

Macro-Financial Volatility under Dispersed Information*

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Abstract

We provide a production-based asset pricing model under dispersed information. In the log-linearized equilibrium system, aggregate output and equity prices depend on the higher-order beliefs about average forecasts of aggregate demand and individual stochastic discount factors, respectively. We prove that the presence of dispersed information reduces aggregate output volatility under very general information structures, provided agents are informationally small. On the other hand, equity volatility can be arbitrarily high as the volatility of the idiosyncratic shock approaches infinity. We show our analytical results using the frequency-domain techniques. Under reasonable calibrations our model can match output and equity volatilities in the data.

Keywords: Dispersed Information, Frequency Domain Analysis, Higher-order Beliefs, Asset Pricing, Business Cycle, Incomplete Markets

JEL Classifications: E32, E44, G12, G14.

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1 Introduction

Finance is littered with puzzles; one prominent and persistent puzzle is the observation by Shiller (1981) that aggregate stock prices are too volatile relative to the expected present value of dividends. Several studies have identified resolutions to the puzzle, including nonstandard preferences and nonstationary dividend processes, that argue the issue is with the inputs to the expectations operator for future dividends.¹ An alternative approach is the dispersed information environment of Kasa, Walker, and Whiteman (2014). In their model the issue is that the wrong expectations operator is used – the relevant expected value is taken using the average expectations operator, which in general does not satisfy a law of iterated expectations.

Another aspect of the equity volatility puzzle is that macroeconomic quantities – aggregate output, consumption, and dividends – are too smooth relative to equity prices. In actual economies all these quantities are endogenous and respond to the same shocks that drive equity price movements. The goal of our paper is to understand whether a production-based asset pricing model is able to deliver both smooth aggregate quantities and volatile equity prices, without introducing complex exogenous shocks or nonstandard preferences. We provide a positive answer to this question by developing a model of a dispersed-information island economy along the lines of Lorenzoni (2009) and Angeletos and La’O (2010, 2013), extended to include a centralized stock market.

Maintaining dynamic and persistent information frictions is crucial for our results. Such frictions often lead to the technical problem of “forecasting the forecasts of others” (Townsend (1983)). That is, the state space for the model solution contains an infinite number of higher-order expectations so that the time-domain methods become largely intractable. Therefore we use the frequency-domain methods to circumvent this obstacle based on the log-linearized equilibrium system and provide analytical characterizations.²

Our first main theoretical result is that higher-order expectations under dispersed information always reduce the volatility of business cycle fluctuations in the real economy. We establish this result by showing that the volatility of aggregate output under full information gives an upper bound for that under dispersed information. The key assumption for this result is that agents in the economy are *informationally small* in the sense that there is a continuum of agents with private information and the idiosyncratic shock component of the private information washes out in the aggregate. Since aggregate output depends on the average forecast of aggregate demand, there is no need for any agent to forecast the behavior of any other particular agent’s action to predict aggregate output. The slow learning effect brought in by signal extraction leads to dampened

¹See Campbell (1999) and Cochrane (2011) for surveys.

²Hansen and Sargent (1980), Futia (1981), Whiteman (1983, 1985), and Taub (1989) are among the first generation of research that employ these methods. Recent developments include Kasa (2000), Walker (2007), Bernhardt, Seiler, and Taub (2010), Makarov and Rytchkov (2012), Kasa, Walker, and Whiteman (2014), Rondina and Walker (2015), Tan and Walker (2015), and Huo and Takayama (2015).

fluctuations, while the speculative effect of the forecasting the forecasts of others in models with finitely many agents completely vanishes due to the law of large numbers.

Our second main theoretical result is that when the idiosyncratic TFP volatility approaches infinity, an endogenous unit root arises in the stochastic processes of investors' shareholdings and the aggregate equity price. Thus the equity price becomes infinitely volatile when the idiosyncratic TFP volatility approaches infinity. This is the most important result of our paper and may seem surprising because aggregate equity prices only respond to aggregate shocks but not idiosyncratic shocks. The key is that the response coefficient endogenously varies with the idiosyncratic TFP volatility – as idiosyncratic volatility rises, a feedback loop emerges and raises the sensitivity of equity prices to aggregate shocks. Our theoretical result has an appealing quantitative implication in that we can choose a relatively low volatility of aggregate shocks to match the low volatility of aggregate consumption and choose a relatively high volatility of idiosyncratic shocks to match the high volatility of equity prices as in the data.

In our incomplete markets model, the equity price is equal to the sum of the discounted average forecast of the individual stochastic discount factors (SDFs) and the discounted average forecast of future dividends. Due to dispersed information, the average forecast of the individual SDFs is not equal to the forecast of the average SDFs. Thus the variation in the distribution of individual consumption matters to equity prices. Since individual shareholdings and labor supply affect individual consumption and SDFs, their responses to idiosyncratic shocks affect equity volatility.³ As a result the effect on the equity price is different from the effect on aggregate output, which depends on the average forecast of aggregate demand instead of individual behavior.

If the idiosyncratic TFP volatility is very large, each agent mistakenly believes that any change in the TFP is driven almost entirely by an idiosyncratic shock and hence he acts as if the source of the shock is known, leading to trading behavior that resembles the choices under full information. We show that individual consumption and shareholdings follow a random walk under full information as in Graham and Wright (2010). Unlike the case under full information in which the permanent shifts cancel out in the cross-sectional aggregation, correlated estimation errors under dispersed information cause the permanent shifts in shareholdings to be transmitted into permanent shifts in equity prices in the limit when the idiosyncratic TFP volatility approaches infinity.

The unit root in the shareholding process does not necessarily generates a unit root in the equity price because investors can learn from the current and past equity prices. The learning effect may cancel out the unit root in the shareholding process. To prevent this to happen, we follow Hassan and Mertens (2017) and assume that investors are near rational and make correlated forecast errors.⁴ Then equity prices fail to aggregate individual information to the same extent as

³This intuition is similar to that in the incomplete markets models of Mankiw (1986) and Constantinides and Duffie (1996).

⁴See Hassan and Mertens (2017) for a discussion on the the psychologically founded mechanisms to generate the

in the noisy rational expectations literature. Equity prices never fully reveal the TFP information even when the idiosyncratic TFP volatility approaches infinity. Thus the learning effect does not cancel out the unit root generated by the shareholding process for near-rational investors.

We establish our preceding results by assuming that the common forecast error follows a special process so that the equilibrium can be characterized by analytic rational functions in the frequency domain or ARMA (p,q) processes in the time domain. By relaxing this assumption, we show that the equilibrium cannot be characterized by rational functions. We then resort to numerical methods using rational functions as approximations. Our numerical solutions show that, unlike output volatility, equity volatility increases quickly with idiosyncratic TFP volatility even for a very small common forecast error. Using reasonably calibrated parameter values for aggregate and idiosyncratic TFP volatilities, we show that our model can match output and equity volatilities in the data.

From a technical point of view, we apply a two-step spectral factorization method in Rozanov (1967) to solve economic problems with non-square signal systems, which is of independent interest. We first derive the Wold representation and then apply the Wiener-Hopf prediction formula to compute conditional expectations. The rest of our solution method follows the classical approach to solving linear rational expectations models (e.g., Whiteman (1983), Kasa, Walker, and Whiteman (2014), Rondina and Walker (2015), and Tan and Walker (2015)). Our procedure extends the existing literature on models with private information to non-square environments with more underlying shocks than signals. The restriction that the numbers of signals and shocks are the same is quite limited in application. Given this restriction, equilibrium will be fully revealing unless there is non-invertibility from signals to shocks.

Our approach complements the state-space approach applied by Huo and Takayama (2015) to deal with non-square signal systems. Their approach is numerically convenient since it can be solved using fast Riccati equation methods and the Kalman filter. One drawback is that it is difficult to find an analytical solution for high-dimensional systems because the Riccati equation typically does not admit an analytical solution. By contrast, our approach is constructive and can deliver analytical solutions in a much wider range of models.

Our paper is related to two strands of the literature. The first strand is on asset pricing under dispersed information (e.g., Bacchetta and van Wincoop (2008), Kasa, Walker, and Whiteman (2014), and Rondina and Walker (2015)).⁵ Bacchetta and van Wincoop (2008) argue that equity volatility is reduced under dispersed information and higher-order expectations. Using the frequency domain methods, Kasa, Walker, and Whiteman (2014) show that equity volatility can be near rational behavior.

⁵Our paper is also related to the literature on noisy rational expectations models, which is too large for us to cite all related papers. Important contributions include Grossman and Stiglitz (1980), Hellwig (1980), Kyle (1985), Wang (1994), Tsyvinski, Mukherji, and Hellwig, (2006), Bernhardt, Seiler, and Taub (2010), Albagli, Hellwig and Tsyvinski (2015), and Albuquerque and Miao (2015).

excessively high given two-types of agents. Their intuition is in a similar spirit of Harrison and Kreps (1978) and Scheinkman and Xiong (2003), in which higher-order beliefs lead to speculative bubbles. Our paper differs from this literature in three important ways. First, in our environment with a continuum of informationally small agents, higher-order beliefs do not lead to high volatilities per se. In fact, they dampen aggregate output volatility rather than amplifying it. It is the higher-order beliefs about the average forecast of individual SDFs that generates massive fluctuations in the financial market. Second, all these papers study endowment economies in which consumption and dividends are exogenously given. They cannot address the issue of why macroeconomic quantities are too smooth relative to equity prices. Finally, many papers in this literature assume constant exogenous SDFs, whereas our SDFs are endogenous, heterogeneous, and time-varying; as noted already, this feature is key for our result.

Our paper is also related to the large literature that incorporates dispersed information into macroeconomics.⁶ Important recent papers include Lorenzoni (2009), Angeletos and La'o (2010, 2013), and Benhabib, Wang, and Wen (2015). Angeletos and La'o (2013) informally argue that dispersed information on its own right may dampen output volatility. Our contribution is to formally prove this result under general assumptions in the frequency domain. Benhabib, Wang, and Wen (2015) show that endogenous information can arise when each firm observes an endogenous private signal about its demand, which in turn depends on the behavior of other firms. This literature typically focuses on business cycle dynamics instead of asset price volatility.

Three recent papers consider both business cycles and asset prices. Benhabib, Liu, and Wang (2016) build an overlapping-generations model to show that exuberant financial market sentiments of high output and high demand for capital increase the price of capital, which signals strong fundamentals of the economy to the real side and consequently leads to an actual boom in real output and employment. Hassan and Mertens (2014, 2017) introduce dispersed information into dynamic stochastic general equilibrium models with physical capital. They use an approximation method in the time domain to solve their models numerically. Hassan and Mertens (2014) introduce noise traders to prevent equilibrium from fully revealing, while Hassan and Mertens (2017) replace noise traders with near rational traders who make small correlated errors. As in Hassan and Mertens (2017), common forecast errors play an important role in our paper. While Hassan and Mertens (2017) focus on the welfare implications of the near-rational behavior, we focus on the output and equity volatilities. Finally, unlike their analysis in the time domain, our novel analytical solutions and limiting results are transparently derived in the frequency domain.

⁶See Angeletos and Lian (2016) for a survey.

2 Basic Intuition

We use a simple two-period model of an endowment economy to illustrate the basic intuition behind our analysis. Suppose that there is a continuum of agents indexed by $i \in I = [0, 1]$ who trade a single stock with a unit supply in period 1. The stock pays random dividends D in period 2. Each agent i is endowed with one unit of the stock and random labor income L_i in period 1. He derives utility from consumption C_{i1} and C_{i2} in the two periods according to the function

$$\mathcal{E}_i \left[\frac{C_{i1}^{1-\gamma}}{1-\gamma} + \beta \frac{C_{i2}^{1-\gamma}}{1-\gamma} \right],$$

where \mathcal{E}_i denotes the subjective expectation operator given agent i 's information, $\beta \in (0, 1)$ is the subjective discount factor, and γ is the risk aversion parameter. His budget constraints are given by

$$C_{i1} + QS_i = Q + L_i, \quad C_{i2} = DS_i,$$

where Q and S_i denote the stock price and shareholdings, respectively.

Suppose that dividends and labor income satisfy

$$\log D = \log \bar{D} + x_d \varepsilon_a, \quad \log L_i = \log \bar{L} + x_l \varepsilon_i,$$

where \bar{D} , x_d , \bar{L} , and x_l are exogenous constants, and ε_a and ε_i are independent random normal variables with means zero and variances σ_a^2 and σ_i^2 . Suppose that the labor income shock is purely idiosyncratic such that $\int_I \varepsilon_i di = 0$.

At the beginning of period 1, each agent i receives an exogenous signal $x_i = \varepsilon_a + \varepsilon_i$, but does not observe ε_a and ε_i separately. All agents do not communicate their signals with each other. Based on his private signal x_i and the equity price Q , each agent i trades on the stock market and chooses consumption C_{i1} . At the beginning of period 2, the random labor income and dividends are realized and agent i chooses consumption C_{i2} out of dividend income. In equilibrium $\int_I S_i di = 1$. It is straightforward to show that the deterministic equilibrium when $\varepsilon_a = \varepsilon_i = 0$ is given by

$$\bar{S}_i = 1, \quad \bar{C}_{i1} = \bar{L}, \quad \bar{C}_{i2} = \bar{D}, \quad \bar{Q} = \beta (\bar{L}/\bar{D})^\gamma \bar{D}.$$

In the stochastic case agent i ' utility maximization leads to the Euler equation

$$Q = \mathcal{E}_i [M_i D].$$

where $M_i = \beta (C_{i1}/C_{i2})^\gamma$ denotes the stochastic discount factor (SDF). Following Hassan and Mertens (2017), we assume that each agent makes a small error when forming his expectation. Specifically, let

$$\mathcal{E}_i(\cdot) = \mathbb{E}_i(\cdot) U_i,$$

where \mathbb{E}_i denotes the rational expectation operator conditional on agent i 's information $\{x_i, Q\}$ and U_i is a small exogenous error that shifts his conditional expectations. Assume that

$$\log U_i = u + v_i,$$

where u and v_i are independent normal random variables with means zero and variances σ_u^2 and σ_v^2 . Here u represents aggregate errors and v_i represents idiosyncratic errors satisfying $\int_I v_i di = 0$. When $U_i = 1$ for all i , agents have full rational expectations.

Introducing near rational forecast errors in the model injects additional noise into the equity price, which prevents information from fully revealing. In the literature, there are many candidate shocks available to serve this purpose, e.g., a noise trader shock to the asset supply. Small deviations from the optimal forecasts will play an important role in the dynamic setting where interactions between the stock price and trading behavior becomes the key to understand the stock price fluctuations.

Now we log-linearize the stochastic equilibrium around the deterministic equilibrium and use a lower case variable to denote its log deviation from its deterministic equilibrium value. We then obtain the log-linearized Euler equation

$$q = \mathbb{E}_i[d] + \mathbb{E}_i[m_i] + u + v_i, \quad m_i = \gamma(c_{i1} - c_{i2}). \quad (1)$$

Next we substitute the log-linearized budget constraints into the SDF and use the Euler equation to derive the log-linearized trading strategy

$$s_i = \frac{\mathbb{E}_i[(1 - \gamma)d] - q + u + v_i}{\gamma(1 + \bar{Q}/\bar{L})} + \frac{\mathbb{E}_i[l_i]}{1 + \bar{Q}/\bar{L}}. \quad (2)$$

This expression is akin to Merton's (1969) result: the trading strategy consists of a mean-variance efficient component and a hedging component against idiosyncratic labor income.

Aggregating equation (2) over $i \in [0, 1]$ and using the log-linearized market-clearing condition $\int_I s_i di = 0$, we obtain

$$q = (1 - \gamma) \bar{\mathbb{E}}_i[d] + \gamma \bar{\mathbb{E}}_i[l_i] + u, \quad (3)$$

where $\bar{\mathbb{E}}_i[\cdot] \equiv \int \mathbb{E}_i[\cdot] di$ denotes the average expectation operator.

To solve the model, we conjecture that the equity price takes the form:

$$q = q_a \epsilon_a + q_u u, \quad (4)$$

where q_a and q_u are nonzero constants to be determined. Then the information set can be normalized to $\{\hat{q}, x_i\}$, where $\hat{q} = \epsilon_a + \frac{q_u}{q_a} u \equiv \epsilon_a + \hat{u}$. The presence of common forecast errors prevents equity prices to fully reveal the aggregate dividend information.

By the Gaussian projection theorem,

$$\mathbb{E}_i[d] = x_d(\tau_q \hat{q} + \tau_x x_i) \implies \bar{\mathbb{E}}_i[d] = \tau_q x_d(\epsilon_a + \hat{u}) + \tau_x x_d \epsilon_a, \quad (5)$$

where the noise-to-signal ratios are defined as

$$\tau_q = \frac{\sigma_a^2 \sigma_i^2}{g^2(\sigma_a^2 \sigma_u^2 + \sigma_u^2 \sigma_i^2) + \sigma_a^2 \sigma_i^2} \in (0, 1), \quad \tau_x = \frac{g^2 \sigma_a^2 \sigma_u^2}{g^2(\sigma_a^2 \sigma_u^2 + \sigma_u^2 \sigma_i^2) + \sigma_a^2 \sigma_i^2} \in (0, 1),$$

and $g \equiv q_u/q_a$ will be determined in the equilibrium.

A direct comparison of the two expectations in (5) implies that, when agents are informationally small, the variance of the market average forecast of aggregate fundamentals is smaller than that of the individual forecast in that $Var(\bar{\mathbb{E}}_i[d]) < Var(\mathbb{E}_i[d])$. It is also easy to check that $Var(\mathbb{E}_i[d]) < Var(d)$. We will show that this dampening result applies to our general dynamic model when aggregate fundamentals are endogenous (see Lemma 2). An immediate implication is that dispersed information does not help generate a large equity volatility in the risk neutral case when $\gamma = 0$. In this case it follows from (3) that equity volatility is bounded by the dividend volatility given the small variations in forecast errors u . Thus we need risk aversion $\gamma > 0$ and hence volatile SDFs.

Consider the second term on the right side of equation (3), which comes from the average forecast of individual SDFs. If agents can communicate with each other so that information is homogenous, this term will vanish $\bar{\mathbb{E}}_i[l_i] = \mathbb{E}_i[\int_I l_i di] = 0$. Under dispersed information without communication, we have

$$\mathbb{E}_i[l_i] = x_l [-\tau_q \hat{q} + (1 - \tau_x) x_i] \implies \bar{\mathbb{E}}_i[l_i] = -\tau_q x_l (\epsilon_a + \hat{u}) + (1 - \tau_x) x_l \epsilon_a. \quad (6)$$

A high equity price \hat{q} may be due to a high dividend shock ϵ_a . Agent i may believe the labor income shock ϵ_i to be low given a fixed signal $x_i = \epsilon_a + \epsilon_i$. This interprets why the coefficient of \hat{q} is negative in (6). Thus learning from prices dampens the impact of idiosyncratic shocks.

Plugging (5) and (6) into (3) yields

$$q = [(1 - \gamma)x_d \tau_a + \gamma x_l \tau_i] \epsilon_a + [(1 - \gamma)g \tau_q x_d - \gamma g \tau_q x_l + 1] u, \quad (7)$$

where $\tau_a = \tau_q + \tau_x \in (0, 1)$ and $\tau_i = 1 - (\tau_q + \tau_x)$. Matching coefficients in (4) yields a cubic equation for g :

$$[(1 - \gamma)x_d \sigma_a^2 \sigma_u^2 + \gamma x_l (\sigma_a^2 \sigma_u^2 + \sigma_u^2 \sigma_i^2)] g^3 - (\sigma_a^2 \sigma_u^2 + \sigma_u^2 \sigma_i^2) g^2 + \gamma x_l \sigma_a^2 \sigma_i^2 g - \sigma_a^2 \sigma_i^2 = 0.$$

Substituting (7) into (2) yields the equilibrium trading strategy

$$s_i = \frac{(1 - \gamma)x_d \tau_x + \gamma x_l (1 - \tau_x)}{\gamma(1 + \bar{Q}/\bar{L})} \epsilon_i + \frac{1}{\gamma(1 + \bar{Q}/\bar{L})} v_i,$$

which only responds to idiosyncratic labor income shocks and idiosyncratic forecast errors. The presence of idiosyncratic forecast errors prevents shareholdings to fully reveal the idiosyncratic labor income information.

Equation (7) shows that small common errors in forecasting leads to non-fundamental deviations in the equilibrium stock price, as emphasized by Hassan and Mertens (2017). They also show that

small common errors in household expectations weaken the stock market’s capacity to aggregate information if information is dispersed. We argue that their results rely on the average forecast of the aggregate shock. Its impact corresponds to the response coefficient τ_a in (7). By contrast, our model features uninsured idiosyncratic labor income shocks and hence the equilibrium equity price also depends on the average forecast of these shocks. The impact corresponds to the response coefficient τ_i in (7).

