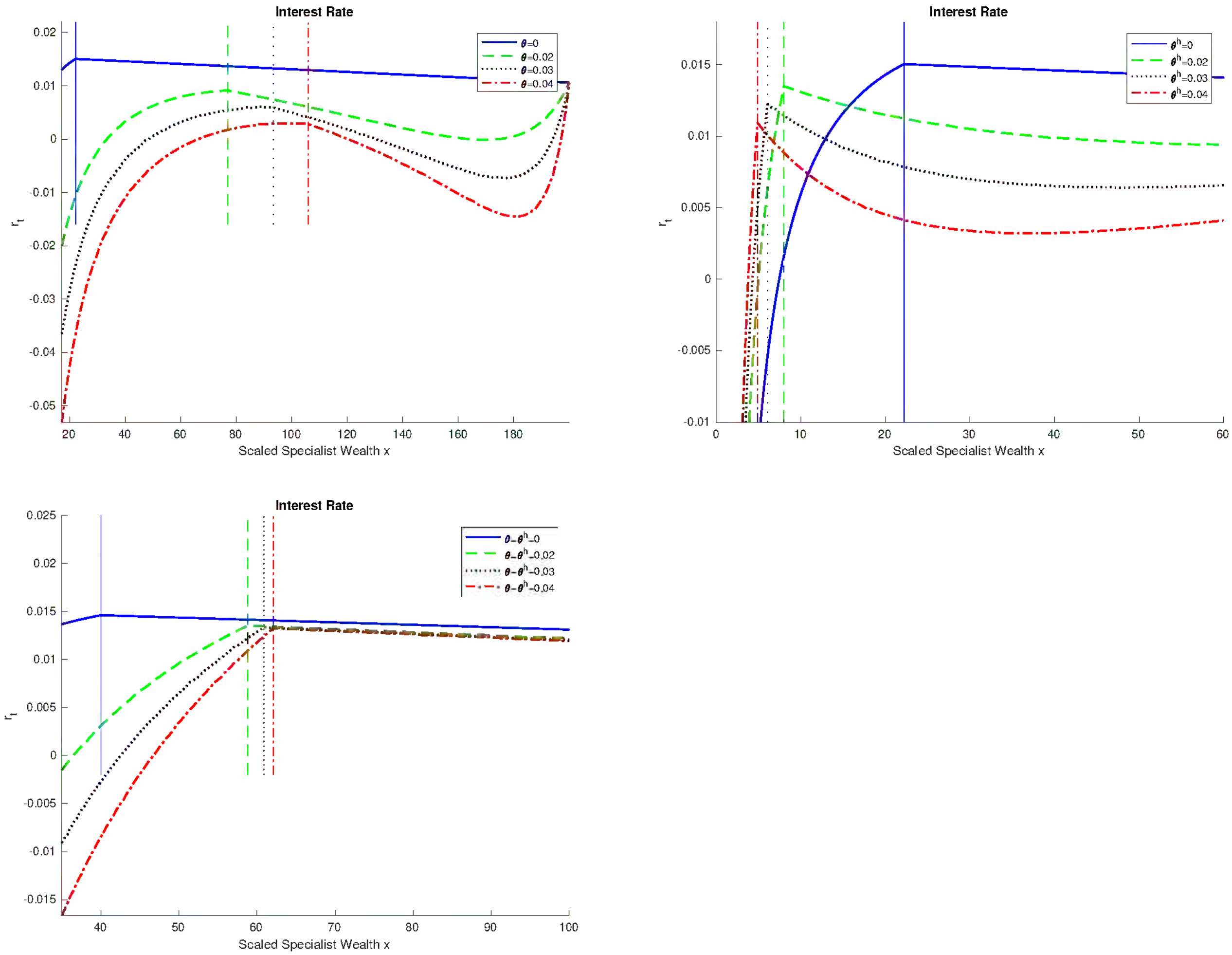

Figure 9: Interest Rate

Interest rate r_t is graphed against the specialist scaled wealth x_t for different ambiguity parameters (θ and θ^h) varying from 0.02 to 0.04. The threshold value x^c (vertical line) separates the constrained (left) and unconstrained (right) region. The solid blue line shuts down the ambiguity. The left panel plots the portfolio share policy function with a fixed $\theta^h = 0.03$ but different values of θ. The middle panel fixes $\theta = 0.03$ with different values of θ^h. The right panel plots the homogeneous ambiguity from two agents ($\theta = \theta^h = \bar{\theta} = 0.3$).