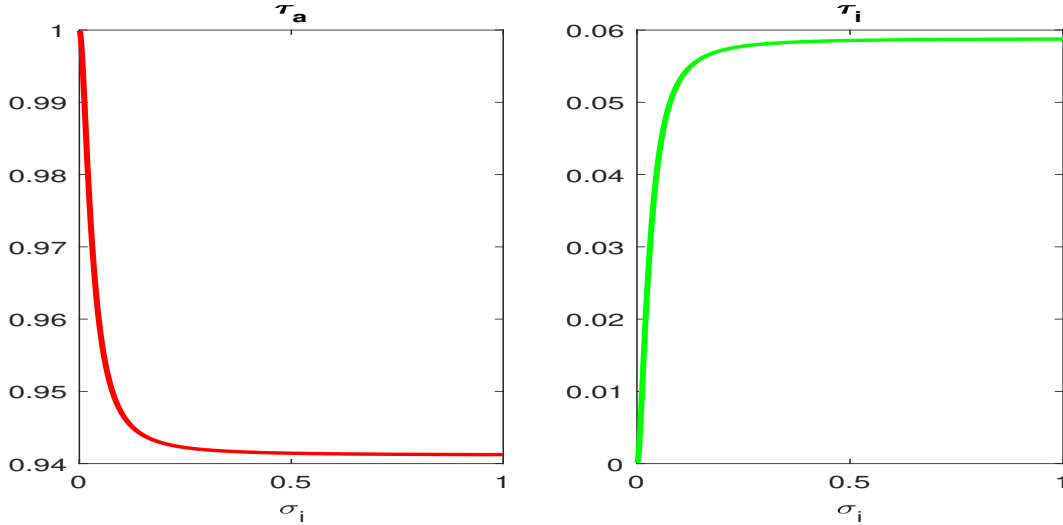


Figure 1: The impact of idiosyncratic volatility σ_i on τ_a and τ_i . Parameter values are $x_l = x_d = 1$, $\gamma = 0.4$, $\sigma_a = 0.1$, and $\sigma_u = 0.01$.

To relate to the results in Hassan and Mertens (2017), we consider the impact of σ_i on τ_a and τ_i , illustrated in Figure 1. The figure shows that τ_a decreases with σ_i as in Hassan and Mertens (2017). However, τ_i increases with σ_i . Intuitively, when agents are unable to distinguish between the aggregate and idiosyncratic shocks and when they make errors in forecasting, the equilibrium price is not fully revealing and agents have to solve a signal extraction problem. When σ_i is higher, the agents put a larger weight on the idiosyncratic labor income shock and a smaller weight on the aggregate dividend shock. However, the additional volatility due to the large idiosyncratic shock only has a limited effect since $\tau_i \in (0, 1)$. Even if idiosyncratic shocks are arbitrarily volatile, aggregation cancels them out and τ_i approaches the upper bound of one. Unless we assume a very high value of x_l , the quantitative effect on equity prices will be small.

In the next section we extend this simple example to an infinite-horizon setup. We will endogenize labor income and dividends by introducing the production side of the economy so that x_d and x_l are endogenous. In the infinite-horizon model individual SDFs depend on future individual consumption which in turn depends on future trading strategies and labor income. Thus equity prices depend on the higher-order beliefs about the average forecasts of future individual shareholdings

and labor income. Interpreted through the lens of the two-period model, this dynamic interaction makes shareholdings and equity prices highly persistent and generates a positive connection between σ_i and x_i that causes equity volatility to increase without bound as $\sigma_i \rightarrow \infty$.

3 Model

We consider a variation of the classical dispersed-information (real) business cycle models of Lorenzoni (2009) and Angeletos and La'O (2010, 2013). The economy consists of a continuum of islands with a Lebesgue measure over $I = [0, 1]$. Information is dispersed across islands. There is a representative household and a representative firm on each island. Each firm is monopolistically competitive and produces a specialized good using labor input only, while households have Dixit-Stiglitz preferences over varieties. Labor is immobile across islands, but consumption goods of all varieties are freely mobile. The equity market is operated through a mutual fund which owns the firms and issues equity shares to households. The stock price therefore reflects the average valuation of firms in the economy. We normalize the aggregate stock supply to one.

3.1 Households

A representative household on each island $i \in I$ derives utility from the composite good consumption $\{C_{it}\}$ and labor supply $\{N_{it}\}$ according to the utility function of Greenwood, Huffman, and Hercowitz (1988):

$$\mathcal{E}_i \left[\sum_{t=0}^{\infty} \beta^t \log \left(C_{it} - \frac{N_{it}^{1+\phi}}{1+\phi} \right) \right],$$

where \mathcal{E}_i denotes household i 's subjective expectation operator, $\beta \in (0, 1)$, $\phi > 0$,

$$C_{it} = \left[\int_I C_{it}(j)^{\frac{\varsigma-1}{\varsigma}} dj \right]^{\frac{\varsigma}{\varsigma-1}},$$

and $C_{it}(j)$ denotes the consumption of good j demanded by the household on island i . Here $\varsigma > 1$ denotes the inter-island elasticity of substitution that determines the degree of strategic complementarity.

The household faces the following intertemporal budget constraint

$$\int_I C_{it}(j) P_t(j) dj + Q_t S_{it+1}^h = S_{it}^h (Q_t + D_t) + W_{it} N_{it}, \quad (8)$$

where $P_t(j)$, Q_t , S_{it}^h , D_t , and W_{it} represent the price of good j , the stock price, share holdings, aggregate dividends, and the wage rate in island i , respectively.

To simplify the forecasting problem, assume that each household i consists of two family members, an investor and a shopper. They have different information sets and do not communicate with each other. In each period t the investor's information set consists of the current and past

TFP shocks A_{it} , wages W_{it} , and stock prices Q_t . Given this information set, the investor chooses labor supply and shareholdings. The first-order conditions are given by

$$W_{it} = N_{it}^\phi, \quad (9)$$

$$\mathcal{E}_{it} [M_{it+1} (Q_{t+1} + D_{t+1})] = Q_t, \quad (10)$$

where the SDF M_{it+1} is given by

$$M_{it+1} = \frac{\beta \left(C_{it} - N_{it}^{1+\phi} / (1 + \phi) \right)}{C_{it+1} - N_{it+1}^{1+\phi} / (1 + \phi)}.$$

Our adopted utility function implies that the labor supply in (9) is independent of C_{it} and hence simplifies our analysis, but it is not crucial for our main results (see Appendix A).

Assume that investors are near rational and each investor i 's subjective expectations satisfy

$$\mathcal{E}_{it} [\cdot] = \mathbb{E}_{it} [\cdot] U_{it},$$

where \mathbb{E}_{it} denotes the rational expectation operator conditional on the investor's information at time t and U_{it} is a small exogenous error that shifts the subjective conditional expectations. Let U_{it} satisfy

$$\log U_{it} = u_t + v_{it}, \quad (11)$$

where the aggregate component u_t satisfies

$$u_t = u(L)\epsilon_{ut}, \quad (12)$$

and the idiosyncratic component satisfies

$$\int_I v_{it} di = 0. \quad (13)$$

Here $u(L)$ is a square-summable, one-sided lag polynomial and ϵ_{ut} and v_{it} are identically and independently distributed random variables drawn from the normal distributions with means zero and variances σ_u^2 and σ_v^2 .

The shopper collects dividends D_t and purchases consumption good $C_{it}(j)$ after observing the product prices $P_t(j)$ for all j and the aggregate price level P_t . He does not face a forecasting problem and the first-order condition is

$$C_{it}(j) = \left[\frac{P_t(j)}{P_t} \right]^{-\varsigma} C_{it}, \quad (14)$$

where the aggregate price index

$$P_t \equiv \left[\int_I P_t(j)^{1-\varsigma} di \right]^{\frac{1}{1-\varsigma}}$$

satisfies

$$\int_I C_{it}(j) P_t(j) dj = P_t C_{it}.$$

We normalize the price index P_t to one so that the budget constraint (8) becomes

$$C_{it} + Q_t S_{it+1}^h = S_{it}^h (Q_t + D_t) + W_{it} N_{it}. \quad (15)$$

Aggregating (14) over $i \in I$ yields the total demand for good $j \in [0, 1]$,

$$Y_{jt} = \int_I C_{it}(j) di = [P_t(j)]^{-\zeta} Y_t, \quad (16)$$

where Y_t denotes aggregate consumption

$$Y_t = \int_I C_{it} di \equiv C_t. \quad (17)$$

3.2 Firms

The representative firm in island $i \in [0, 1]$ has a production function

$$Y_{it} = A_{it} N_{it}^\alpha, \quad \alpha \in (0, 1), \quad (18)$$

where A_{it} satisfies

$$A_{it} = A_t \exp(\epsilon_{it}). \quad (19)$$

Here A_t represents the aggregate component that affects all firms in all islands and ϵ_{it} represents the idiosyncratic component that is independent of A_t and affects the firm in island i only. Investors on island i observe A_{it} at time t , but cannot distinguish between the aggregate and idiosyncratic components. Let

$$\log A_t = a(L)\epsilon_{at}, \quad (20)$$

where ϵ_{at} and ϵ_{it} are identically and independently distributed over time and drawn from the normal distributions with means zero and variances σ_a^2 and σ_i^2 , respectively. Here $a(L)$ denotes a one-sided, square-summable lag polynomial. Moreover, assume that the law of large number (LLN) holds for ϵ_{it} so that

$$\int_I \epsilon_{it} di = 0. \quad (21)$$

In each period t the firm's information set consists of the current and past TFP shocks A_{it} , wages W_{it} , and stock prices Q_t . Given this information set it solves the static profit maximization problem

$$\pi_{it} = \max_{N_{it}} \mathbb{E}_{it} [P_t(i)] Y_{it} - W_{it} N_{it}$$

subject to the demand schedule in (16) for $j = i$. Since the production and labor demand choice is made before observing the output price $P_t(i)$, the firm needs to form static conditional expectation

about the price $P_t(i)$ given the infinite history of signals. Since Y_{it} and N_{it} are observable choice variables, the firm essentially forms conditional expectations about the aggregate demand Y_t . Simple algebra yields the labor demand condition

$$\alpha \left(1 - \frac{1}{\varsigma}\right) \frac{Y_{it}^{(1-\frac{1}{\varsigma})} \mathbb{E}_{it} \left[Y_t^{\frac{1}{\varsigma}} \right]}{N_{it}} = W_{it}. \quad (22)$$

For simplicity we assume that firms are fully rational and do not make forecasting errors. Introducing forecasting errors affects profits π_{it} and hence dividends and stock prices. The firms' forecasting errors play a similar role to the households' forecasting errors reflected by equation (10).

It follows from equations (18) and (22) that observing the local wage W_{it} is equivalent to observing the local productivity shock A_{it} . Thus we can write the information set in the conditional expectation operators \mathbb{E}_{it} and \mathcal{E}_{it} as $\{X_{i,t-k}\}_{k=0}^{\infty}$, where the signal vector is $X_{it} = [A_{it}, Q_t]^T$.

3.3 Equilibrium Characterization in the Time Domain

There is one aggregate mutual fund that issues equity shares and collects dividends from individual islands. The aggregate dividend satisfies $D_t = \int_I \pi_{it} di$ and aggregate output satisfies $Y_t = \int_I Y_{it} di$. The mutual fund distributes the dividend to households. The market-clearing condition for the stock is given by

$$\int_I S_{it+1}^h di = 1, \forall t \quad (23)$$

-

A competitive equilibrium with dispersed information is characterized by a system of 9 equations (9), (10), (14), (15), (16), (17), (18), (22), and (23) for 9 variables W_{it} , N_{it} , S_{it}^h , C_{it} , $C_{it}(j)$, Y_{it} , $P_t(j)$, Q_t , and Y_t , where D_t satisfies

$$\int_I W_{it} N_{it} di + D_t = Y_t. \quad (24)$$

This equation follows from aggregating (15) using (17) and (23).

Since the equilibrium system is nonlinear and does not admit an explicit solution, we follow the standard method in the literature on dispersed information and derive a log-linearized system (see Appendix A). We use a lower case variable to denote the log deviation from the non-stochastic steady state. We impose the following assumption on the parameters so that there exists a unique deterministic steady-state equilibrium.

Assumption 1 *The parameter values satisfy $\alpha, \beta \in (0, 1)$, $\phi > 0$, $\varsigma > 1$.*

We first use (9), (22), and (18) to eliminate W_{it} and N_{it} to derive

$$y_{it} = \frac{1}{\xi} a_{it} + \theta \mathbb{E}_{it} [y_t], \quad (25)$$

and

$$y_{it} = a_{it} + \alpha n_{it}, \quad (26)$$

where we define

$$\xi \equiv \frac{1 + \phi - \alpha(1 - 1/\varsigma)}{1 + \phi} > 0, \quad \theta \equiv \frac{\alpha}{\alpha + (1 - \alpha + \phi)\varsigma} \in (0, 1).$$

The parameter θ describes the degree of strategic complementarity (see Angeletos and La'O (2013) and Huo and Takayama (2015)). Aggregating (25) over $I = [0, 1]$, we have

$$y_t = \frac{1}{\xi} \int_I a_{it} di + \theta \bar{\mathbb{E}}_{it} [y_t], \quad (27)$$

where the average conditional expectation operator is defined as

$$\bar{\mathbb{E}}_{it} [\cdot] \equiv \int_I \mathbb{E}_{it} [\cdot] di.$$

Log-linearizing (10) and (15) yields

$$q_t = \mathbb{E}_{it} [m_{it+1}] + \mathbb{E}_{it} [\beta q_{t+1} + (1 - \beta) d_{t+1}] + u_t + v_{it}, \quad (28)$$

where

$$\mathbb{E}_{it} [m_{it+1}] = \alpha_2 s_{it}^h - \alpha_1 s_{it+1}^h + \mathbb{E}_{it} [\alpha_3 s_{it+2}^h + \Delta b_{it}], \quad (29)$$

and

$$b_{it} = \alpha_4 d_t + \alpha_5 n_{it}, \quad \Delta b_{it+1} \equiv b_{it} - b_{it+1}. \quad (30)$$

Notice that s_{it}^h and s_{it+1}^h are in agent i 's information set at time t . Unlike the two-period model, agent i 's Euler equation depends on his future consumption so that his expected SDF depends on his forecast of his future shareholdings, labor income, and dividends. Using (9) and (24) we obtain

$$\alpha_6 d_t + \alpha_7 n_t = y_t, \quad (31)$$

where $n_t = \int_I n_{it} di$. Here the coefficients $\alpha_1, \alpha_2, \dots, \alpha_7$ are defined in Appendix A. Define the parameter $\lambda_s \equiv \alpha_2/\alpha_1$. In Appendix A we show the following lemma, which is important for our unit root results and also holds for general utility functions.

Lemma 1 *Under Assumption 1, $\alpha_1, \alpha_2, \dots, \alpha_7 > 0$, $\lambda_s \equiv \alpha_2/\alpha_1 \in (1/2, 1)$, and $\alpha_1 = \alpha_2 + \alpha_3$.*

Aggregating (28) and using (23) and (29), we show that equity prices satisfy

$$q_t = \bar{\mathbb{E}}_{it} [\alpha_3 s_{it+2}^h + \Delta b_{it+1}] + \bar{\mathbb{E}}_{it} [\beta q_{t+1} + (1 - \beta) d_{t+1}] + u_t. \quad (32)$$

The first term on the right-hand side of the second equality is the average forecast of the individual SDFs, which depend on future aggregate dividends, individual shareholdings, and individual labor

income. Iterating (32) forward, we find that the equity price is determined by an infinite number of forward-looking higher-order expectations about aggregate dividends and individual shareholdings and labor income.

In summary, we characterize the log-linearized equilibrium by a system of 6 equations (25), (26), (27), (28), (31), and (32) for 6 variables $y_{it}, n_{it}, y_t, s_{it}^h, d_t$, and q_t . We are looking for causal covariance stationary equilibrium processes.

3.4 Full Information Benchmark

Before solving for the equilibrium under dispersed information, we present the equilibrium under full information. In this case all agents have the same information about all shocks. They have rational expectations except when forecasting future stock market conditions because they make small forecast errors due to the near rational shock. Hence equations (27) and (32) become

$$y_t = \frac{1}{\xi} a_t + \theta \mathbb{E}_t [y_t], \quad (33)$$

$$q_t = \mathbb{E}_t [\Delta b_{t+1}] + \mathbb{E}_t [\beta q_{t+1} + (1 - \beta) d_{t+1}] + u_t, \quad (34)$$

where \mathbb{E}_t denotes the rational expectation operator given all available information. It follows that

$$c_t^{FI} = y_t^{FI} = \frac{1}{(1 - \theta) \xi} a_t, \quad (35)$$

where a variable with a superscript ‘‘FI’’ denotes its full information value.

We then use (25) and (31) to derive

$$n_t^{FI} = \frac{1 - (1 - \theta) \xi}{\alpha (1 - \theta) \xi} a_t, \quad d_t^{FI} = \frac{\alpha - \alpha_7 [1 - (1 - \theta) \xi]}{\alpha \alpha_6 (1 - \theta) \xi} a_t.$$

Applying the method of undetermined coefficients and the Hansen and Sargent (1980) prediction formula to (34) yields

$$q_t^{FI} = c_t^{FI} + \frac{u(L)L - \beta u(\beta)}{L - \beta} \epsilon_{ut}.$$

Thus, given a small forecast error u_t , the model under full information cannot simultaneously generate smooth consumption (output) and highly volatile equity prices. For example, $q_t^{FI} = c_t^{FI} + u_t$ when $u_t = \epsilon_{ut}$.

Next we investigate individual trading behavior, which lies on the heart of our model mechanism. A subtle but important observation in the full information case is that the processes of individual consumption and shareholdings contain a unit root. Applying the method of undetermined coefficients to (28) under full information and using Lemma 1 yield

$$s_{it+1}^{h,FI} = s_{it}^{h,FI} + \chi_s \epsilon_{it} + \frac{1}{\alpha_2} v_{it}, \quad \chi_s \equiv \frac{\alpha_5 (1/\xi - 1)}{\alpha \alpha_2}.$$

This in turn implies that individual consumption possesses contain a random walk component using the log-linearized budget constraint:

$$c_{it}^{FI} = c_{it-1}^{FI} + y_t^{FI} - y_{t-1}^{FI} + \chi_c \epsilon_{it} + \left(\frac{D}{C} \chi_s - \chi_c \right) \epsilon_{it-1} - \frac{Q}{\alpha_2 C} v_{it} + \frac{Q+D}{\alpha_2 C} v_{it-1},$$

where $\chi_c \equiv \frac{WN}{C}(1+\phi)\frac{1/\xi-1}{\alpha} - \chi_s \frac{Q}{C}$, and W , N , Q , D , and C are steady state values given in Appendix A.

This result is similar to that in Graham and Wright (2010), where the LLN condition (21) and the full-information assumption ensure that permanent shifts in idiosyncratic consumption and shareholdings cancel out in the aggregate. In particular,

$$\int_I \mathbb{E}_t [s_{it+2}^h] di = \mathbb{E}_t \int_I [s_{it+2}^h] di = 0.$$

Under dispersed information, however, such interchange of integration operators is invalid because agents have different information sets, and the interconnection between shareholding choices and the equity price leads to our key results for the financial market.

4 Business Cycle Volatility

In this section we show that output volatility under dispersed information is lower than that under full information. This result may seem counterintuitive because speculation due to dispersed information might be expected to generate high volatility. We will show that our result is quite general and can be established without explicitly solving the model.

To analyze the log-linearized equilibrium system under dispersed information, we need to deal with the problem of forecasting the forecast of others as revealed by equations (27) and (32). To see this point, iterating (27) yields

$$y_t = \frac{1}{\xi} \sum_{k=0}^{\infty} \theta^k \overline{\mathbb{E}}_{it}^{(k)} \left[\int_I a_{it} di \right] + \lim_{k \rightarrow \infty} \theta^k \overline{\mathbb{E}}_{it}^{(k)} [y_t],$$

where the k -order average expectation is the repeated integral

$$\overline{\mathbb{E}}_{it}^{(k)} [\cdot] = \underbrace{\int_I \mathbb{E}_{it} \int_I \mathbb{E}_{it} \cdots \int_I \mathbb{E}_{it} [\cdot] di \cdots di}_k.$$

Under homogeneous information, higher-order expectations collapse to first-order expectations. Under dispersed information, aggregate output depends on an infinite number of higher-order expectations. Solving these higher-order expectations in the time domain is challenging. Therefore we adopt the frequency domain approach discussed in Appendices E and F.

Conjecture that the solution for output in island i takes the following form

$$y_{it} = M_y^a(L) \epsilon_{at} + M_y^i(L) \epsilon_{it} + M_y^u(L) \epsilon_{ut}, \quad (36)$$

where the corresponding z -transforms $M_y^a(z)$, $M_y^u(z)$, and $M_y^i(z)$ are some analytic functions in $\mathbf{H}^2(\mathbb{D})$.⁷ Then aggregate output satisfies

$$y_t = \int_I y_{it} di = M_y^a(L) \epsilon_{at} + M_y^u(L) \epsilon_{ut}. \quad (37)$$

We first present a lemma characterizing the property of the variance of higher-order expectations, which is central for determining business cycle volatility when information is dispersed.

Lemma 2 *Under Assumption 1, we have*

$$\text{Var}(\overline{\mathbb{E}}_{it}[y_t]) < \text{Var}(\mathbb{E}_{it}[y_t]) \leq \text{Var}(y_t).$$

The second inequality is merely the orthogonality condition associated with the conditional expectation. The nontrivial part is the first inequality. It shows that the variance of the average expectations about aggregate output is smaller than the variance of individual expectations about aggregate output, when individual agents' effect on the aggregate equilibrium is infinitesimal so that the LLN can be applied. This feature is in sharp contrast with models that assume finitely-many uninformed agents, such as Kasa, Walker, and Whiteman (2014).

Using the preceding lemma, we show in Appendix B that the business cycle volatility is dampened under dispersed information relative to a full-information environment.

Theorem 1 *Under Assumption 1, the variance of output under dispersed information is bounded above by the variance under full information*

$$\text{Var}(y_t^{FI}) > \text{Var}(y_t).$$

The proof is a simple application of the triangle inequality in Hilbert spaces. Note that this theorem is applicable to general information structures, exogenous or endogenous, univariate or multivariate. Adding confidence, noise, or higher-order sentiment shocks would also not change the results. This is because the critical step in the proof is Lemma 2, which applies to general high-dimensional non-square signals. Bacchetta and van Wincoop (2008) and Angeletos and La'O (2013) infer similar results based on the variance bound in the time domain without a formal proof, but our theorem is the first formal statement of the result and its proof uses the frequency domain methods.

Contrary to the common intuition, here the presence of higher-order beliefs and the forecasting the forecasts of others problem dampens business cycle fluctuations. To understand the economic rationale behind this result, we note that the effect of dispersed information and higher-order expectations works through two channels. The first channel is associated with slow learning of the

⁷Here $\mathbf{H}^2(\mathbb{D})$ denotes the Hardy space for the open unit disk \mathbb{D} of the complex space and $\|\cdot\|_{\mathbf{H}^2}$ denotes its norm. See Appendix E.

unobserved states. Slow learning creates inertia in endogenous variables, and more importantly in the higher-order average expectations of endogenous and exogenous variables, which leads to low volatility. The second channel is associated with the forecasting the forecasts of others. Agents have a speculative motive if other agents overreact to news. This channel is strong for informationally-influential participants in models with finitely many agents (Kasa, Walker, Whiteman (2014)). It is also at work in the heterogeneous prior setup (Scheinkman and Xiong (2003)). When each agent is informationally negligible as in our model, the second channel completely vanishes since there is no need to forecast any particular agent's forecast. What matters is the forecast of the average. Thus the first channel dominates and leads to the volatility bounds we deliver above.

5 Equity Price Volatility

We now turn to the financial side of the model. The main result of this section is that equity volatility will converge to infinity as the variance of the idiosyncratic TFP shock converges to infinity. In contrast to the previous section, we need to derive an explicit model solution to establish this result. We will also prove the existence and uniqueness of equilibrium by extensively using the frequency domain methods described in Appendixes E and F.

5.1 Equilibrium Solution

We rewrite (32) as

$$q_t = \int_I \chi_{it} di + u_t, \quad (38)$$

where we define

$$\chi_{it} \equiv \mathbb{E}_{it} [\alpha_3 s_{it+2}^h + \Delta b_{it}] + \mathbb{E}_{it} [\beta q_{t+1} + (1 - \beta) d_{t+1}]. \quad (39)$$

The information set consists of the history of signals $X_{it} = [a_{it}, q_t]^\top$. Conjecture that

$$\chi_{it} = \pi_1(L) a_{it} + \pi_2(L) q_t, \quad (40)$$

where the analytic functions $\pi_1(z)$ and $\pi_2(z)$ are endogenously determined in $\mathbf{H}^2(\mathbb{D})$.

It follows from (38) that

$$q_t = \frac{\pi_1(L) a(L)}{1 - \pi_2(L)} \epsilon_{at} + \frac{u(L)}{1 - \pi_2(L)} \epsilon_{ut}. \quad (41)$$

Intuitively, the lag polynomial $\pi_1(L)$ characterizes how the dispersed information about TFP shocks affects equity prices, while $\frac{1}{1 - \pi_2(L)}$ characterizes the impact of endogenous learning from equity prices.

To verify the conjecture in (40), we use (41) and the Wiener-Hopf prediction formula to compute the conditional expectations in (40). To apply this formula, we write the signal representation as

$$X_{it} = H(L)\eta_{it} \equiv \begin{bmatrix} a(L) & 1 & 0 \\ \frac{\pi_1(L)a(L)}{1-\pi_2(L)} & 0 & \frac{u(L)}{1-\pi_2(L)} \end{bmatrix} \begin{bmatrix} \epsilon_{at} \\ \epsilon_{it} \\ \epsilon_{ut} \end{bmatrix}, \quad (42)$$

which is a non-square system containing endogenous functions. To derive transparent analytical solutions, we impose the following assumption:

Assumption 2 *Let $u(z) = \pi_1(z)$ and $a(z) = 1$.*

The assumption of IID TFP shocks is for simplicity and can be easily relaxed. The assumption of $u(z) = \pi_1(z)$ follows from Taub (1989) and Rondina and Walker (2015). It is crucial to simplify the computation of the spectral factorization and the Wold representation for the preceding non-square signal system. We can then express equilibrium conditions as a system of linear functional equations for $\pi_1(z)$ and $\pi_2(z)$, allowing us to establish the equilibrium existence and uniqueness and analyze the key model mechanism transparently in the frequency domain. In the next section we relax Assumption 2 and derive numerical results.

Conjecture that the equilibrium individual shareholdings satisfy

$$s_{it+1}^h = M_s^i(L) \epsilon_{it} + M_s^v(L) v_{it}, \quad (43)$$

where $M_s^i(z), M_s^v(z) \in \mathbf{H}^2(\mathbb{D})$. Individual shareholdings only responds to idiosyncratic TFP shocks and idiosyncratic forecast errors. The following result delivers the link between equity prices and individual shareholdings.

Lemma 3 *Under Assumptions 1 and 2, we have*

$$M_s^i(z) = \frac{\pi_1(z)}{\alpha_1 - \alpha_2 z}. \quad (44)$$

This lemma shows that an investor's shareholdings exposure to the idiosyncratic TFP shock is closely related to the equity price exposure to the aggregate TFP shock due to investor's dispersed information about the two components of the shocks. Thus, if investors make large adjustments of their shareholding positions, the response of equity prices to the aggregate TFP shock will also be large. However, this relation vanishes under full-information as analyzed in Section 3.4, because the cross-sectional aggregation neutralizes the impact of individual trading decisions on equity prices.

Theorem 2 *Under Assumptions 1 and 2, there is a unique equilibrium under dispersed information in which $\pi_1(z)$ and $\pi_2(z)$ are rational analytical functions if the function $\frac{\pi_1(z)}{1-\pi_2(z)} \in \mathbf{H}^2(\mathbb{D})$ has no roots in the open unit disk.*

In Appendix C we provide an explicit solution to the equilibrium. In particular the equilibrium can be characterized by rational analytic functions $\pi_1(z)$ and $\pi_2(z)$ in the closed unit disk, which corresponds to ARMA(p,q) representations in the time domain. Despite the presence of the infinite number of higher-order expectations formed by agents, the ARMA(p,q) representation allows us to compute the equity price volatility in closed-form via the integral method using the Parseval theorem. More importantly, the explicit expression also highlights some crucial analytical properties of the equity price fluctuations under dispersed information. We are particularly interested in the limit property as $\sigma_i \rightarrow \infty$.

5.2 Equity Volatility

We decompose the equity price in (41) as $q_t = q_t^f + q_t^n$, where

$$q_t^f \equiv \frac{\pi_1(L)}{1 - \pi_2(L)} \epsilon_t^a \text{ and } q_t^n = \frac{u(L)}{1 - \pi_2(L)} \epsilon_{ut}$$

represent the components driven by the fundamental TFP shock and the common forecast error, respectively. In Appendix C we prove the following result.

Theorem 3 *Under Assumptions 1 and 2, we have*

$$\lim_{\sigma_i \rightarrow \infty} \pi_1(1) = \infty,$$

and

$$\lim_{\sigma_i \rightarrow \infty} \text{Var} \left(q_t^f \right) = \sigma_a^2 \lim_{\sigma_i \rightarrow \infty} \left\| \frac{\pi_1(z)}{1 - \pi_2(z)} \right\|_{\mathbf{H}^2}^2 = \infty. \quad (45)$$

This theorem is somewhat surprising. Although idiosyncratic TFP shocks have no effect on the movement of the equity price, the equity price becomes arbitrarily volatile as the volatility of the idiosyncratic shock approaches infinity for any finite $\sigma_a > 0$ and $\sigma_u > 0$. Therefore, our model has the potential to generate a highly volatile equity price, because the idiosyncratic TFP volatility is much larger than the aggregate TFP volatility in the data.

To understand the economic mechanism generating the high equity price volatility, we rewrite (32) as

$$q_t = \int_I \mathbb{E}_{it} [\beta q_{t+1} + (1 - \beta) d_{t+1}] di + \int_I \mathbb{E}_{it} m_{it+1} di + u_t,$$

where we can show that

$$\int_I \mathbb{E}_{it} [m_{it+1}] di = \int_I \mathbb{E}_{it} [\alpha_3 s_{it+2}^h + \Delta b_{it+1}] di. \quad (46)$$

Iterating forward gives

$$\begin{aligned} q_t &= (\bar{\mathbb{E}}_{it} [m_{it+1}] + \beta \bar{\mathbb{E}}_{it} \bar{\mathbb{E}}_{it+1} [m_{it+2}] + \beta^2 \bar{\mathbb{E}}_{it} \bar{\mathbb{E}}_{it+1} \bar{\mathbb{E}}_{it+2} [m_{it+3}] + \dots) \\ &\quad + (1 - \beta) (\bar{\mathbb{E}}_{it} [d_{t+1}] + \beta \bar{\mathbb{E}}_{it} \bar{\mathbb{E}}_{it+1} [d_{t+2}] + \beta^2 \bar{\mathbb{E}}_{it} \bar{\mathbb{E}}_{it+1} \bar{\mathbb{E}}_{it+2} [d_{t+3}] + \dots) \\ &\quad + (u_t + \beta \bar{\mathbb{E}}_{it} [u_{t+1}] + \beta^2 \bar{\mathbb{E}}_{it} \bar{\mathbb{E}}_{it+1} [u_{t+2}] + \dots). \end{aligned}$$

Thus the equity price consists of a present-value component under a constant SDF, i.e., the infinite sum of higher-order expectations about future aggregate dividends, a component of the infinite sum of higher-order expectations about individual SDFs, and a nonfundamental component due to common forecast errors.

By the intuition developed in Section 2 and 4, the present value component cannot generate a large volatility, as the higher-order expectations about future aggregate dividends are smoother than aggregate dividends. In other words, higher-order expectations about aggregate variables and the failure of the law of iterate expectations do not lead to excess volatility per se. We thus focus on the second component, which depends on the average forecast of future individual shareholdings and labor income by (46). Unlike in the case of full information studied in Section 3.5, the average forecast of individual shareholdings is not equal to the forecast of the average shareholdings,

$$\int_I \mathbb{E}_{it} \left[s_{it+2}^h \right] di \neq \mathbb{E}_{it} \int_I \left[s_{it+2}^h \right] di = 0.$$

Thus individual shareholding choices affect aggregate equity prices. The individual equity trading decisions only respond to idiosyncratic TFP shocks instead of aggregate TFP shocks. Investors interpret a change in the TFP signal as an idiosyncratic shock to their investment opportunities. When the idiosyncratic TFP volatility σ_i tends to infinity, the individual shareholdings volatility also tends to infinity because the shareholding process contains a unit root as in the full information case. This unit root is transmitted to the equity price in response to the aggregate TFP shock by Lemma 3. Formally, $M_s^i(1) \rightarrow \infty$ if and only if $\pi_1(1) \rightarrow \infty$.

This unit root does not necessarily mean that equity volatility tends to infinity because investors will learn from equity prices which may reveal private information about TFP shocks. By (40) and (41) the learning effect is reflected by the denominator $1 - \pi_2(L)$. If $1 - \pi_2(1) \rightarrow \infty$ as $\sigma_i \rightarrow \infty$, the unit root for $\pi_1(z)$ would cancel out. It will not cancel out when near-rational investors make forecast errors. In this case the price signal fails to aggregate dispersed information to the same extent as in the fully rational case, as pointed out by Hassan and Mertens (2017). The forecast error in equity prices contaminates TFP shocks. Thus, even when $\sigma_i \rightarrow \infty$, equity prices still do not fully reveal private information. The learning effect from equity prices is always weaker than the information conveyed by the TFP signal so that the unit root associated with the TFP shock will survive. Therefore when the aggregate TFP shock hits the economy, investors' expectations about future trading decisions adjust in a simultaneous and persistent manner, leading to large equity volatility.

Formally, by the Wiener-Hopf prediction formula, we have

$$\begin{aligned} \mathbb{E}_{it} \left[s_{it+2}^h \right] &= \frac{\tau_1}{\alpha_3 L} \left[\frac{(1 - \lambda_s) \pi_1(L)}{1 - \lambda_s L} - (1 - \lambda_s) \pi_1(0) \right] a_{it} \\ &\quad - \frac{\tau_2}{\alpha_3 L} \left[\frac{(1 - \lambda_s) \pi_1(L)}{1 - \lambda_s L} - (1 - \lambda_s) \pi_1(0) \right] \frac{1 - \pi_2(L)}{\pi_1(L)} q_t, \end{aligned}$$

where τ_1 and τ_2 are the signal-to-noise ratios (see equations (C.2) and (C.3) in Appendix C):

$$\tau_1 = \frac{\sigma_i^2}{\sigma_i^2 + (\sigma_a^{-2} + \sigma_u^{-2})^{-1}}, \quad \tau_2 = \tau_1 \frac{\sigma_a^2}{\sigma_a^2 + \sigma_u^2}.$$

Due to the common forecast error $\sigma_u > 0$, we have $\tau_2 < \tau_1$. When $\sigma_i \rightarrow \infty$, we have $\tau_1 \rightarrow 1$, but $\tau_2 \rightarrow \sigma_a^2 / (\sigma_a^2 + \sigma_u^2) \in (0, 1)$. Thus the expression on the second line of the equation above associated with learning from prices does not fully offset the expression on the first line.

6 Extension and Numerical Results

One side effect of the assumption of $u(z) = \pi_1(z)$ is that the volatility of the nonfundamental component of the equity price also approaches infinity as $\sigma_i \rightarrow \infty$. To isolate this effect, we relax Assumption 2 by assuming that u_t and a_t follow independent AR(1) processes.

Assumption 3 $u(z) = 1/(1 - \rho_u z)$ and $a(z) = 1/(1 - \rho_a z)$, where $\rho_a, \rho_u \in [0, 1)$.

Then the equilibrium system cannot be reduced to a system of linear functional equations for $\pi_1(z)$ and $\pi_2(z)$ and hence $\pi_1(z)$ and $\pi_2(z)$ cannot be represented by analytic rational functions. It is well known that any non-rational analytic functions can be approximated by rational functions with arbitrary accuracy (Rudin (1987)). Using this fact, we compute the model numerically by using rational functions to approximate $\pi_1(z)$ and $\pi_2(z)$. In other words we use finite ARMA(p,q) processes to approximate MA(∞) processes in the time domain. In Appendix D we provide the equilibrium system and the numerical methods.

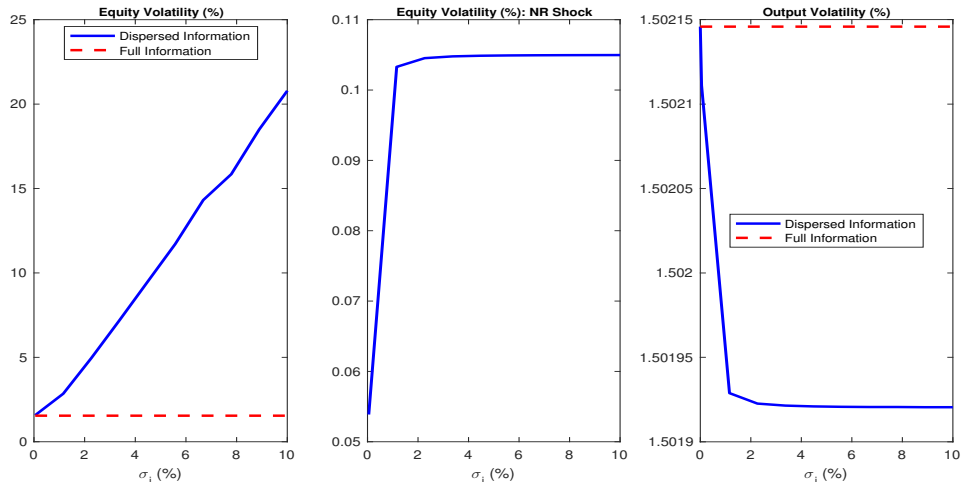


Figure 2: The impact of idiosyncratic TFP volatility σ_i on equity and output volatilities.

To derive quantitative implications, we calibrate the model parameters at quarterly frequency. As is standard in the business cycle literature, we set the subjective discount factor $\beta = 0.99$,

the labor share of output $\alpha = 0.67$, the persistence of the aggregate TFP shock $\rho_a = 0.8$, and the volatility of the aggregate TFP shock $\sigma_a = 0.7\%$. We also set the inter-island elasticity of substitution $\varsigma = 9$ to match the steady-state markup at 12.5%, and set the $\phi = 2$ to match the Frisch elasticity of labor supply at 0.5, which are consistent with Angeletos and La'O (2013) and King and Rebelo (2000). As baseline values, we set the idiosyncratic volatility $\sigma_i = 5\%$ and the persistence and volatility of the common forecast error $\rho_u = 0.05$, $\sigma_u = 0.04\%$. The implied ratio of the unconditional volatility of the common forecast error and the unconditional total volatility of the aggregate and idiosyncratic TFP shocks is 0.65%. This small forecast error is consistent with the estimate in Hassan and Mertens (2017). In Appendix D we show that aggregate output and equity volatilities are independent of the idiosyncratic forecast error volatility. We thus do not need to assign a value for σ_v for our numerical solutions. Our baseline calibration gives quarterly output volatility of 1.5% and quarterly equity volatility of 10.5%, which are in line with the US data.

Figure 2 presents the impact of the idiosyncratic TFP volatility. When $\sigma_i = 0$, the model with dispersed information reduces to the one with full information. As σ_i increases from 0 to 10%, equity volatility rises quickly, but output volatility declines slowly. The component (q_t^n) of equity volatility contributed by the common forecast error increases with σ_i and accounts for a very small fraction of total equity volatility (less than 1%). Thus a very small near-rational error can cause a large equity volatility for reasonable values of σ_i .

7 Conclusion

We have developed a model of a production economy with dispersed information that features smooth aggregate consumption (output) dynamics and highly volatile equity prices. The key elements of our model are not assumptions on nonstandard preferences, bubbles, or sentiments, but the introduction of dispersed information, near rational expectations, incomplete markets, and the endogeneity of SDFs that are time-varying and heterogeneous across population. The key for our model result is due to the different impact of the higher-order beliefs about the average forecasts of aggregate demand and the individual SDFs, together with the dynamic interaction between shareholdings and equity prices. From a technical point of view, we have applied a two-step spectral factorization method in the frequency domain, which can be applied to many other contexts that involves solving signal extraction problems with non-square systems.