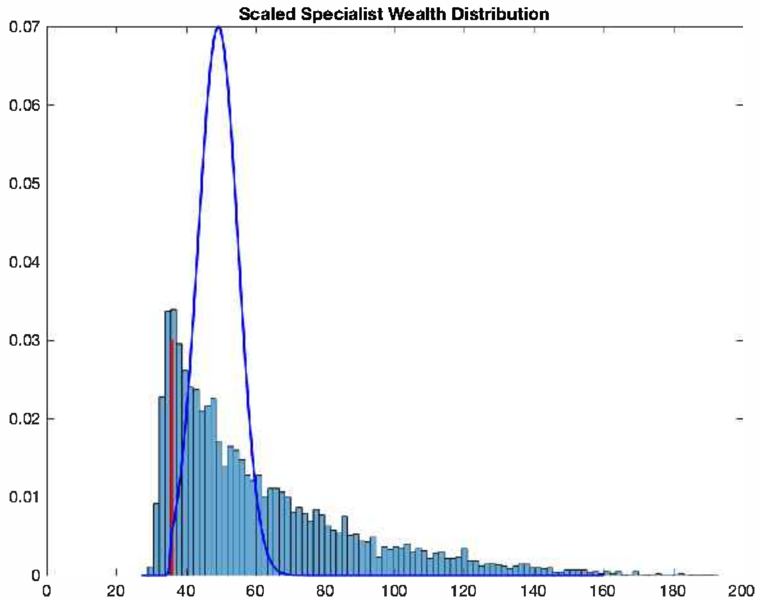

Figure 10: Stationary Distribution
This figure plots the stationary distribution of specialist scaled wealth. The histogram displays the simulated results and the solid blue line plots the theoretical density function. The vertical red line separates the constrained (left) and unconstrained (right) region.

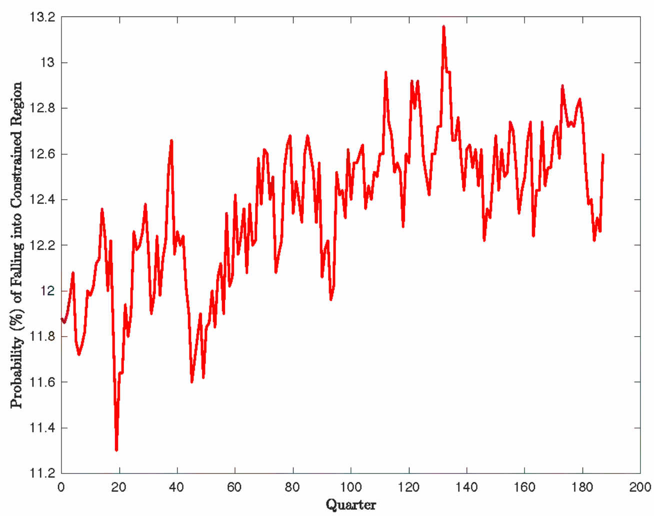

Figure 11: Probability of Constraint Binds
This figure plots the time path for the probability of falling into constrained region.