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Appendix

A Proofs of Results in Section 3

We consider a general utility function

$$\mathcal{E}_i \left[\sum_{t=0}^{\infty} \beta^t U(C_{it}, N_{it}) \right],$$

where U is twice continuously differentiable and satisfy the usual concavity, monotonicity, and Inada conditions for consumption C_{it} and labor N_{it} . Then the optimality conditions from utility maximization give

$$W_{it} = -\frac{U_n(C_{it}, N_{it})}{U_c(C_{it}, N_{it})},$$

$$Q_t = \mathcal{E}_{it} [M_{it+1}(Q_{t+1} + D_{t+1})], \quad M_{it+1} = \frac{\beta U_c(C_{it+1}, N_{it+1})}{U_c(C_{it}, N_{it})}.$$

The (symmetric) deterministic steady state is characterized by the following nonlinear system

$$\begin{aligned} Y_i &= C_i = C = Y = N^\alpha, \\ W_i &= W = \left(1 - \frac{1}{\varsigma}\right) \alpha N^{\alpha-1}, \\ D &= \left(1 - \left(1 - \frac{1}{\varsigma}\right) \alpha\right) N^\alpha, \\ Q &= \frac{\beta}{1-\beta} D, \quad S_i^h = 1, \end{aligned}$$

and

$$-\frac{U_n(N^\alpha, N)}{U_c(N^\alpha, N)} = \alpha \left(1 - \frac{1}{\varsigma}\right) N^{\alpha-1}. \quad (\text{A.1})$$

Suppose that equation (A.1) has a unique solution $N > 0$. We then obtain a unique deterministic steady state.

Now we consider the log-linear approximation around the deterministic steady state. We use a lower case variable to denote its log deviation from the deterministic steady state. We derive

$$\begin{aligned} u_c(c_{it}, n_{it}) &= -u_1 c_{it} + u_2 n_{it}, \\ u_n(c_{it}, n_{it}) &= u_3 c_{it} + u_4 n_{it}, \end{aligned}$$

where u_1 , u_2 , u_3 , and u_4 are functions of steady-state values as well as the preference parameters

$$\begin{aligned} u_1 &= \frac{-CU_{cc}}{U_c} > 0, \quad u_2 = \frac{U_{cn}N}{U_c}, \\ u_3 &= \frac{CU_{nc}}{U_n}, \quad u_4 = \frac{U_{nn}N}{U_n}. \end{aligned}$$

The Euler equation can be log-linearized as

$$q_t = \mathbb{E}_{it} [\beta q_{t+1} + (1 - \beta) d_{t+1}] + \mathbb{E}_{it} [m_{it+1}] + u_t + v_{it}. \quad (\text{A.2})$$

where the stochastic discount factor has the general form

$$m_{it+1} = u_1(c_{it} - c_{it+1}) - u_2(n_{it} - n_{it+1}).$$

The wage rate satisfies

$$w_{it} = (u_3 + u_1)c_{it} + (u_4 - u_2)n_{it}. \quad (\text{A.3})$$

Substituting this equation into the log-linearized budget constraint yields

$$\begin{aligned} Cc_{it} + Qs_{it+1}^h &= (Q + D)s_{it}^h + Dd_t + WN(n_{it} + w_{it}) \\ &= (Q + D)s_{it}^h + Dd_t + WN(1 + u_4 - u_2)n_{it} + WN(u_3 + u_1)c_{it}, \end{aligned} \quad (\text{A.4})$$

which in turn implies

$$\begin{aligned} c_{it} &= -\frac{Q}{C - WN(u_3 + u_1)}s_{it+1}^h + \left(\frac{Q + D}{C - WN(u_3 + u_1)}\right)s_{it}^h \\ &\quad + \frac{D}{C - WN(u_3 + u_1)}d_t + \frac{WN(1 + u_4 - u_2)}{C - WN(u_3 + u_1)}n_{it}. \end{aligned}$$

Substituting this expression for c_{it} into the log-linearized SDF and Euler equation yields

$$\begin{aligned} \frac{u_1(2Q + D)}{C - WN(u_3 + u_1)}s_{it+1}^h &= \frac{u_1(Q + D)}{C - WN(u_3 + u_1)}s_{it}^h + \mathbb{E}_{it} \left[\frac{u_1Q}{C - WN(u_3 + u_1)}s_{it+2}^h + \Delta b_{it+1} \right] \\ &\quad + \mathbb{E}_{it} [\beta q_{t+1} + (1 - \beta) d_{t+1}] - q_t. \end{aligned}$$

where $\Delta b_{it+1} \equiv b_{it} - b_{it+1}$ and

$$b_{it} \equiv \frac{u_1D}{C - WN(u_3 + u_1)}d_t + \left(u_1 \frac{WN(1 + u_4 - u_2)}{C - WN(u_3 + u_1)} - u_2 \right) n_{it}.$$

Define

$$\begin{aligned} \alpha_1 &= \frac{u_1(2Q + D)}{C - WN(u_3 + u_1)}, \quad \alpha_2 = \frac{u_1(Q + D)}{C - WN(u_3 + u_1)}, \quad \alpha_3 = \frac{u_1Q}{C - WN(u_3 + u_1)}, \\ \alpha_4 &= \frac{u_1D}{C - WN(u_3 + u_1)}, \quad \alpha_5 = \left(u_1 \frac{WN(1 + u_4 - u_2)}{C - WN(u_3 + u_1)} - u_2 \right). \end{aligned}$$

We then obtain equation (28) and can verify that

$$\begin{aligned} \alpha_1 &= \alpha_2 + \alpha_3, \\ \lambda_s &\equiv \frac{\alpha_2}{\alpha_1} = \frac{Q + D}{2Q + D} \in (1/2, 1). \end{aligned}$$

Thus we have proven Lemma 1.

Log-linearizing (22), (18) and using (A.3) to eliminate w_{it} and n_{it} , we derive

$$\left[\frac{1}{\alpha}(u_4 - u_2 + 1) - \left(1 - \frac{1}{\varsigma}\right) \right] y_{it} + (u_3 + u_1)c_{it} = \left[\frac{1}{\alpha}(u_4 - u_2 + 1) \right] a_{it} + \frac{1}{\varsigma} \mathbb{E}_{it}[y_t].$$

Aggregating this equation yields (25), where

$$\xi = \frac{\frac{1}{\alpha}(u_4 - u_2 + 1) - \left(1 - \frac{1}{\varsigma}\right) + (u_3 + u_1)}{\frac{1}{\alpha}(u_4 - u_2 + 1)},$$

and

$$\theta = \frac{1}{\varsigma \left(\frac{1}{\alpha}(u_4 - u_2 + 1) - \left(1 - \frac{1}{\varsigma}\right) + (u_3 + u_1) \right)}.$$

To ensure a stationary solution, we need to impose assumptions on technology and utility such that $\theta \in (0, 1)$.

For the utility function of Greenwood, Huffman, and Hercowitz (1988) used in our paper, we can simplify the computation significantly. In particular, we can derive deterministic steady state in an explicit form:

$$\begin{aligned} N_i &= N = \left(\alpha \left(1 - \frac{1}{\varsigma} \right) \right)^{\frac{1}{\phi - \alpha + 1}}, \\ Y_i &= C_i = C = Y = \left(\alpha \left(1 - \frac{1}{\varsigma} \right) \right)^{\frac{\alpha}{\phi - \alpha + 1}}, \\ D &= \left(1 - \left(1 - \frac{1}{\varsigma} \right) \alpha \right) \left(\alpha \left(1 - \frac{1}{\varsigma} \right) \right)^{\frac{\alpha}{\phi - \alpha + 1}}, \\ Q &= \frac{\beta}{1 - \beta} D, \quad S_i^h = 1, \\ W_i &= W = \left(1 - \frac{1}{\varsigma} \right) \alpha N^{\alpha - 1}. \end{aligned}$$

Given Assumption 1, all equilibrium variables are positive and

$$C - \frac{N^{1+\phi}}{1+\phi} > 0.$$

Log-linearizing equation (9) yields $w_{it} = \phi n_{it}$. We can also compute that

$$b_{it} = \frac{D}{C - \frac{N^{1+\phi}}{1+\phi}} dt + \frac{[WN(1+\phi) - N^{\phi+1}]}{C - \frac{N^{1+\phi}}{1+\phi}} n_{it},$$

and

$$\begin{aligned} \alpha_1 &= \frac{2Q + D}{C - \frac{N^{1+\phi}}{1+\phi}} > 0, \quad \alpha_2 = \frac{Q + D}{C - \frac{N^{1+\phi}}{1+\phi}} > 0, \\ \alpha_3 &= \frac{Q}{C - \frac{N^{1+\phi}}{1+\phi}} > 0, \quad \alpha_4 = \frac{D}{C - \frac{N^{1+\phi}}{1+\phi}} > 0, \\ \alpha_5 &= \frac{(1+\phi)WN - N^{\phi+1}}{C - \frac{N^{1+\phi}}{1+\phi}} = \frac{\phi N^{\phi+1}}{C - \frac{N^{1+\phi}}{1+\phi}} > 0. \end{aligned}$$

Log-linearizing (18) yields

$$y_{it} = a_{it} + \alpha n_{it} \implies n_{it} = \frac{1}{\alpha} (y_{it} - a_{it}).$$

Aggregating leads to $n_t = \frac{1}{\alpha} (y_t - a_t)$. Log-linearizing (24) yields $y_t = \alpha_6 d_t + \alpha_7 n_t$, where

$$\alpha_6 = \frac{D}{Y} > 0, \quad \alpha_7 = \frac{(1 + \phi) WN}{Y} > 0.$$

B Proofs of Results in Section 4

Proof of Lemma 2: By the Wiener-Hopf prediction formula,

$$\mathbb{E}_{it} [y_t] = a_y^a(L) a_{it} + a_y^q(L) q_t = a_y^a(L) (a_t + \epsilon_{it}) + a_y^q(L) q_t,$$

where $a_y^a(z)$ and $a_y^q(z)$ can be computed using (F.3). By the LLN (21),

$$\bar{\mathbb{E}}_{it} [y_t] = a_y^a(L) \left(a_t + \int_I \epsilon_{it} di \right) + a_y^q(L) q_t = a_y^a(L) a(L) \epsilon_{at} + a_y^q(L) q_t.$$

In the stationary equilibrium the equity price can be represented as

$$q_t = M_q^a(L) \epsilon_{at} + M_q^u(L) \epsilon_{ut},$$

where $M_q^a(z)$ and $M_q^u(z)$ are some analytic functions in $H^2(\mathbb{D})$. By the Parseval theorem,

$$\begin{aligned} \text{Var}(\bar{\mathbb{E}}_{it} [y_t]) &= \|a_y^a(z) a(z) + a_y^q(z) M_q^a(z)\|_{\mathbf{H}^2}^2 \sigma_a^2 + \|a_y^q(z) M_q^u(z)\|_{\mathbf{H}^2}^2 \sigma_u^2 \\ &< \|a_y^a(z) a(z) + a_y^q(z) M_q^a(z)\|_{\mathbf{H}^2}^2 \sigma_a^2 + \|a_y^q(z) M_q^u(z)\|_{\mathbf{H}^2}^2 \sigma_u^2 + \|a_y^a(z)\|_{\mathbf{H}^2}^2 \sigma_i^2 \\ &= \text{Var}(\mathbb{E}_{it} [y_t]). \end{aligned}$$

We can write $\mathbb{E}_{it} [y_t] + e_t = y_t$, where e_t is uncorrelated with $\mathbb{E}_{it} [y_t]$. Thus

$$\text{Var}(y_t) \geq \text{Var}(\mathbb{E}_{it} [y_t]).$$

Combining the two inequalities above gives us the desired result. Q.E.D.

Proof of Theorem 1: By equation (27),

$$\text{Var}(y_t) = \text{Var}\left(\frac{a_t}{\xi} + \theta \bar{\mathbb{E}}_{it} [y_t]\right).$$

Using the triangular inequality and Lemma 2, we have

$$\sqrt{\text{Var}(y_t)} \leq \sqrt{\text{Var}(a_t/\xi)} + \theta \sqrt{\text{Var}(\bar{\mathbb{E}}_{it} [y_t])} < \frac{\|a(z)\|_{\mathbf{H}^2} \sigma_a}{\xi} + \theta \sqrt{\text{Var}(y_t)}.$$

Thus

$$\sqrt{\text{Var}(y_t)} < \frac{\|a(z)\|_{\mathbf{H}^2} \sigma_a}{(1 - \theta) \xi}.$$

Using (35), we obtain the desired result. Q.E.D.

C Proofs of Results in Section 5

Proof of Lemma 3: By equation (28) and (39), we obtain

$$\alpha_1 s_{it+1}^h = \alpha_2 s_{it}^h - q_t + \chi_{it} + u_t + v_{it}. \quad (\text{C.1})$$

Plugging equations (40), (41), and (43) into the equation above, we obtain

$$\begin{aligned} & \alpha_1 M_s^i(L) \epsilon_{it} + \alpha_1 M_s^v(L) v_{it} \\ = & \alpha_2 L M_s^i(L) \epsilon_{it} + \alpha_2 L M_s^v(L) v_{it} + \pi_1(L) (\epsilon_{at} + \epsilon_{it}) \\ & + [\pi_2(L) - 1] \left(\frac{\pi_1(L)}{1 - \pi_2(L)} \epsilon_{at} + \frac{u(L)}{1 - \pi_2(L)} \epsilon_{ut} \right) \\ & + u(L) \epsilon_{ut} + v_{it}. \end{aligned}$$

Matching coefficients on the two sides of the equation yields

$$\begin{aligned} \alpha_1 M_s^i(z) &= \alpha_2 z M_s^i(z) + \pi_1(z), \\ \alpha_1 M_s^v(z) &= \alpha_2 z M_s^v(z) + 1. \end{aligned}$$

We then establish this lemma and obtain $M_s^v(z) = \frac{1}{\alpha_1 - \alpha_2 z}$. Q.E.D.

Proof of Theorem 2: Consider the equilibrium conjecture in (41). Given the assumption that $u(z) = \pi_1(z)$, it follows that

$$q_t = \frac{\pi_1(L)}{1 - \pi_2(L)} \epsilon_{at} + \frac{\pi_1(L)}{1 - \pi_2(L)} \epsilon_{ut}.$$

For q_t and u_t to be causal stationary processes, we need $\frac{\pi_1(z)}{1 - \pi_2(z)}$ and $\pi_1(z)$ to be in the Hardy space $\mathbf{H}^2(\mathbb{D})$. We will verify this condition later.

Step 1. We start by deriving the Wold representation for the signal process $\{X_{it}\}$ given in (42). We compute the covariance generating function

$$S_x(z) = H(z) \Sigma_\eta H(z^{-1})^\top = \begin{bmatrix} \sigma_a^2 + \sigma_i^2 & \frac{\pi_1(z^{-1})}{1 - \pi_2(z^{-1})} \sigma_a^2 \\ \frac{\pi_1(z)}{1 - \pi_2(z)} \sigma_a^2 & \frac{\pi_1(z) \pi_1(z^{-1})}{(1 - \pi_2(z))(1 - \pi_2(z^{-1}))} (\sigma_a^2 + \sigma_u^2) \end{bmatrix},$$

where

$$\Sigma_\eta = \begin{bmatrix} \sigma_a^2 & 0 & 0 \\ 0 & \sigma_i^2 & 0 \\ 0 & 0 & \sigma_u^2 \end{bmatrix}$$

is the covariance matrix for the innovation vector $\eta_{it} = [\epsilon_{at}, \epsilon_{it}, \epsilon_{ut}]'$. We wish to derive the spectral factorization

$$S_x(z) = \Gamma(z) \Gamma(z^{-1})^\top.$$

Applying the triangular factorization method described in Appendix F,⁸ we obtain

$$\Gamma(z) = \begin{bmatrix} \sigma_e & \frac{\sigma_a^2}{\sigma_p} \\ 0 & \frac{\pi_1(z)}{1-\pi_2(z)}\sigma_p \end{bmatrix}, \quad \Gamma^{-1}(z) = \begin{bmatrix} \frac{1}{\sigma_e} & -\frac{\sigma_a^2}{\sigma_p^2\sigma_e} \frac{1-\pi_2(z)}{\pi_1(z)} \\ 0 & \frac{1-\pi_2(z)}{\pi_1(z)\sigma_p} \end{bmatrix},$$

where we define

$$\sigma_e^2 \equiv \sigma_i^2 + \frac{\sigma_a^2\sigma_u^2}{\sigma_a^2 + \sigma_u^2}, \quad \sigma_p^2 \equiv \sigma_a^2 + \sigma_u^2.$$

Note that

$$\det \Gamma(z) = \sigma_p\sigma_e \frac{\pi_1(z)}{1-\pi_2(z)}.$$

By Theorem 4.6.11 in Lindquist and Picci (2015), $\Gamma(z)$ is a Wold spectral factor if and only if $\frac{\pi_1(z)}{1-\pi_2(z)}$ has no roots in the open unit disk. We shall make this assumption and then obtain the Wold representation $X_{it} = \Gamma(L) e_{it}$, where e_{it} is a two-dimensional Wold fundamental innovation vector with zero mean and identity covariance matrix.

Step 2. We next solve for the equilibrium quantities. We conjecture that $y_{it} = M_y(L)\eta_{it}$, where $M_y(z) = [M_y^a(z), M_y^i(z), M_y^u(z)]$ and $M_y^a(z)$, $M_y^i(z)$, and $M_y^u(z)$ are all in $\mathbf{H}^2(\mathbb{D})$. Aggregation leads to aggregate output $y_t = M_y(z)I_y\eta_{it}$, where

$$I_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the Wiener-Hopf prediction formula, we derive that $\mathbb{E}_{it}[y_t] = [\psi_y(L)]_+ \Gamma^{-1}(L)X_{it}$, where $[\cdot]_+$ is the annihilation operator and the z -transform of the operator ψ_y is

$$\psi_y(z) = S_{yx}(z) (\Gamma^{-1}(z^{-1}))^\top.$$

The cross-spectrum is given by

$$\begin{aligned} S_{yx}(z) &= M_y(z)I_y\Sigma_\eta H^\top(z^{-1}) \\ &= [M_y^a(z), 0, M_y^u(z)] \begin{bmatrix} \sigma_a^2 & 0 & 0 \\ 0 & \sigma_i^2 & 0 \\ 0 & 0 & \sigma_u^2 \end{bmatrix} \begin{bmatrix} 1 & \frac{\pi_1(z^{-1})}{1-\pi_2(z^{-1})} \\ 1 & 0 \\ 0 & \frac{\pi_1(z^{-1})}{1-\pi_2(z^{-1})} \end{bmatrix} \\ &= \left[M_y^a(z) \sigma_a^2, \frac{\pi_1(z^{-1})}{1-\pi_2(z^{-1})} (M_y^a(z) \sigma_a^2 + M_y^u(z) \sigma_u^2) \right] \end{aligned}$$

Routine algebra reveals that

$$\psi_y(z) = S_{yx}(z) (\Gamma^{-1}(z^{-1}))^\top = \left[\left(\frac{\sigma_a^2}{\sigma_e} - \frac{\sigma_a^4}{\sigma_p^2\sigma_e} \right) M_y^a(z) - \frac{\sigma_a^2\sigma_u^2}{\sigma_p^2\sigma_e} M_y^u(z), \frac{M_y^a(z)\sigma_a^2 + M_y^u(z)\sigma_u^2}{\sigma_p} \right].$$

⁸Following Rondina and Walker (2015), we transform the lower-triangular matrix to the upper triangular form by right multiplication of an unitary matrix, which ease the algebra.

Since $M_y^a(z), M_y^u(z) \in \mathbf{H}^2(\mathbb{D})$, both components of $\psi_y(z)$ are in $\mathbf{H}^2(\mathbb{D})$. Thus $[\psi_y(z)]_+ = \psi_y(z)$. In the innovation form, we have

$$\begin{aligned}\mathbb{E}_{it}[y_t] &= [\psi_y(L)]_+ \Gamma^{-1}(L)H(L)\eta_{it} \\ &= [(h_1 + h_3)M_y^a(L) + (h_4 - h_2)M_y^u(L), h_1M_y^a(L) - h_2M_y^u(L), h_3M_y^a(L) + h_4M_y^u(L)] \eta_{it},\end{aligned}$$

where we define

$$\begin{aligned}h_1 &\equiv \frac{\sigma_a^2}{\sigma_e^2} - \frac{\sigma_a^4}{\sigma_p^2\sigma_e^2}, \quad h_2 \equiv \frac{\sigma_a^2\sigma_u^2}{\sigma_p^2\sigma_e^2}, \\ h_3 &\equiv \frac{\sigma_a^2}{\sigma_p^2} + \frac{\sigma_a^6}{\sigma_p^4\sigma_e^2} - \frac{\sigma_a^4}{\sigma_p^2\sigma_e^2}, \quad h_4 \equiv \frac{\sigma_a^4\sigma_u^2}{\sigma_p^4\sigma_e^2} + \frac{\sigma_u^2}{\sigma_p^2}.\end{aligned}$$

Plugging $y_{it} = M_y(L)\eta_{it}$ and the preceding conditional expectation $\mathbb{E}_{it}[y_t]$ into (25) and matching coefficients, we obtain a system of linear equations

$$\begin{aligned}M_y^a(z) &= \frac{1}{\xi} + \theta [(h_1 + h_3)M_y^a(z) + (h_4 - h_2)M_y^u(z)], \\ M_y^u(z) &= \theta [h_3M_y^a(z) + h_4M_y^u(z)], \\ M_y^i(z) &= \frac{1}{\xi} + \theta [h_1M_y^a(z) - h_2M_y^u(z)],\end{aligned}$$

which yields the solution

$$\begin{aligned}M_y^a(z) &= \frac{1}{\xi m_1}, \quad M_y^u(z) = \frac{m_2}{\xi m_1}, \\ M_y^i(z) &= \frac{1}{\xi} + \theta \frac{h_1 - h_2 m_2}{\xi m_1},\end{aligned}$$

where we define

$$m_1 \equiv 1 - \frac{(h_4 - h_2)h_3\theta}{1 - \theta h_4} - \theta(h_1 + h_3), \quad m_2 \equiv \frac{h_3\theta}{1 - \theta h_4}.$$

The preceding solution is independent of z , confirming our previous conjecture.

Step 3. We proceed to the financial side of the model and compute the conditional expectations χ_{it} in (39). Using equation (43) and the Wiener-Hopf prediction formula, we compute the conditional expectation

$$\mathbb{E}_{it} [s_{it+2}^h] = [\psi_s(L)]_+ \Gamma^{-1}(L)X_{it}.$$

where the z -transform of the operator ψ_s is given by

$$\psi_s(z) = z^{-1}S_{sx}(z) (\Gamma^{-1}(z^{-1}))^\top$$

and the cross-spectrum is given by

$$\begin{aligned}S_{sx}(z) &= [0, M_s^i(z), 0] \begin{bmatrix} \sigma_a^2 & 0 & 0 \\ 0 & \sigma_i^2 & 0 \\ 0 & 0 & \sigma_u^2 \end{bmatrix} \begin{bmatrix} 1 & \frac{\pi_1(z^{-1})}{1 - \pi_2(z^{-1})} \\ 1 & 0 \\ 0 & \frac{\pi_1(z^{-1})}{1 - \pi_2(z^{-1})} \end{bmatrix} \\ &= [M_s^i(z) \sigma_i^2, 0].\end{aligned}$$

Thus

$$\psi_s(z) = \frac{1}{z} [S_{sx}(z) (\Gamma^{-1}(z^{-1}))^\top] = \frac{1}{z} \left[\frac{\sigma_i^2}{\sigma_e} M_s^i(z), 0 \right].$$

There is a pole at zero. Using a lemma in the Appendix A of Hansen and Sargent (1980) we can compute that

$$[\psi_s(z)]_+ = \psi_s(z) - \frac{\lim_{z \rightarrow 0} z \psi_s(z)}{z} = \frac{1}{z} \left[\frac{\sigma_i^2}{\sigma_e} (M_s^i(z) - M_s^i(0)), 0 \right].$$

It follows that

$$[\psi_s(z)]_+ \Gamma^{-1}(z) = \left[\tau_1 \frac{M_s^i(z) - M_s^i(0)}{z}, -\tau_2 \frac{1 - \pi_2(z)}{\pi_1(z)} \frac{M_s^i(z) - M_s^i(0)}{z} \right],$$

where we define the signal-to-noise ratios

$$\tau_1 \equiv \frac{\sigma_i^2}{\sigma_e^2} \in (0, 1), \quad \tau_2 \equiv \tau_1 \frac{\sigma_a^2}{\sigma_p^2} \in (0, 1). \quad (\text{C.2})$$

By Lemmas 1 and 3, we derive

$$M_s^i(z) - M_s^i(0) = \frac{\pi_1(z)}{\alpha_1 - \alpha_2 z} - \frac{\pi_1(0)}{\alpha_1} = \frac{1}{\alpha_3} \left[\frac{(1 - \lambda_s) \pi_1(z)}{1 - \lambda_s z} - (1 - \lambda_s) \pi_1(0) \right].$$

Thus

$$\mathbb{E}_{it} [s_{it+2}^h] = \frac{1}{\alpha_3 L} \left[\frac{(1 - \lambda_s) \pi_1(L)}{1 - \lambda_s L} - (1 - \lambda_s) \pi_1(0) \right] \left[\tau_1, -\tau_2 \frac{1 - \pi_2(L)}{\pi_1(L)} \right] X_{it}. \quad (\text{C.3})$$

Note that $\tau_2 < \tau_1$ reflects the fact that equity prices do not fully aggregate information due to near-rational forecast errors.

Now we conjecture that $d_t = M_d(L) \eta_{it}$, $n_{it} = M_n(L) \eta_{it}$, and $b_{it} = M_b(L) \eta_{it}$, where

$$\begin{aligned} M_d(z) &= [M_d^a(z), 0, M_d^u(z)], \\ M_n(z) &= [M_n^a(z), M_n^i(z), M_n^u(z)], \\ M_b(z) &= [M_b^a(z), M_b^i(z), M_b^u(z)], \end{aligned}$$

and each component of these vectors is in $\mathbf{H}^2(\mathbb{D})$. Plugging these equations and (36) into equations (30), (31), and $n_{it} = \frac{1}{\alpha} (y_{it} - a_{it})$, and matching coefficients, we can derive that

$$\begin{aligned} M_d(z) &= \left[\frac{1}{\alpha_6} \left(1 - \frac{\alpha_7}{\alpha} \right) M_y^a(z) + \frac{\alpha_7}{\alpha \alpha_6}, 0, \frac{1}{\alpha_6} \left(1 - \frac{\alpha_7}{\alpha} \right) M_y^u(z) \right], \\ M_n(z) &= \frac{1}{\alpha} [M_y^a(z) - 1, M_y^i(z) - 1, M_y^u(z)], \\ M_b(z) &= \alpha_4 M_d(z) + \alpha_5 M_n(z). \end{aligned}$$

Note that we have used the assumption that $a_{it} = \epsilon_{at} + \epsilon_{it}$. Since we have shown above that $M_y(z)$ is independent of z , $M_d(z)$, $M_n(z)$ and $M_b(z)$ are all independent of z .

Using the Wiener-Hopf prediction formula, we compute

$$\begin{aligned}\mathbb{E}_{it} [\Delta b_{it}] &= \mathbb{E}_{it} [(1 - L^{-1}) b_{it}] = \mathbb{E}_{it} [(1 - L^{-1}) M_b(L) \eta_{it}] \\ &= [\psi_b(L)]_+ \Gamma^{-1}(L) X_{it},\end{aligned}$$

where the z -transform of the operator ψ_b is given by

$$\psi_b(z) = \frac{z-1}{z} S_{bx}(z) (\Gamma^{-1}(z^{-1}))^\top,$$

and the cross-spectrum $S_{bx}(z)$ is given by

$$\begin{aligned}S_{bx}(z) &= M_b(z) \Sigma_\eta H(z^{-1})^\top \\ &= \left[M_b^a(z) \sigma_a^2 + M_b^i(z) \sigma_i^2, \frac{\pi_1(z^{-1})}{1 - \pi_2(z^{-1})} (M_b^a(z) \sigma_a^2 + M_b^u(z) \sigma_u^2) \right].\end{aligned}$$

It follows that

$$\begin{aligned}\psi_b(z) &= \frac{z-1}{z} S_{bx}(z) (\Gamma^{-1}(z^{-1}))^\top \\ &= \frac{z-1}{z} \left[\left(\frac{\sigma_a^2}{\sigma_e} - \frac{\sigma_a^4}{\sigma_p^2 \sigma_e} \right) M_b^a(z) - \frac{\sigma_a^2 \sigma_u^2}{\sigma_p^2 \sigma_e} M_b^u(z) + \frac{\sigma_i^2}{\sigma_e} M_b^i(z), \frac{M_b^a(z) \sigma_a^2 + M_b^u(z) \sigma_u^2}{\sigma_p} \right]\end{aligned}$$

The complex function $\psi_b(z)$ has a first-order pole at $z = 0$. Following Hansen and Sargent (1980), the annihilation operation is given by

$$[\psi_b(z)]_+ = \psi_b(z) - \frac{\lim_{z \rightarrow 0} z \psi_b(z)}{z}.$$

It follows immediately that

$$\begin{aligned}[\psi_b(z)]_+ &= \psi_b(z) - \frac{(-1)}{z} \left[\left(\frac{\sigma_a^2}{\sigma_e} - \frac{\sigma_a^4}{\sigma_p^2 \sigma_e} \right) M_b^a(0) - \frac{\sigma_a^2 \sigma_u^2}{\sigma_p^2 \sigma_e} M_b^u(0) + \frac{\sigma_i^2}{\sigma_e} M_b^i(0), \frac{M_b^a(0) \sigma_a^2 + M_b^u(0) \sigma_u^2}{\sigma_p} \right] \\ &= \psi_b(z) + \frac{1}{z} \left[h_1 \sigma_e M_b^a(0) - h_2 \sigma_e M_b^u(0) + \frac{\sigma_i^2}{\sigma_e} M_b^i(0), \frac{M_b^a(0) \sigma_a^2 + M_b^u(0) \sigma_u^2}{\sigma_p} \right].\end{aligned}$$

We can then derive that

$$\begin{aligned}\mathbb{E}_{it} [\Delta b_{it}] &= [\psi_b(L)]_+ \Gamma^{-1}(L) X_{it} \\ &= \frac{1}{z} \left[G_b^{(1)}(L) - G_b^{(1)}(0), \frac{1 - \pi_2(L)}{\pi_1(L)} (G_b^{(2)}(L) - G_b^{(2)}(0)) \right] X_{it}\end{aligned}\tag{C.4}$$

where we define the functions

$$\begin{aligned}G_b^{(1)}(z) &\equiv (z-1) [h_1 M_b^a(z) - h_2 M_b^u(z) + \tau_1 M_b^i(z)], \\ G_b^{(2)}(z) &\equiv (z-1) [h_3 M_b^a(z) + h_4 M_b^u(z) - \tau_2 M_b^i(z)].\end{aligned}$$

Using the same method, we can compute the conditional expectation of future dividends

$$\mathbb{E}_{it} [d_{t+1}] = \frac{1}{L} \left[G_d^{(1)}(L) - G_d^{(1)}(0), \frac{1 - \pi_2(L)}{\pi_1(L)} (G_d^{(2)}(L) - G_d^{(2)}(0)) \right] X_{it},\tag{C.5}$$

where we define the functions

$$\begin{aligned} G_d^{(1)}(z) &\equiv h_1 M_d^a(z) - h_2 M_d^u(z), \\ G_d^{(2)}(z) &\equiv h_3 M_d^a(z) + h_4 M_d^u(z). \end{aligned}$$

Since $M_b(z)$, $M_d(z)$, and $M_y(z)$ are constant independent of z , it follows from the previous construction that $G_d^{(1)}(z)$ and $G_d^{(2)}(z)$ are constant independent of z , but $G_b^{(1)}(z)$ and $G_b^{(2)}(z)$ are linear functions of z .

We use the Wiener-Kolmogorov formula to compute

$$\mathbb{E}_{it} [q_{t+1}] = [\psi_q(L)]_+ \Gamma^{-1}(L) X_{it},$$

where the z -transform of the operator ψ_q is given by

$$\psi_q(z) = \frac{1}{z} [0, \quad 1] S_x(z) (\Gamma^{-1}(z^{-1}))^\top = \frac{1}{z} [0, \quad 1] \Gamma(z) = \frac{1}{z} \left[0, \quad \frac{\pi_1(z)}{1-\pi_2(z)} \sigma_p \right],$$

where the second equality follows from the previous definition of $S_x(z)$. Thus

$$\begin{aligned} [\psi_q(z)]_+ \Gamma^{-1}(z) &= \left[\frac{1}{z} \left[0, \quad \frac{\pi_1(z)}{1-\pi_2(z)} \sigma_p \right] \right]_+ \Gamma^{-1}(z) \\ &= \frac{1}{z} \left[0, \quad \frac{\pi_1(z)}{1-\pi_2(z)} \sigma_p \right] \Gamma^{-1}(z) - \frac{1}{z} \left[0, \quad \frac{\pi_1(0)}{1-\pi_2(0)} \sigma_p \right] \Gamma^{-1}(z) \\ &= \left[0, \quad \frac{1}{z} \left(1 - \frac{1-\pi_2(z)}{\pi_1(z)} \frac{\pi_1(0)}{1-\pi_2(0)} \right) \right], \end{aligned}$$

and

$$\mathbb{E}_{it} [q_{t+1}] = \left[0, \quad \frac{1}{z} \left(1 - \frac{1-\pi_2(z)}{\pi_1(z)} \frac{\pi_1(0)}{1-\pi_2(0)} \right) \right] X_{it}. \quad (\text{C.6})$$

Step 4. Derive the solution for $\pi_1(z)$ and $\pi_2(z)$. Plugging the expressions for the conditional expectations (C.3), (C.4), (C.5), and (C.6) derived in Step 3 into equation (39), we obtain an expression for χ_{it} . Matching coefficients of $X_{it} = [a_{it}, q_t]'$ with those in (40), we construct the following equilibrium conditions:

$$\begin{aligned} z\pi_1(z) &= \frac{1-\lambda_s}{1-\lambda_s z} \tau_1 \pi_1(z) - (1-\lambda_s) \tau_1 \pi_1(0) \\ &\quad + (1-\beta) \left[G_d^{(1)}(z) - G_d^{(1)}(0) \right] + G_b^{(1)}(z) - G_b^{(1)}(0), \end{aligned} \quad (\text{C.7})$$

and

$$\begin{aligned} z\pi_2(z) &= \frac{1-\pi_2(z)}{\pi_1(z)} \left\{ -\frac{1-\lambda_s}{1-\lambda_s z} \tau_2 \pi_1(z) + (1-\lambda_s) \tau_2 \pi_1(0) - \frac{\beta \pi_1(0)}{1-\pi_2(0)} \right. \\ &\quad \left. + (1-\beta) \left[G_d^{(2)}(z) - G_d^{(2)}(0) \right] + G_b^{(2)}(z) - G_b^{(2)}(0) \right\} + \beta. \end{aligned} \quad (\text{C.8})$$

Simplifying equation (C.7) yields

$$\pi_1(z) = \frac{(1 - \lambda_s z) [x(z) - (1 - \lambda_s)\tau_1\pi_1(0)]}{P_1(z)}, \quad (\text{C.9})$$

where we define the functions

$$P_1(z) \equiv -\lambda_s z^2 + z - (1 - \lambda_s)\tau_1,$$

and

$$x(z) \equiv (1 - \beta) \left[G_d^{(1)}(z) - G_d^{(1)}(0) \right] + G_b^{(1)}(z) - G_b^{(1)}(0). \quad (\text{C.10})$$

By the analysis in Step 3, $x(z)$ is a linear function of z .

Since $\lambda_s \in (1/2, 1)$ by Lemma 1 and $\tau_1 \in (0, 1)$, we have $P_1(0) = -(1 - \lambda_s)\tau_1 < 0$, $P_1(1) = (1 - \lambda_s)(1 - \tau_1) > 0$, and $\lim_{z \rightarrow +\infty} P_1(z) = -\infty$. Thus $P_1(z) = 0$ has two real roots, denoted by $\gamma_1 \in (0, 1)$ and $\gamma_2 > 1$. We can then write

$$\pi_1(z) = \frac{(1 - \lambda_s z)}{-\lambda_s(z - \gamma_2)(z - \gamma_1)} [x(z) - (1 - \lambda_s)\tau_1\pi_1(0)].$$

To remove the pole at γ_1 , we set $\pi_1(0)$ such that

$$x(\gamma_1) - (1 - \lambda_s)\tau_1\pi_1(0) = 0,$$

which implies that

$$\pi_1(0) = \frac{x(\gamma_1)}{(1 - \lambda_s)\tau_1}.$$

We then collect terms and simplify expressions to derive

$$\pi_1(z) = \frac{(1 - \lambda_s z) [x(z) - x(\gamma_1)]}{-\lambda_s(z - \gamma_1)(z - \gamma_2)}. \quad (\text{C.11})$$

Since the pole $|\gamma_1| < 1$ is removed and $x(z)$ is a linear function of z , we deduce that $\pi_1(z) \in \mathbf{H}^2(\mathbb{D})$.

Next consider the equilibrium condition (C.8). It is straightforward to show that

$$\frac{\pi_1(z)}{1 - \pi_2(z)} = \frac{\kappa(z) - \beta\pi_1(0) / (1 - \pi_2(0))}{z - \beta}, \quad (\text{C.12})$$

where we define the function

$$\begin{aligned} \kappa(z) \equiv & (1 - \beta)[G_d^{(2)}(z) - G_d^{(2)}(0)] + G_b^{(2)}(z) - G_b^{(2)}(0) \\ & - \left[\frac{(1 - \lambda_s)\tau_2}{1 - \lambda_s z} - z \right] \pi_1(z) + (1 - \lambda_s)\tau_2\pi_1(0). \end{aligned} \quad (\text{C.13})$$

Since $\pi_1(z) \in \mathbf{H}^2(\mathbb{D})$, $\lambda_s \in (1/2, 1)$ by Lemma 1, $G_d^{(2)}(z)$ is a constant, and $G_b^{(2)}(z)$ is linear in z , it follows that $\kappa(z) \in \mathbf{H}^2(\mathbb{D})$.

As mentioned earlier, we need $\frac{\pi_1(z)}{1-\pi_2(z)}$ to be analytical in the unit disk. Thus we should remove the pole at $z = \beta$ by setting the constant $\pi_2(0)$ such that $\kappa(\beta) - \beta\pi_1(0) / (1 - \pi_2(0)) = 0$. Solving this equation yields

$$\pi_2(0) = 1 - \frac{\pi_1(0)\beta}{\kappa(\beta)}.$$

We can then rewrite (C.12) as

$$\frac{\pi_1(z)}{1 - \pi_2(z)} = \frac{\kappa(z) - \kappa(\beta)}{z - \beta}. \quad (\text{C.14})$$

Since we have removed the pole at $z = \beta$ and $\kappa(z) \in \mathbf{H}^2(\mathbb{D})$, it follows that $\frac{\pi_1(z)}{1-\pi_2(z)} \in \mathbf{H}^2(\mathbb{D})$.

By our constructive proof above, we conclude that the equilibrium solution is characterized by unique rational functions of z in the frequency domain. As mentioned earlier to ensure the spectral factorization to be valid, we need to impose the assumption that the equation

$$\frac{\pi_1(z)}{1 - \pi_2(z)} = \frac{\kappa(z) - \kappa(\beta)}{z - \beta} = 0$$

has no roots inside the open unit disk. The proof is then complete. Q.E.D.

Proof of Theorem 3: We first show that the denominator of the expression for $\pi_1(z)$ in (C.9) has a unit root as $\sigma_i \rightarrow \infty$. Consider the quadratic function,

$$P_1(z) \equiv -\lambda_s z^2 + z - (1 - \lambda_s)\tau_1.$$

Since

$$\lim_{\sigma_i \rightarrow \infty} \tau_1 = \lim_{\sigma_i \rightarrow \infty} \frac{\sigma_i^2}{\sigma_e^2} = 1,$$

we have

$$\lim_{\sigma_i \rightarrow \infty} P_1(z) = -\lambda_s z^2 + z - (1 - \lambda_s) = -\lambda_s(z - 1) \left(z - \frac{1 - \lambda_s}{\lambda_s} \right).$$

Since $\lambda_s \in (1/2, 1)$, the root $\frac{1-\lambda_s}{\lambda_s}$ is located inside the unit circle. We know that $P_1(z)$ has one root inside the unit circle and the other outside the unit circle. By the continuous dependence of roots on coefficients, the larger root γ_2 of $P_1(z)$ gradually converges to the unit root as $\sigma_i \rightarrow \infty$.