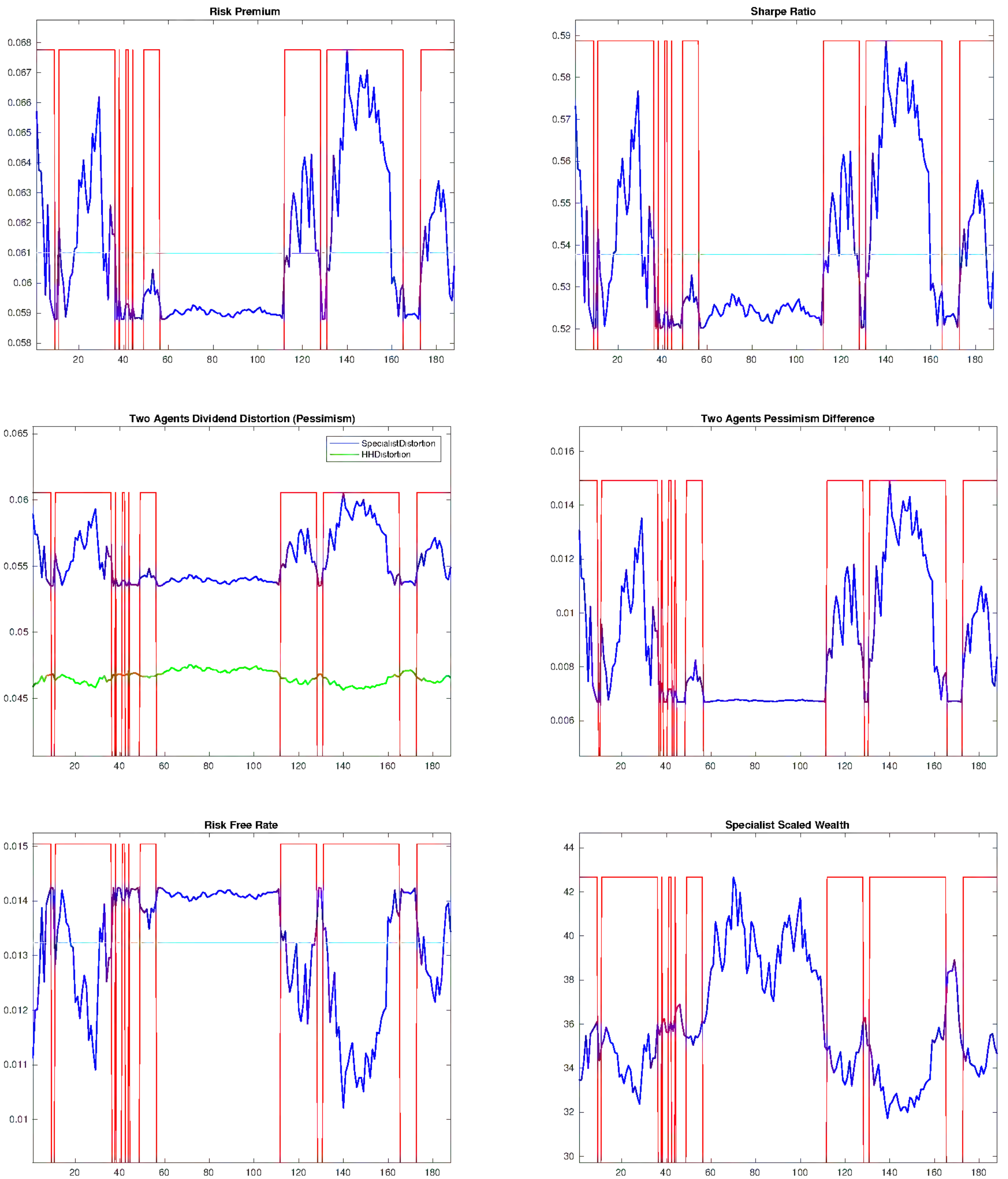

Figure 12:

This figure plots one sample path using calibrated values. Red rectangular regions indicate the constrained region. The light blue horizontal lines in the first two figures are unconditional means for risk premium and Sharpe Ratio, respectively.