We next show that the numerator of $\pi_1(z)$ in (C.9) or (C.11) does have a zero at $z = 1$ when $\sigma_i \rightarrow \infty$. By (C.11), it suffices to show that the analytic function $x(z) - x(\gamma_1)$ does not have a zero at $z = 1$. Using the result derived in the proof of Theorem 2, we can show that $\lim_{\sigma_i \rightarrow \infty} h_1 = \lim_{\sigma_i \rightarrow \infty} h_2 = 0$, $\lim_{\sigma_i \rightarrow \infty} G_d^{(1)}(z) = 0$, and $\lim_{\sigma_i \rightarrow \infty} G_b^{(1)}(z) = \frac{\alpha_5}{\alpha} \left(\frac{1}{\xi} - 1 \right) (z - 1)$. It follows from (C.10) that

$$\lim_{\sigma_i \rightarrow \infty} x(z) = \frac{\alpha_5(1 - \xi)}{\alpha\xi} z.$$

Therefore,

$$\lim_{\sigma_i \rightarrow \infty} [x(1) - x(\gamma_1)] = \frac{\alpha_5(1 - \xi)}{\alpha\xi} \left(1 - \lim_{\sigma_i \rightarrow \infty} \gamma_1 \right) = \frac{\alpha_5(1 - \xi)}{\alpha\xi} \left(1 - \frac{1 - \lambda_s}{\lambda_s} \right).$$

By Assumption 1, we know that

$$\xi \equiv \frac{1 + \phi - \alpha(1 - 1/\varsigma)}{1 + \phi} \in (0, 1),$$

as $\phi > 0$ and $\varsigma > 1$. It follows from $\lambda_s \in (\frac{1}{2}, 1)$ that

$$\lim_{\sigma_i \rightarrow \infty} [x(1) - x(\gamma_1)] \neq 0.$$

Hence, $\pi_1(z)$ does not converge to zero at $z = 1$ when $\sigma_i \rightarrow \infty$, but it has a pole at $z = \lim_{\sigma_i \rightarrow \infty} \gamma_2 = 1$. Since $\pi_1(z)$ is rational in the frequency domain, a pole at the unit circle is sufficient to ensure that

$$\lim_{\sigma_i \rightarrow \infty} \|\pi_1(z)\|_{\mathbf{H}^2} \rightarrow \infty.$$

Now we wish to show that the analytic function $\frac{\pi_1(z)}{1 - \pi_2(z)}$ has no zero at $z = 1$ as $\sigma_i \rightarrow \infty$. We only need to consider the equation $\kappa(z) - \kappa(\beta) = 0$ by (C.14). We can rewrite the expression for $\kappa(z)$ in (C.13) as

$$\kappa(z) = A(z) - \left[\frac{(1 - \lambda_s)\tau_2}{1 - \lambda_s z} - z \right] \pi_1(z) + (1 - \lambda_s)\tau_2\pi_1(0),$$

where $A(z)$ is a linear function of z . Plugging (C.11) into this equation, we can derive

$$\begin{aligned} \kappa(z) - \kappa(\beta) &= A(z) + (1 - \lambda_s)\tau_2\pi_1(0) - \kappa(\beta) \\ &\quad + \frac{[x(z) - x(\gamma_1)][(1 - \lambda_s)\tau_2 - (1 - \lambda_s z)z]}{\lambda_s(z - \gamma_1)(z - \gamma_2)}. \end{aligned}$$

The linear function $A(z)$ is bounded in the closed unit disk, $\sup_{|z| \leq 1} |A(z)| < \infty$. Moreover, we know that $\pi_1(z)$ is analytic and rational inside the open unit disk $|z| < 1$, even when $\sigma_i \rightarrow \infty$. Thus $\pi_1(0)$ and $\kappa(\beta)$ are finite for $0 < \beta < 1$. It follows that the expression on the first line of the right-hand side of the equation above is bounded at $z = 1$ when $\sigma_i \rightarrow \infty$.

Consider the expression on the second line of the equation above. The denominator converges to zero at $z = 1$ as $\sigma_i \rightarrow \infty$. For the numerator, we have

$$\begin{aligned} &\lim_{\sigma_i \rightarrow \infty} [x(z) - x(\gamma_1)][(1 - \lambda_s)\tau_2 - (1 - \lambda_s z)z]|_{z=1} \\ &= \left[\frac{\alpha_5(1 - \xi)}{\alpha\xi} \left(1 - \frac{1 - \lambda_s}{\lambda_s} \right) \right] (1 - \lambda_s) \left(\frac{\sigma_a^2}{\sigma_p^2} - 1 \right) \\ &\neq 0, \end{aligned}$$

where we have used the previous definition of σ_e^2 to derive

$$\lim_{\sigma_i \rightarrow \infty} \tau_2 = \lim_{\sigma_i \rightarrow \infty} \frac{\sigma_a^2 \sigma_i^2}{\sigma_e^2 \sigma_p^2} = \frac{\sigma_a^2}{\sigma_p^2} \in (0, 1).$$

We conclude that $\kappa(1) - \kappa(\beta)$ converges to infinity as $\sigma_i \rightarrow \infty$.

Therefore, $\frac{\pi_1(z)}{1-\pi_2(z)}$ does not have a zero at $z = 1$ when $\sigma_i \rightarrow \infty$, but it has a pole at $z = 1$. This implies that the rational function $\frac{\pi_1(z)}{1-\pi_2(z)}$ will have infinite norm at the limit,

$$\lim_{\sigma_i \rightarrow \infty} \left\| \frac{\pi_1(z)}{1-\pi_2(z)} \right\|_{\mathbf{H}^2} \rightarrow \infty.$$

This completes the proof. Q.E.D.

D Analysis in Section 6

D.1 Equilibrium System

As in Section 5, we derive the equilibrium system in four steps.

Step 1. Derive the Wold representation for the signal system under Assumption 3. Given the AR(1) processes for a_t and u_t , the signal representation follows

$$X_{it} = H(L)\eta_{it} \equiv \begin{bmatrix} \frac{1}{1-\rho_a L} & 1 & 0 \\ \frac{\pi_1(L)}{(1-\pi_2(L))(1-\rho_a L)} & 0 & \frac{1}{(1-\pi_2(L))(1-\rho_u L)} \end{bmatrix} \begin{bmatrix} \epsilon_{at} \\ \epsilon_{it} \\ \epsilon_{ut} \end{bmatrix}, \quad (\text{D.1})$$

and so the spectral density for the signal is

$$S_x(z) = H(z)\Sigma_\eta H(z^{-1})^\top = \begin{bmatrix} \frac{\sigma_a^2}{(1-\rho_a z)(1-\rho_a z^{-1})} + \sigma_i^2 & \frac{\pi_1(z^{-1})}{1-\pi_2(z^{-1})} \frac{1}{(1-\rho_a z)(1-\rho_a z^{-1})} \sigma_a^2 \\ \frac{\pi_1(z)}{1-\pi_2(z)} \frac{1}{(1-\rho_a z)(1-\rho_a z^{-1})} \sigma_a^2 & \frac{\pi_1(z)\pi_1(z^{-1})}{(1-\rho_a z)(1-\rho_a z^{-1})} \sigma_a^2 + \frac{\sigma_u^2}{(1-\rho_u z)(1-\rho_u z^{-1})} \end{bmatrix}.$$

Using the method presented in Appendix F, we can first factorize the spectral density in a lower triangular form

$$\tilde{\Gamma}(z) = \begin{bmatrix} \sigma_w \frac{z-\lambda_w}{1-\rho_a z} & 0 \\ \frac{\sigma_a^2}{\sigma_w} \frac{\pi_1(z)z}{(1-\pi_2(z))(1-\lambda_w z)(1-\rho_a z)} & \frac{1}{\sigma_w} \frac{\tilde{\pi}_1(z)}{1-\pi_2(z)} \frac{1-\rho_a z}{1-\lambda_w z} \end{bmatrix},$$

where the constants $\lambda_w \in (0, 1)$ and σ_w are determined by the univariate spectral factorization of the first signal a_{it} in the frequency domain,

$$\sigma_w^2 \frac{(1-\lambda_w z)(1-\lambda_w z^{-1})}{(1-\rho_a z)(1-\rho_a z^{-1})} = \frac{\sigma_a^2}{(1-\rho_a z)(1-\rho_a z^{-1})} + \sigma_i^2.$$

It follows that

$$\sigma_w^2 (1-\lambda_w z)(1-\lambda_w z^{-1}) = \sigma_a^2 + \sigma_i^2 (1-\rho_a z)(1-\rho_a z^{-1}).$$

Matching coefficients on the two sides of the equality yields

$$\lambda_w = \frac{1}{2\rho_a} \left[(1 + \tau + \rho_a^2) - \sqrt{\tau^2 + 2\tau + 2\tau\rho_a^2 + 1 - 2\rho_a^2 + \rho_a^4} \right],$$

and

$$\sigma_w^2 = \frac{\rho_a \sigma_i^2}{\lambda_w}.$$

Here $\tau \equiv \sigma_a^2 / \sigma_i^2 \in (0, \infty)$ denotes the relative volatility of the aggregate shock to the idiosyncratic shock. It is easy to verify that $0 < \lambda_w < \rho_a < 1$ and $\lim_{\sigma_i \rightarrow \infty} \lambda_w = \rho_a$.

Define the function $\tilde{\pi}_1(z)$ by the following equation

$$\begin{aligned} \tilde{\pi}_1(z) \tilde{\pi}_1(z^{-1}) &= \frac{\pi_1(z) \pi_1(z^{-1})}{(1 - \rho_a z)(1 - \rho_a z^{-1})} \sigma_a^2 \sigma_i^2 \\ &\quad + \frac{(1 - \lambda_w z)(1 - \lambda_w z^{-1})}{(1 - \rho_a z)(1 - \rho_a z^{-1})(1 - \rho_u z)(1 - \rho_u z^{-1})} \sigma_u^2 \sigma_w^2. \end{aligned} \quad (\text{D.2})$$

A stationary equilibrium requires that the endogenous function $\pi_1 \in \mathbf{H}^2(\mathbb{D})$. It is then clear that the right-hand side of equation (D.2) is a well-defined spectral density supported by a stationary process. Then by the Paley-Wiener Theorem (e.g. Lindquist and Picci, 2015, Theorem 4.4.1), there exists a Wold spectral factor $\tilde{\pi}_1(z) \in \mathbf{H}^2(\mathbb{D})$ that satisfies the factorization (D.2). Using a similar argument, we can show that the function $\frac{\tilde{\pi}_1(z)}{1 - \pi_2(z)} \in \mathbf{H}^2(\mathbb{D})$. Hence, the matrix $\tilde{\Gamma}(z)$ is a valid spectral factor in $\mathbf{H}^2(\mathbb{D})$ that satisfies

$$S_x(z) = \tilde{\Gamma}(z) \tilde{\Gamma}^\top(z^{-1}).$$

The determinant of $\tilde{\Gamma}(z)$ is given by

$$\det \tilde{\Gamma}(z) = \frac{\tilde{\pi}_1(z)}{1 - \pi_2(z)} \frac{z - \lambda_w}{1 - \lambda_w z}.$$

As in Section 5, we restrict our attention to the equilibrium such that $\frac{\tilde{\pi}_1(z)}{1 - \pi_2(z)}$ has no roots in the open unit disk. To derive the wold fundamental representation, we need to remove the root at $z = \lambda_w \in (0, 1)$. Using the Blaschke matrix $B(z)$ by Proposition 2 in Appendix F, we set

$$\Gamma(z) = \tilde{\Gamma}(z) V^{-1} B(z),$$

where

$$V = \begin{bmatrix} \sqrt{\frac{h^2}{1+h^2}} & \sqrt{\frac{1}{1+h^2}} \\ \sqrt{\frac{1}{1+h^2}} & -\sqrt{\frac{h^2}{1+h^2}} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{bmatrix}, \quad B(z) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1 - \lambda_w z}{z - \lambda_w} \end{bmatrix}.$$

Here the constant

$$h \equiv \frac{\pi_1(\lambda_w) \lambda_w \sigma_a^2}{\tilde{\pi}_1(\lambda_w) (1 - \rho_a \lambda_w)^2}$$

is endogenous and will be determined in equilibrium. The unitary matrix V is symmetric and satisfies $V = V^\top = V^{-1}$, and $\det V = -1$. We then obtain the Wold fundamental matrix

$$\Gamma(z) = \begin{bmatrix} \sigma_w \frac{z - \lambda_w}{1 - \rho_a z} V_{11} & \sigma_w \frac{1 - \lambda_w z}{1 - \rho_a z} V_{12} \\ \Gamma_\pi^{(1)}(z) & \Gamma_\pi^{(2)}(z) \end{bmatrix},$$

where we define

$$\begin{aligned}\Gamma_{\pi}^{(1)}(z) &\equiv \frac{\sigma_a^2}{\sigma_w} \frac{\pi_1(z)z}{(1-\pi_2(z))(1-\lambda_w z)(1-\rho_a z)} V_{11} + \frac{1}{\sigma_w} \frac{\tilde{\pi}_1(z)}{1-\pi_2(z)} \frac{1-\rho_a z}{1-\lambda_w z} V_{12}, \\ \Gamma_{\pi}^{(2)}(z) &\equiv \frac{\sigma_a^2}{\sigma_w} \frac{\pi_1(z)z}{(1-\pi_2(z))(z-\lambda_w)(1-\rho_a z)} V_{12} + \frac{1}{\sigma_w} \frac{\tilde{\pi}_1(z)}{1-\pi_2(z)} \frac{1-\rho_a z}{z-\lambda_w} V_{22}.\end{aligned}$$

We compute that

$$\Gamma^{-1}(z) = \begin{bmatrix} G_1(z) \frac{\sigma_a^2}{\sigma_w} \frac{\pi_1(z)}{\tilde{\pi}_1(z)} + G_2(z) \frac{1}{\sigma_w} & -\frac{1-\pi_2(z)}{\tilde{\pi}_1(z)} \sigma_w G_3(z) \\ -\left[G_4(z) \frac{\sigma_a^2}{\sigma_w} \frac{\pi_1(z)}{\tilde{\pi}_1(z)} + G_5(z) \frac{1}{\sigma_w} \right] & \frac{1-\pi_2(z)}{\tilde{\pi}_1(z)} \sigma_w G_6(z) \end{bmatrix},$$

where we define

$$\begin{aligned}G_1(z) &= -V_{12} \frac{z}{(z-\lambda_w)(1-\rho_a z)}, & G_2(z) &= -V_{22} \frac{1-\rho_a z}{z-\lambda_w}, \\ G_3(z) &= -V_{12} \frac{1-\lambda_w z}{1-\rho_a z}, & G_4(z) &= -V_{11} \frac{z}{(1-\lambda_w z)(1-\rho_a z)}, \\ G_5(z) &= -V_{12} \frac{1-\rho_a z}{1-\lambda_w z}, & G_6(z) &= -V_{11} \frac{z-\lambda_w}{1-\rho_a z}.\end{aligned}$$

Note that all $G_1(z), \dots, G_6(z)$ are independent of the endogenous price signal except for the constant in V . We also define the following functions that will be repeatedly used later:

$$\begin{aligned}\Gamma_I^{(1)}(z) &= G_1(z) \frac{\sigma_a^2}{\sigma_w} \frac{\pi_1(z)}{\tilde{\pi}_1(z)} + G_2(z) \frac{1}{\sigma_w}, \\ \Gamma_I^{(2)}(z) &= G_4(z) \frac{\sigma_a^2}{\sigma_w} \frac{\pi_1(z)}{\tilde{\pi}_1(z)} + G_5(z) \frac{1}{\sigma_w}, \\ \Gamma_I^{(3)}(z) &\equiv \sigma_w G_3(z) \frac{\pi_1(z)}{\tilde{\pi}_1(z)}, \\ \Gamma_I^{(4)}(z) &\equiv \sigma_w G_6(z) \frac{\pi_1(z)}{\tilde{\pi}_1(z)}.\end{aligned}$$

By the Paley-Wiener Theorem and the fact that $\tilde{\pi}_1(z)$ is analytic in the open unit disk and Wold fundamental, these functions are analytic in the open unit disk.⁹

Step 2. Solve for the equilibrium quantities. We conjecture that $y_{it} = M_y(L)\eta_{it}$, where $M_y(z) = [M_y^a(z), M_y^i(z), M_y^u(z)]$ and $M_y^a(z)$, $M_y^i(z)$, and $M_y^u(z)$ are all in $\mathbf{H}^2(\mathbb{D})$. Aggregation leads to aggregate output $y_t = M_y(z)I_y\eta_{it}$, where I_y is defined earlier.

Using the Wiener-Hopf prediction formula, we derive that

$$\mathbb{E}_{it}[y_t] = \left[\psi_y^{(1)}(L) \ \psi_y^{(2)}(L) \right]_+ \Gamma^{-1}(L)H(L)\eta_{it},$$

⁹Sayed and Kailath (2001) summarized the property of the Wold fundamental matrix implied by the Paley-Wiener theorem.

in terms of innovations, where the z -transform of the operator $\psi_y = [\psi_y^{(1)} \ \psi_y^{(2)}]$ is given by

$$\psi_y(z) = z^{-1} S_{yx}(z) (\Gamma^{-1}(z^{-1}))^\top. \quad (\text{D.3})$$

The annihilation is given by $[\psi_y^{(1)}(z)]_+ = \psi_y^{(1)}(z) - P_y^{(1)}(z)$ and $[\psi_y^{(2)}(z)]_+ = \psi_y^{(2)}(z) - P_y^{(2)}(z)$, where $P_y^{(1)}(z)$ and $P_y^{(2)}(z)$ denote the negative powers of z in the Laurent series expansions of $\psi_y^{(1)}(z)$ and $\psi_y^{(2)}(z)$, respectively. There are no explicit formulas for $P_y^{(1)}(z)$ and $P_y^{(2)}(z)$ in general.

Using (D.3), $y_t = M_y(z) I_y \eta_{it}$, and the cross-spectrum

$$S_{yx} = M_y(z) I_y \Sigma_\eta H^\top(z^{-1}) = [M_y^a, 0, M_y^u] \begin{bmatrix} \sigma_a^2 & & \\ & \sigma_u^2 & \\ & & \sigma_u^2 \end{bmatrix} \begin{bmatrix} \frac{1}{1-\rho_a z^{-1}} & \frac{\pi_1(z^{-1})}{(1-\pi_2(z^{-1}))(1-\rho_a z^{-1})} \\ 1 & 0 \\ 0 & \frac{1}{(1-\pi_2(z^{-1}))(1-\rho_u z^{-1})} \end{bmatrix},$$

we can derive

$$\begin{aligned} \psi_y^{(1)}(z) &= M_y^a(z) \sigma_a^2 A_n^{(1)}(z) - M_y^u(z) \sigma_u^2 A_n^{(2)}(z), \\ \psi_y^{(2)}(z) &= -M_y^a(z) \sigma_a^2 A_n^{(3)}(z) + M_y^u(z) \sigma_u^2 A_n^{(4)}(z), \end{aligned}$$

where we define

$$\begin{aligned} A_n^{(1)}(z) &= \frac{1}{1-\rho_a z^{-1}} [\Gamma_I^{(1)}(z^{-1}) - \Gamma_I^{(3)}(z^{-1})], \\ A_n^{(2)}(z) &= \frac{1}{1-\rho_u z^{-1}} \frac{1}{\pi_1(z^{-1})} \Gamma_I^{(3)}(z^{-1}), \\ A_n^{(3)}(z) &= \frac{1}{1-\rho_a z^{-1}} [\Gamma_I^{(2)}(z^{-1}) - \Gamma_I^{(4)}(z^{-1})], \\ A_n^{(4)}(z) &= \frac{1}{1-\rho_u z^{-1}} \frac{1}{\pi_1(z^{-1})} \Gamma_I^{(4)}(z^{-1}). \end{aligned}$$

Substituting the preceding expression for $\mathbb{E}_{it}[y_t]$ into (25) and matching coefficients for η_{it} , we obtain

$$M_y^a(z) = \frac{1}{\xi} \frac{1}{1-\rho_a z} + \frac{1}{1-\rho_a z} [G_y^{(1)}(z) - A_y^{(1)}(z) + G_y^{(2)}(z) - A_y^{(2)}(z)] \theta, \quad (\text{D.4})$$

$$M_y^i(z) = \frac{1}{\xi} + [G_y^{(1)}(z) - A_y^{(1)}(z)] \theta, \quad (\text{D.5})$$

$$M_y^u(z) = \frac{1}{1-\rho_u z} \frac{\theta}{\pi_1(z)} [G_y^{(2)}(z) - A_y^{(2)}(z)], \quad (\text{D.6})$$

where we define

$$\begin{aligned} G_y^{(1)}(z) &= \psi_y^{(1)}(z) \Gamma_I^{(1)}(z) - \psi_y^{(2)} \Gamma_I^{(2)}(z), \\ A_y^{(1)}(z) &= P_y^{(1)}(z) \Gamma_I^{(1)}(z) - P_y^{(2)} \Gamma_I^{(2)}(z), \\ G_y^{(2)}(z) &= \psi_y^{(2)}(z) \Gamma_I^{(4)}(z) - \psi_y^{(1)}(z) \Gamma_I^{(3)}(z), \\ A_y^{(2)}(z) &= P_y^{(2)}(z) \Gamma_I^{(4)}(z) - P_y^{(1)} \Gamma_I^{(3)}(z). \end{aligned}$$

Here $\Gamma_I^{(1)}(z), \dots, \Gamma_I^{(4)}(z)$ are defined earlier.

Using equations (D.4) and (D.6) and the definition of $G_y^{(1)}(z)$ and $G_y^{(2)}(z)$, we can derive that

$$\begin{bmatrix} Q_1(z) & Q_2(z) \\ Q_3(z) & Q_4(z) \end{bmatrix} \begin{bmatrix} M_y^a(z) \\ M_y^u(z) \end{bmatrix} = \begin{bmatrix} \frac{1}{\xi} - A_y^{(1)}(z)\theta - A_y^{(2)}(z)\theta \\ -A_y^{(2)}(z)\theta \end{bmatrix}, \quad (\text{D.7})$$

where we define

$$\begin{aligned} Q_1(z) &= (1 - \rho_a z) - \theta \sigma_a^2 H_a(z), \\ Q_2(z) &= \theta \sigma_u^2 H_u(z), \\ Q_3(z) &= \theta \sigma_a^2 H_d(z), \\ Q_4(z) &= (1 - \rho_u z) \pi_1(z) - \theta \sigma_u^2 H_c(z), \end{aligned}$$

and

$$\begin{aligned} H_a(z) &= A_n^{(1)}(z) \left(\Gamma_I^{(1)}(z) - \Gamma_I^{(3)}(z) \right) + A_n^{(3)}(z) \left(\Gamma_I^{(2)}(z) - \Gamma_I^{(4)}(z) \right), \\ H_u(z) &= A_n^{(2)}(z) \left(\Gamma_I^{(1)}(z) - \Gamma_I^{(3)}(z) \right) + A_n^{(4)}(z) \left(\Gamma_I^{(2)}(z) - \Gamma_I^{(4)}(z) \right), \\ H_c(z) &= A_n^{(4)}(z) \Gamma_I^{(4)}(z) + A_n^{(2)} \Gamma_I^{(3)}(z), \\ H_d(z) &= A_n^{(3)}(z) \Gamma_I^{(4)}(z) + A_n^{(1)} \Gamma_I^{(3)}(z). \end{aligned}$$

Once $\pi_1(z)$ and $\pi_2(z)$ are known, we can use the system (D.7) to determine $M_y^a(z)$ and $M_y^u(z)$. Equation (D.5) then determines $M_y^i(z)$.

As in the proof of Theorem 2, we deduce that $d_t = M_d(L) \eta_{it}$, $n_{it} = M_n(L) \eta_{it}$, and $b_{it} = M_b(L) \eta_{it}$, where

$$M_d(z) = \left[\frac{1}{\alpha_6} \left(1 - \frac{\alpha_7}{\alpha} \right) M_y^a(z) + \frac{\alpha_7}{\alpha \alpha_6} \frac{1}{1 - \rho_a z}, \quad 0, \quad \frac{1}{\alpha_6} \left(1 - \frac{\alpha_7}{\alpha} \right) M_y^u(z) \right], \quad (\text{D.8})$$

$$M_n(z) = \frac{1}{\alpha} \left[M_y^a(z) - \frac{1}{1 - \rho_a z}, \quad M_y^i(z) - 1, \quad M_y^u(z) \right], \quad (\text{D.9})$$

$$M_b(z) = \alpha_4 M_d(z) + \alpha_5 M_n(z). \quad (\text{D.10})$$

Each component of these vectors is in $\mathbf{H}^2(\mathbb{D})$.

Step 3. We proceed to the financial side of the model. We need to compute several conditional expectations for χ_{it} in (39). First, we use the Wiener-Hopf formula to derive

$$\alpha_3 \mathbb{E}_{it} \left[s_{it+2}^h \right] = \alpha_3 [\psi_s(L)]_+ \Gamma^{-1}(L) X_{it},$$

where the z -transform of the operator ψ_s is given by

$$\psi_s(z) = z^{-1} S_{sx}(z) (\Gamma^{-1}(z^{-1}))^\top,$$

and

$$\begin{aligned}\alpha_3 \left[\psi_s^{(1)}(z) \right]_+ &= \alpha_3 \psi_s^{(1)}(z) - P_s^{(1)}(z), \\ \alpha_3 \left[\psi_s^{(2)}(z) \right]_+ &= \alpha_3 \psi_s^{(2)}(z) - P_s^{(2)}(z).\end{aligned}$$

Here $P_s^{(1)}(z)$ and $P_s^{(2)}(z)$ denote the negative powers of z in the Laurent series expansions of $\alpha_3 \psi_s^{(1)}(z)$ and $\alpha_3 \psi_s^{(2)}(z)$, respectively. It follows that

$$\alpha_3 \left[\psi_s^{(1)}(z), \psi_s^{(2)}(z) \right]_+ \Gamma^{-1}(z) = \left[G_s^{(1)}(z) - A_s^{(1)}(z), \frac{1 - \pi_2(z)}{\pi_1(z)} \left(G_s^{(2)}(z) - A_s^{(2)}(z) \right) \right],$$

where

$$\begin{aligned}G_s^{(1)}(z) &= \sigma_i^2 z^{-1} \alpha_3 M_s^i(z) \left[\Gamma_I^{(1)}(z) \Gamma_I^{(1)}(z^{-1}) + \Gamma_I^{(2)}(z) \Gamma_I^{(2)}(z^{-1}) \right], \\ G_s^{(2)}(z) &= \sigma_i^2 z^{-1} \alpha_3 M_s^i(z) \left[-\Gamma_I^{(3)}(z) \Gamma_I^{(1)}(z^{-1}) - \Gamma_I^{(4)}(z) \Gamma_I^{(2)}(z^{-1}) \right],\end{aligned}$$

and

$$\begin{aligned}A_s^{(1)}(z) &= P_s^{(1)}(z) \Gamma_I^{(1)}(z) - P_s^{(2)}(z) \Gamma_I^{(2)}(z), \\ A_s^{(2)}(z) &= P_s^{(2)}(z) \Gamma_I^{(4)}(z) - P_s^{(1)}(z) \Gamma_I^{(3)}(z).\end{aligned}$$

It is easy to verify that Lemma 3 continues to hold, which implies

$$\begin{aligned}G_s^{(1)}(z) &= \sigma_i^2 \frac{1 - \lambda_s}{z(1 - \lambda_s z)} \pi_1(z) \left[\Gamma_I^{(1)}(z) \Gamma_I^{(1)}(z^{-1}) + \Gamma_I^{(2)}(z) \Gamma_I^{(2)}(z^{-1}) \right], \\ G_s^{(2)}(z) &= \sigma_i^2 \frac{1 - \lambda_s}{z(1 - \lambda_s z)} \pi_1(z) \left[-\Gamma_I^{(3)}(z) \Gamma_I^{(1)}(z^{-1}) - \Gamma_I^{(4)}(z) \Gamma_I^{(2)}(z^{-1}) \right].\end{aligned}$$

Second, the Wiener-Hopf formula gives

$$\mathbb{E}_{it} [q_{t+1}] = [\psi_q(L)]_+ \Gamma^{-1}(L) X_{it},$$

where the z -transform of the operator ψ_q is given by

$$\psi_q(z) = \frac{1}{z} [0, 1] S_x(z) (\Gamma^{-1}(z^{-1}))^\top = \frac{1}{z} [0, 1] \Gamma(z) = z^{-1} \left[\Gamma_\pi^{(1)}(z) \quad \Gamma_\pi^{(2)}(z) \right],$$

where $\Gamma_\pi^{(1)}(z)$ and $\Gamma_\pi^{(2)}(z)$ are defined earlier. Since $z = 0$ is the only inside pole of $\psi_q(z)$, it follows from the lemma in Appendix A of Hansen and Sargent (1980) that

$$[\psi_q(L)]_+ \Gamma^{-1}(z) = z^{-1} [0 \ 1] - P_q(z) \Gamma^{-1}(z),$$

where

$$P_q(z) = z^{-1} \left[\frac{1}{\sigma_w} \frac{\tilde{\pi}_1(0)}{1 - \pi_2(0)} V_{12}, \quad \frac{1}{\sigma_w} \frac{\tilde{\pi}_1(0)}{1 - \pi_2(0)} \left(-\frac{1}{\lambda_w} \right) V_{22} \right].$$

Thus

$$\begin{aligned}\mathbb{E}_{it} [q_{t+1}] &= -z^{-1} \frac{1}{\sigma_w} \frac{\tilde{\pi}_1(0)}{1 - \pi_2(0)} \left[V_{12} \Gamma_I^{(1)}(z) + \frac{1}{\lambda_w} V_{22} \Gamma_I^{(2)}(z) \right] a_{it} \\ &\quad + z^{-1} \frac{1}{\sigma_w} \frac{\tilde{\pi}_1(0)}{1 - \pi_2(0)} \left[V_{12} \Gamma_I^{(3)}(z) + \frac{1}{\lambda_w} V_{22} \Gamma_I^{(4)}(z) \right] \frac{1 - \pi_2(z)}{\pi_1(z)} q_t.\end{aligned}$$

Third, the Wiener-Hopf formula gives

$$\mathbb{E}_{it} [d_{t+1}] = [\psi_d(L)]_+ \Gamma^{-1}(L) X_{it},$$

where the z -transform of the operator ψ_d is given by

$$\psi_d(z) = \left[\psi_d^{(1)}(z), \psi_d^{(2)}(z) \right] = z^{-1} S_{dx}(z) (\Gamma^{-1}(z^{-1}))^\top,$$

and $\left[\psi_d^{(1)}(z) \right]_+ = \psi_d^{(1)}(z) - P_d^{(1)}(z)$, $\left[\psi_d^{(2)}(z) \right]_+ = \psi_d^{(2)}(z) - P_d^{(2)}(z)$. Here $P_d^{(1)}(z)$ and $P_d^{(2)}(z)$ denote the negative powers of z in the Laurent series expansions of $\psi_d^{(1)}(z)$ and $\psi_d^{(2)}(z)$, respectively. As in Step 2 we can compute that

$$\begin{aligned}\psi_d^{(1)}(z) &= z^{-1} \left[M_d^a(z) A_n^{(1)}(z) \sigma_a^2 - M_d^u(z) A_n^{(2)}(z) \sigma_u^2 \right], \\ \psi_d^{(2)}(z) &= z^{-1} \left[-M_d^a(z) A_n^{(3)}(z) \sigma_a^2 + M_d^u(z) A_n^{(4)}(z) \sigma_u^2 \right].\end{aligned}$$

It follows that

$$\mathbb{E}_{it} [d_{t+1}] = \left[G_d^{(1)}(L) - A_d^{(1)}(L), \frac{1 - \pi_2(z)}{\pi_1(z)} \left(G_d^{(2)}(L) - A_d^{(2)}(L) \right) \right] X_{it},$$

where

$$\begin{aligned}G_d^{(1)}(z) &= \psi_d^{(1)}(z) \Gamma_I^{(1)}(z) - \psi_d^{(2)}(z) \Gamma_I^{(2)}(z), \\ G_d^{(2)}(z) &= \psi_d^{(2)}(z) \Gamma_I^{(4)}(z) - \psi_d^{(1)}(z) \Gamma_I^{(3)}(z),\end{aligned}$$

and

$$\begin{aligned}A_d^{(1)}(z) &= P_d^{(1)}(z) \Gamma_I^{(1)}(z) - P_d^{(2)}(z) \Gamma_I^{(2)}(z), \\ A_d^{(2)}(z) &= P_d^{(2)}(z) \Gamma_I^{(4)}(z) - P_d^{(1)}(z) \Gamma_I^{(3)}(z).\end{aligned}$$

Finally, the Wiener-Hopf formula gives

$$\mathbb{E}_{it} [\Delta b_{it}] = [\psi_b(L)]_+ \Gamma^{-1}(L) X_{it},$$

where the z -transform of the operator ψ_b is given by

$$\psi_b(z) = \left[\psi_b^{(1)}(z), \psi_b^{(2)}(z) \right] = z^{-1} (z - 1) S_{bx}(z) (\Gamma^{-1}(z^{-1}))^\top,$$

and $\left[\psi_b^{(1)}(z)\right]_+ = \psi_b^{(1)}(z) - P_b^{(1)}(z)$, $\left[\psi_b^{(2)}(z)\right]_+ = \psi_b^{(2)}(z) - P_b^{(2)}(z)$. Here $P_b^{(1)}(z)$ and $P_b^{(2)}(z)$ denote the negative powers of z in the Laurent series expansions of $\psi_b^{(1)}(z)$ and $\psi_b^{(2)}(z)$, respectively. It follows that

$$\left[\psi_b^{(1)}(z), \psi_b^{(2)}(z)\right]_+ \Gamma^{-1}(z) = \left[G_b^{(1)}(z) - A_b^{(1)}(z), \frac{1 - \pi_2(z)}{\pi_1(z)} \left(G_b^{(2)}(z) - A_b^{(2)}(z)\right)\right],$$

where

$$\begin{aligned} G_b^{(1)}(z) &= \psi_b^{(1)}(z)\Gamma_I^{(1)}(z) - \psi_b^{(2)}\Gamma_I^{(2)}(z), \\ G_b^{(2)}(z) &= \psi_b^{(2)}(z)\Gamma_I^{(4)}(z) - \psi_b^{(1)}(z)\Gamma_I^{(3)}(z), \end{aligned}$$

and

$$\begin{aligned} A_b^{(1)}(z) &= P_b^{(1)}(z)\Gamma_I^{(1)}(z) - P_b^{(2)}(z)\Gamma_I^{(2)}(z), \\ A_b^{(2)}(z) &= P_b^{(2)}(z)\Gamma_I^{(4)}(z) - P_b^{(1)}(z)\Gamma_I^{(3)}(z). \end{aligned}$$

As in Step 2 we can also derive that

$$\begin{aligned} \psi_b^{(1)}(z) &= z^{-1}(z-1) \left[M_b^a(z)A_n^{(1)}(z)\sigma_a^2 - M_b^u(z)A_n^{(2)}(z)\sigma_u^2 + \Gamma_I^{(1)}(z^{-1})M_b^i(z)\sigma_i^2 \right], \\ \psi_b^{(2)}(z) &= z^{-1}(z-1) \left[-M_b^a(z)A_n^{(3)}(z)\sigma_a^2 + M_b^u(z)A_n^{(4)}(z)\sigma_u^2 - \Gamma_I^{(2)}(z^{-1})M_b^i(z)\sigma_i^2 \right]. \end{aligned}$$

Step 4. Derive the equilibrium system for $\pi_1(z)$ and $\pi_2(z)$. By Step 3 we obtain an expression for χ_{it} . Matching coefficients of $X_{it} = [a_{it}, q_i]^\top$ with those in (40), we obtain the following equilibrium conditions for $\pi_1(z)$ and $\pi_2(z)$:

$$\pi_1(z) = \frac{(1 - \lambda_s)}{z(1 - \lambda_s z)} \left[\Gamma_I^{(1)}(z)\Gamma_I^{(1)}(z^{-1}) + \Gamma_I^{(2)}(z)\Gamma_I^{(2)}(z^{-1}) \right] \sigma_i^2 \pi_1(z) - A_s^{(1)}(z) + \frac{R^{(1)}(z)}{z(1 - \lambda_s z)}, \quad (\text{D.11})$$

and

$$\begin{aligned} \pi_2(z) &= \frac{1 - \pi_2(z)}{z(1 - \lambda_s z)\pi_1(z)} \left\{ (\lambda_s - 1) \left[\Gamma_I^{(1)}(z^{-1})\Gamma_I^{(3)}(z) + \Gamma_I^{(2)}(z^{-1})\Gamma_I^{(4)}(z) \right] \sigma_i^2 \pi_1(z) \right. \\ &\quad \left. - z(1 - \lambda_s z)A_s^{(2)}(z) + R^{(2)}(z) \right\} + z^{-1}\beta, \end{aligned} \quad (\text{D.12})$$

where $R^{(1)}(z)$ and $R^{(2)}(z)$ are defined as

$$\begin{aligned} R^{(1)}(z) &= \left\{ -\beta \frac{1}{\sigma_w} \frac{\tilde{\pi}_1(0)}{1 - \pi_2(0)} z^{-1} \left(V_{12}\Gamma_I^{(1)}(z) + \frac{1}{\lambda_w} V_{22}\Gamma_I^{(2)}(z) \right) \right. \\ &\quad \left. + (1 - \beta) \left[G_d^{(1)}(z) - A_d^{(1)}(z) \right] + \left[G_b^{(1)}(z) - A_b^{(1)}(z) \right] \right\} z(1 - \lambda_s z) \end{aligned}$$

and

$$R^{(2)}(z) = \left\{ \beta \frac{1}{\sigma_w} \frac{\tilde{\pi}_1(0)}{1 - \pi_2(0)} z^{-1} \left(V_{12} \Gamma_I^{(3)}(z) + \frac{1}{\lambda_w} V_{22} \Gamma_I^{(4)}(z) \right) + (1 - \beta) \left[G_d^{(1)}(z) - A_d^{(1)}(z) \right] + \left[G_b^{(2)}(z) - A_b^{(2)}(z) \right] \right\} z(1 - \lambda_s z).$$

Define an operator \mathcal{T} that maps the vector of functions $[\pi_1(z), \pi_2(z)]$ to the vector of functions that are equal to the expressions on the right-hand sides of equations (D.11) and (D.12). Since the signal system contains endogenous prices, many variables in these expressions depend on $[\pi_1(z), \pi_2(z)]$ in a complicated way. Thus the operator \mathcal{T} is nonlinear in general. The equilibrium functions $\pi_1(z)$ and $\pi_2(z)$ correspond to the fixed point of \mathcal{T} in $\mathbf{H}^2(\mathbb{D})$. Moreover, we use (D.12) to derive that

$$\frac{\pi_1(z)}{1 - \pi_2(z)} = \frac{1}{(1 - \lambda_s z)(z - \beta)} \left\{ -z(1 - \lambda_s z) A_s^{(2)}(z) + R^{(2)}(z) + \left[z(1 - \lambda_s z) - (1 - \lambda_s) \left(\Gamma_I^{(1)}(z^{-1}) \Gamma_I^{(3)}(z) + \Gamma_I^{(2)}(z^{-1}) \Gamma_I^{(4)}(z) \right) \sigma_i^2 \right] \pi_1(z) \right\}. \quad (\text{D.13})$$

We also have to ensure that $\frac{\pi_1(z)}{1 - \pi_2(z)} \in \mathbf{H}^2(\mathbb{D})$ in equilibrium.

D. 2 Numerical Methods

The equilibrium is characterized by the fixed point of the operator \mathcal{T} . Due to the endogeneity of the price signal, this operator is nonlinear and thus the model does not admit a solution in the form of rational functions. We now approximate the true model solution, which is in the form of $\text{MA}(\infty)$, by finite-order $\text{ARMA}(p, q)$ processes in the time domain or by rational functions in the frequency domain. Rational functions also allow us to evaluate the annihilation operator tractably using the lemma in Appendix A of Hansen and Sargent (1980). The numerical method involves the following steps.

Step 1. We begin by an initial guess for $\pi_1(z)$ in the form of an irreducible rational function:

$$\pi_1(z) = \sigma_\pi \frac{\prod_{i=1}^q (1 + \theta_i z)}{\prod_{j=1}^p (1 - \rho_j z)}, \quad (\text{D.14})$$

where p and q are the orders of the ARMA representation and σ_π , θ_i , and $|\rho_j| < 1$ are constants. Given the initial guess, we solve for the canonical factorization equation (D.2) to obtain

$$\tilde{\pi}_1(z) = \sigma_{\tilde{\pi}} \frac{\prod_{i=1}^{m+1} (1 + \hat{\theta}_i z)}{(1 - \rho_a z)(1 - \rho_u z) \prod_{j=1}^p (1 - \rho_j z)}, \quad (\text{D.15})$$

where $m = \max(p, q)$ and $\sigma_{\bar{\pi}}$ and $\hat{\theta}_i$ are determined by the factorization:

$$\begin{aligned} \sigma_{\bar{\pi}}^2 \prod_{i=1}^{m+1} (1 + \hat{\theta}_i z) (1 + \hat{\theta}_i z^{-1}) &= \sigma_a^2 \sigma_i^2 \sigma_{\pi}^2 \prod_{i=1}^q (1 + \theta_i z) (1 + \theta_i z^{-1}) (1 - \rho_u z) (1 - \rho_u z^{-1}) \\ &\quad + \sigma_u^2 \sigma_w^2 (1 - \lambda_w z) (1 - \lambda_w z^{-1}) \prod_{j=1}^p (1 - \rho_j z) (1 - \rho_j z^{-1}). \end{aligned} \quad (\text{D.16})$$

In particular, set $|\hat{\theta}_i| < 1, \forall i = 1, 2, \dots, m+1$.