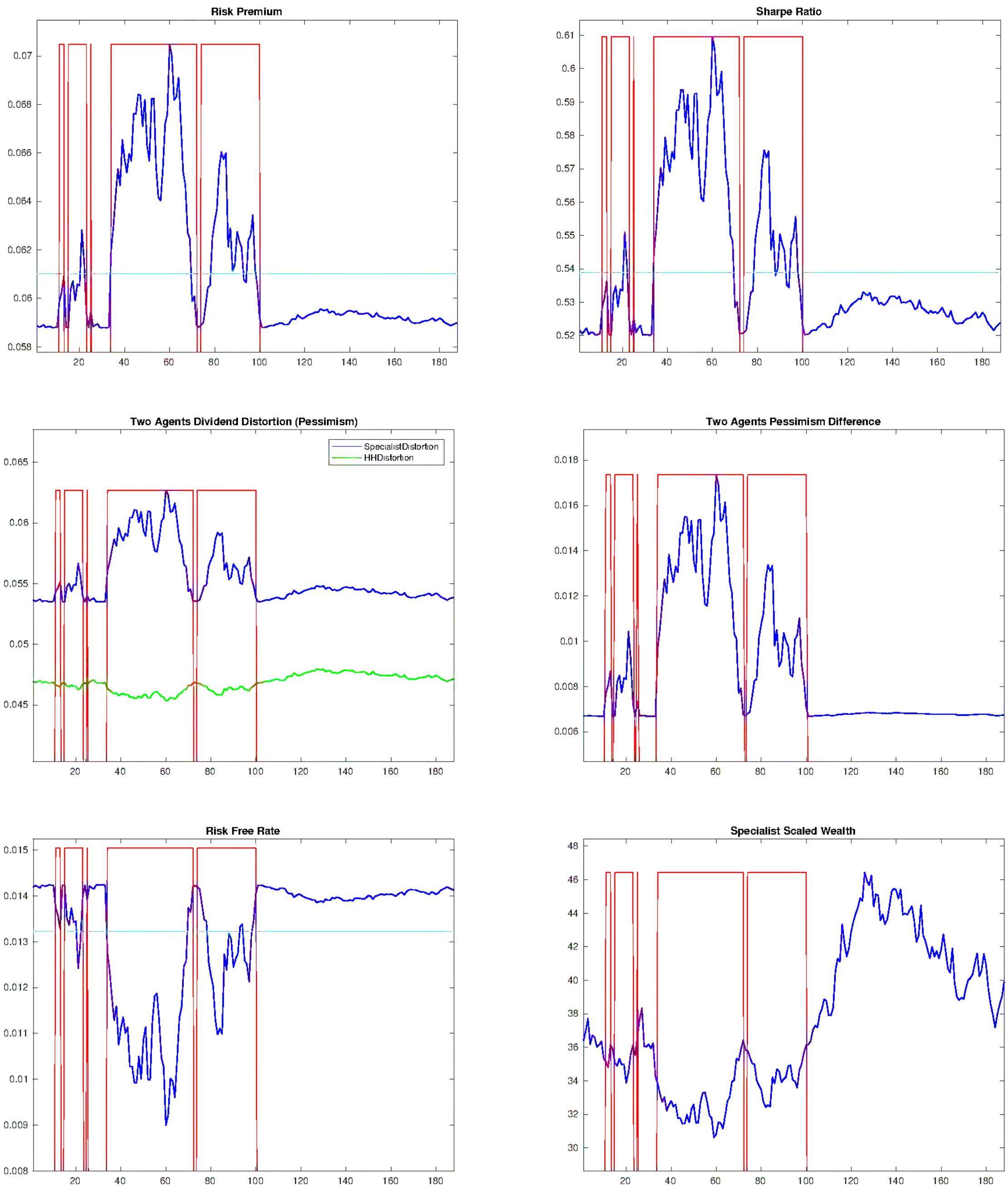

Figure 13:

This figure plots one sample path using calibrated values. Red rectangular regions indicate the constrained region. The light blue horizontal lines in the first two figures are unconditional means for risk premium and Sharpe Ratio, respectively.

Table 1: Parameters and Targets

Panel A. Preferences		
ρ	Time discount rate of specialist	0.005
ρ^h	Time discount rate of household	0.01
θ	Ambiguity attitude of specialist	0.03
θ^h	Ambiguity attitude of household	0.03
Panel B. Intermediation		
m	intermediation multiplier	4
$\bar{g}$	Mean Dividend growth rate	0.02
σ	Dividend volatility	0.12

Table 2: Measurements

	Data	Model	
θ		0.0001	0.03
θ^h		0.0001	0.03
γ		1.02	7
γ^h		1.01	4
Risk Premium (%)	6.93	0.92	6.22
Sharpe Ratio (%)	42.86	9.59	55.92
Interest Rate (%)	1.02	1.59	1.31
Interest Rate Volatility (%)	2.96	0.31	0.04
Return Volatility (%)	16.17	9.40	11.13
Portfolio Share		1	1.0064
Specialist Scaled Wealth Mean		200.00	60.95
Specialist Scaled Wealth Volatility		0.00	5.69
Probability of Sharpe Ratio Exceed 30 % of the Mean (%)		0	33.76
Probability of Sharpe Ratio Exceed 40 % of the Mean (%)		0	4.20
Probability of Constraint Binds (%)		0	3.09
Specialist Detection Error Probability		0.25	0.29
Household Detection Error Probability		0.25	0.28

This table reports the unconditional simulated results. We simulate 5000 years and 5000 sample paths with quarterly frequency. To match the data from 1970-2017, we report 47 years simulated results in stationary distribution.