In addition, we take an initial guess for the constant $\frac{\bar{\pi}_1(0)}{1 - \pi_2(0)}$.

Step 2. Solve for the decision rules for quantities on the real side of the economy. We use (D.7) to derive $M_y^a(z)$ and $M_y^u(z)$. We need to compute $P_y^{(1)}(z)$ and $P_y^{(2)}(z)$ by using the lemma in Hansen and Sargent (1980). Given the guess for $\pi_1(z)$ in (D.14), (D.15), and the expressions for $\psi_y^{(1)}(z)$ and $\psi_y^{(2)}(z)$ derived in Step 2 of Section D. 1, we deduce that $-\hat{\theta}_1, \dots$, and $-\hat{\theta}_{m+1}$ are the poles of $\psi_y^{(1)}(z)$ and $\psi_y^{(2)}(z)$ that are inside the unit disk. Thus we have

$$\begin{aligned} P_y^{(1)}(z) &= \sum_{k=1}^{m+1} \frac{\psi_{k,y}}{z + \hat{\theta}_k}, \quad P_y^{(2)}(z) = \sum_{k=1}^{m+1} f_k \frac{\psi_{k,y}}{z + \hat{\theta}_k}, \\ f_k &\equiv \frac{V_{11}}{V_{12}} \frac{1 + \lambda_w \hat{\theta}_k}{\hat{\theta}_k + \lambda_w}, \quad k = 1, 2, \dots, m+1, \end{aligned}$$

where each $\psi_{k,y}$ is a constant defined as

$$\psi_{k,y} = \lim_{z \rightarrow -\hat{\theta}_k} (z + \hat{\theta}_k) \left[M_y^a(z) \sigma_a^2 A_n^{(1)}(z) - M_y^u(z) \sigma_u^2 A_n^{(2)}(z) \right],$$

provided that all poles $\{-\hat{\theta}_k\}_{k=1}^{m+1}$ inside the unit disk are distinct. No constant $\psi_{k,y}$ can be solved numerically using the preceding formula because $M_y^a(z)$ and $M_y^u(z)$ are unknown functions to be determined. We will use the method below to determine all $\psi_{k,y}$.

Plugging the guess for $\pi_1(z)$ and the expressions above for $P_y^{(1)}(z)$ and $P_y^{(2)}(z)$ (taking all unknown constant $\psi_{k,y}$ as given) into (D.7), we obtain the following linear system:

$$\begin{bmatrix} Q_1(z) & Q_2(z) \\ \tilde{Q}_3(z) & \tilde{Q}_4(z) \end{bmatrix} \begin{bmatrix} M_y^a(z) \\ M_y^u(z) \end{bmatrix} = \begin{bmatrix} \frac{1}{\xi} - A_y^{(1)}(z)\theta - A_y^{(2)}(z)\theta \\ -\prod_{j=1}^p (1 - \rho_j z) A_y^{(2)}(z)\theta \end{bmatrix} \equiv \begin{bmatrix} C_y^{(1)}(z) \\ C_y^{(2)}(z) \end{bmatrix}, \quad (\text{D.17})$$

where

$$\begin{aligned} \tilde{Q}_3(z) &= \theta \sigma_a^2 \prod_{j=1}^p (1 - \rho_j z) H_d(z), \\ \tilde{Q}_4(z) &= \prod_{i=1}^q (1 + \theta_i z) (1 - \rho_u z) \sigma_{\pi} - \theta \sigma_u^2 \prod_{j=1}^p (1 - \rho_j z) H_c(z). \end{aligned}$$

Solving this linear system yields

$$\begin{bmatrix} M_y^a(z) \\ M_y^u(z) \end{bmatrix} = \frac{1}{D_1^2(z) [\tilde{Q}_4(z)Q_1(z) - Q_2(z)\tilde{Q}_3(z)]} \begin{bmatrix} D_1(z)\tilde{Q}_4(z)C_y^{(1)}(z) - D_1(z)Q_2(z)C_y^{(2)}(z) \\ -D_1(z)\tilde{Q}_3(z)C_y^{(1)}(z) + D_1(z)Q_1(z)C_y^{(2)}(z) \end{bmatrix},$$

where we define

$$D_1(z) = \prod_{i=1}^{m+1} (1 + \hat{\theta}_i z) (z + \hat{\theta}_i).$$

We can verify that the above solutions for $M_y^a(z)$ and $M_y^u(z)$ are irreducible rational functions. That is, the numerator and denominator are pure polynomial functions.

The denominator function $D_y(z) \equiv D_1^2(z) [\tilde{Q}_4(z)Q_1(z) - Q_2(z)\tilde{Q}_3(z)]$ determines the existence and uniqueness of a stationary equilibrium. The necessary condition for the existence requires that $D_y(z)$ has precisely $m+1$ roots inside the open unit disk. We verify this condition in every iteration in our numerical computations. Let $\{z_j\}_{j=1}^{m+1}$ denote all the inside roots of $D_y(z)$. To pin down the vector of constants $\psi_y = [\psi_{1,y}, \dots, \psi_{m+1,y}]^\top$, we use the following system of $m+1$ equations:

$$D_1(z_j)\tilde{Q}_4(z_j)C_y^{(1)}(z_j) - D_1(z_j)Q_2(z_j)C_y^{(2)}(z_j) = 0, \quad j = 1, 2, \dots, m+1,$$

which gives a linear system for ψ_y :

$$A^c \psi_y = C^c,$$

where A^c is an $(m+1) \times (m+1)$ matrix of constants, and C^c is an $(m+1)$ dimensional vector of constants. We derive this system by substituting $P_y^{(1)}(z)$ and $P_y^{(2)}(z)$ (which depend on ψ_y) into $A_y^{(i)}(z)$ and $C_y^{(i)}(z)$, $i = 1, 2$. For simplicity, we omit the detailed algebra here. The idea is that the solution for ψ_y must remove the poles of $D_y(z)$ inside the open unit disk so that the solutions for $M_y^a(z)$ and $M_y^u(z)$ are analytic inside the open unit disk. If the matrix A^c is invertible, the solution is unique. We verify this condition in every iteration of our numerical computations. Given the solutions for $M_y^a(z)$ and $M_y^u(z)$, we solve for $M_y^i(z)$ using (D.5). We can also solve for $M_b(z)$, $M_n(z)$, and $M_d(z)$ using the formulas derived in Step 2 of Section D. 1.

Step 3. We compute all annihilated functions of negative powers of z on the financial side of the model using the Hansen-Sargent lemma. Let $\{z_k\}_{k=1}^{m+2} = \{0, -\hat{\theta}_1, \dots, -\hat{\theta}_{m+1}\}$ denote the set of poles inside the unit disk. Provided that all poles are distinct, we have

$$P_s^{(1)}(z) = \sum_{k=1}^{m+2} \frac{\psi_{k,s}^{(1)}}{z - z_k}, \quad P_s^{(2)}(z) = - \sum_{k=1}^{m+2} \frac{\psi_{k,s}^{(2)}}{z - z_k},$$

$$P_d^{(1)}(z) = \sum_{k=1}^{m+2} \frac{\psi_{k,d}^{(1)}}{z - z_k}, \quad P_d^{(2)}(z) = \sum_{k=1}^{m+2} \frac{\psi_{k,d}^{(2)}}{z - z_k},$$

$$P_b^{(1)}(z) = \sum_{k=1}^{m+2} \frac{\psi_{k,b}^{(1)}}{z - z_k}, \quad P_b^{(2)}(z) = \sum_{k=1}^{m+2} \frac{\psi_{k,b}^{(2)}}{z - z_k},$$

where the constants are given by

$$\psi_{k,s}^{(1)} = \lim_{z \rightarrow z_k} (z - z_k) \left[z^{-1} \alpha_3 M_s^i(z) \Gamma_I^{(1)}(z^{-1}) \right] \sigma_i^2,$$

$$\psi_{k,s}^{(2)} = \lim_{z \rightarrow z_k} (z - z_k) \left[z^{-1} \alpha_3 M_s^i(z) \Gamma_I^{(2)}(z^{-1}) \right] \sigma_i^2,$$

$$\psi_{k,d}^{(1)} = \lim_{z \rightarrow z_k} (z - z_k) z^{-1} \left[M_d^a(z) A_n^{(1)}(z) \sigma_a^2 - M_d^u(z) A_n^{(2)}(z) \sigma_u^2 \right],$$

$$\psi_{k,d}^{(2)} = \lim_{z \rightarrow z_k} (z - z_k) z^{-1} \left[M_d^u(z) A_n^{(4)}(z) \sigma_u^2 - M_d^a(z) A_n^{(3)}(z) \sigma_a^2 \right],$$

$$\psi_{k,b}^{(1)} = \lim_{z \rightarrow z_k} (z - z_k)(z - 1) z^{-1} \left[M_b^a(z) A_n^{(1)}(z) \sigma_a^2 - M_b^u(z) A_n^{(2)}(z) \sigma_u^2 + M_b^i(z) \Gamma_I^{(1)}(z^{-1}) \sigma_i^2 \right],$$

$$\psi_{k,b}^{(2)} = \lim_{z \rightarrow z_k} (z - z_k)(z - 1) z^{-1} \left[M_b^u(z) A_n^{(4)}(z) \sigma_u^2 - M_b^a(z) A_n^{(3)}(z) \sigma_a^2 - M_b^i(z) \Gamma_I^{(2)}(z^{-1}) \sigma_i^2 \right].$$

Given the guess of $\pi_1(z)$ in (D.14) and the solutions for $M_y(z)$, $M_d(z)$, $M_n(z)$, and $M_b(z)$ in the previous step, we can compute the constants $\psi_{k,d}^{(1)}$, $\psi_{k,d}^{(2)}$, $\psi_{k,b}^{(1)}$, and $\psi_{k,b}^{(2)}$ for $k = 1, 2, \dots, m + 2$. The other constants $\psi_{k,s}^{(1)}$ and $\psi_{k,s}^{(2)}$ will be solved in the next step. We cannot use the formulas above to determine $\psi_{k,s}^{(1)}$ and $\psi_{k,s}^{(2)}$ because $M_s^i(z)$ is unknown function to be determined in equilibrium. We can verify that

$$\psi_{k,s}^{(2)} = h_k \psi_{k,s}^{(1)},$$

$$h_k = \begin{cases} \frac{V_{12}}{V_{22}} \frac{1 - \lambda_w z_k}{z_k - \lambda_w}, & \text{if } z_k = 0, \\ \frac{V_{11}}{V_{12}} \frac{1 - \lambda_w z_k}{z_k - \lambda_w}, & \text{else.} \end{cases}$$

Thus we only need to solve for $\psi_{k,s}^{(1)}$, $k = 1, \dots, m + 2$.

Step 4. Solve for the update of $\pi_1(z)$ and $\pi_2(z)$ using equations (D.11) and (D.12). Given the guess for $\pi_1(z)$ in (D.14), we can verify that $R^{(1)}(z)$ is an analytic rational function. Let $R_D^{(1)}(z)$ denote the denominator polynomial function of $R^{(1)}(z)$ in its irreducible form. Since $R^{(1)}(z)$ is analytic, $R_D^{(1)}(z) \neq 0$ inside the open unit disk. We can write

$$R_D^{(1)}(z) = R_D^{(1)}(0) \prod_{i=1}^g (1 + z_i z),$$

where g denotes the degree of $R_D^{(1)}(z)$ and $-z_i^{-1}, \dots, -z_g^{-1}$ are the g roots of $R_D^{(1)}(z)$ that are outside the open unit disk. Using the definition of the unitary matrix V , we can show that the denominator

of the rational function $z(1-\lambda_s z) - (1-\lambda_s)\sigma_i^2 \left[\Gamma_I^{(1)}(z)\Gamma_I^{(1)}(z^{-1}) + \Gamma_I^{(2)}(z)\Gamma_I^{(2)}(z^{-1}) \right]$ in the irreducible form is given by

$$D_1(z) = \prod_{k=1}^{m+1} \left(1 + \hat{\theta}_k z \right) \left(z + \hat{\theta}_k \right).$$

Notice that some factors in $D_1(z)$ and $R_D^{(1)}(z)$ may be identical. We define $D_2(z)$ as their least common multiple.

We now rewrite (D.11) as

$$\pi_1(z) = \frac{D_2(z) \left[R^{(1)}(z) - z(1 - \lambda_s z) A_s^{(1)}(z) \right]}{D_2(z) \left[z(1 - \lambda_s z) - (1 - \lambda_s)\sigma_i^2 \left[\Gamma_I^{(1)}(z)\Gamma_I^{(1)}(z^{-1}) + \Gamma_I^{(2)}(z)\Gamma_I^{(2)}(z^{-1}) \right] \right]}, \quad (\text{D.18})$$

where both the numerator and the denominator are pure polynomial functions. Let $\pi_1^D(z)$ denote the denominator function. The existence and uniqueness of a stationary equilibrium solution for $\pi_1(z)$ is determined by the roots of $\pi_1^D(z)$. More specifically, to determine the $m+2$ dimensional vector of unknown constants $\psi_s = \left[\psi_{1,s}^{(1)}, \dots, \psi_{m+2,s}^{(1)} \right]^\top$, we need $\pi_1^D(z)$ to have precisely $m+2$ distinct roots inside the open unit disk. We verify this condition in every iteration of the numerical computation. Without risk of confusion, let $\{\hat{z}_k\}_{k=1}^{m+2}$ denote the set of distinct roots of $\pi_1^D(z)$ that are inside the open unit disk.

We then pin down ψ_s by removing the poles $\{\hat{z}_k\}_{k=1}^{m+2}$ and evaluating the numerator polynomial

$$D_2(\hat{z}_k) \left[R^{(1)}(\hat{z}_k) - \hat{z}_k(1 - \lambda_s \hat{z}_k) A_s^{(1)}(\hat{z}_k) \right] = 0, \quad \forall k = 1, 2, \dots, m+2,$$

which leads to the linear system

$$A^\pi \psi_s = C^\pi,$$

where we have used the definition of $A_s^{(1)}(z)$ and the expression of $P_s^{(1)}(z)$ derived in Step 2. We deduce that A^π is an $(m+2) \times (m+2)$ matrix with elements given by

$$A^\pi(k, i) = \frac{\Gamma_I^{(1)}(\hat{z}_k) D_2(\hat{z}_k)}{\hat{z}_k - z_i} + \frac{\Gamma_I^{(2)}(\hat{z}_k) D_2(\hat{z}_k)}{\hat{z}_k - z_i} h_i,$$

for $k = 1, 2, \dots, m+2$ and $i = 1, 2, \dots, m+2$, and $z_i \in \left\{ 0, -\hat{\theta}_1, \dots, -\hat{\theta}_{m+1} \right\}$. The k^{th} element of $(m+2) \times 1$ vector C^π is given by

$$C^\pi(k) = R^{(1)}(\hat{z}_k) D_2(\hat{z}_k), \quad \forall k = 1, 2, \dots, m+2.$$

If A^π is full rank, the solution is indeed unique. Again, we verify this condition in every iteration.

Once determining ψ_s , we update the guess for $\pi_1(z)$ using the solution in (D.18). Given this solution for $\pi_1(z)$, we use (D.13) to solve for $\frac{\pi_1(z)}{1-\pi_2(z)}$. Observe that the numerator on the right-hand

side of (D.13) is analytic inside the open unit disk, but we still need to remove the pole at $z = \beta$. We set the constant $\frac{\tilde{\pi}_1(0)}{1-\pi_2(0)}$ to remove this pole. That is,

$$\phi(\beta)\pi_1(\beta) - \beta(1 - \lambda_s\beta)A_s^{(2)}(\beta) + R^{(2)}(\beta) = 0,$$

where

$$\phi(z) = z(1 - \lambda_s z) - (1 - \lambda_s) \left[\Gamma_I^{(1)}(z^{-1})\Gamma_I^{(3)}(z) + \Gamma_I^{(2)}(z^{-1})\Gamma_I^{(4)}(z) \right] \sigma_i^2.$$

This leads to the following solution for the constant

$$\frac{\tilde{\pi}_1(0)}{1 - \pi_2(0)} = \frac{\sigma_w}{\beta(1 - \lambda_s\beta) \left[V_{12}\Gamma_I^{(3)}(\beta) + \frac{1}{\lambda_w}V_{22}\Gamma_I^{(4)}(\beta) \right]} \times \left\{ \beta(1 - \lambda_s\beta) \left(A_s^{(2)}(\beta) - (1 - \beta) \left[G_d^{(1)}(\beta) - A_d^{(1)}(\beta) \right] - \left[G_b^{(2)}(\beta) - A_b^{(2)}(\beta) \right] \right) - \phi(\beta)\pi_1(\beta) \right\}. \quad (\text{D.19})$$

We use this solution to update the initial guess for $\frac{\tilde{\pi}_1(0)}{1-\pi_2(0)}$. Finally, we iterate until convergence.

In summary, we employ the following iterative algorithm to solve the model.

Algorithm 1 Numerical Approximation of Equilibrium

Step 0: Begin with a guess for $p, q, \sigma_\pi, \pi_c \equiv \frac{\tilde{\pi}_1(0)}{1-\pi_2(0)}, \{\theta_i\}_{i=1}^q, \{\rho_j\}_{j=1}^p$ with $|\rho_j| < 1, \forall j$.

Step 1: Set $m = \max\{p, q\}$ and compute $\sigma_{\hat{\pi}}$ and $\{\hat{\theta}_i\}_{i=1}^m$ using (D.16).

Step 2: Solve for the functions $M_y(z), M_d(z), M_b(z),$ and $M_n(z)$.

Step 3: Let $\pi_1^A(z)$ and π_c^+ be the expressions on the right-hand sides of (D.18) and (D.19), respectively.

Step 4: Update the initial guess using

$$\pi_1^+(z) = \sigma_\pi^+ \frac{\prod_{i=1}^q (1 + \theta_i^+ z)}{\prod_{j=1}^p (1 - \rho_j^+ z)},$$

where $\sigma_\pi^+, \theta_i^+, \rho_j^+$ are the solution to the problem

$$\min_{\sigma_\pi^+, \theta_i^+, \rho_j^+} \sum_{n=1}^N |\pi_1^+(n) - \pi_1^A(n)|^2,$$

where $\pi_1^+(n)$ and $\pi_1^A(n)$ are the coefficients of the moving average expansion of $\pi_1^+(z)$ and $\pi_1^A(z)$, with $N = 70$.

Step 5: Iterate Steps 0–4 until $\max\{|\rho_j^+ - \rho_j|, |\theta_i^+ - \theta_i|, |\sigma_\pi^+ - \sigma_\pi|\} < 10^{-3}, \forall i, j$

Step 6: Compute $\epsilon = \max\{|\|\pi_1^+(z) - \pi_1^A(z)\|_{\mathbf{H}^2}, |\pi_c^+ - \pi_c|\}$; if $\epsilon < 10^{-5}$, stop; otherwise, set $p := p + 1, q := q + 1$ and repeat Steps 0–5.

E Frequency Domain Methods

In this section we introduce some mathematical background for the frequency domain methods. We study casual covariance stationary real-valued equilibrium processes that have an $\text{MA}(\infty)$ representation. For example, the aggregate output process in the model of Section 3 can be written as

$$y_t = \sum_{j=0}^{\infty} M_j \varepsilon_{a,t-j}, \quad (\text{E.1})$$

where $\{M_j\}_{j=0}^\infty$ is square summable, i.e., $\sum_{j=0}^\infty |M_j|^2 < \infty$. Solving for the infinite sequence of $\{M_j\}_{j=0}^\infty$ is a daunting task. The idea of the frequency domain method is to transform this problem into an equivalent problem of solving for an analytical function in the Hardy space. To define this space, we recall that \mathbb{C} denotes the complex plan, \mathbb{T} denotes the unit circle, and \mathbb{D} denotes the open unit disk.

Definition 1 *The Hardy space $\mathbf{H}^2(\mathbb{D})$ is the class of analytical functions g in the unit disk \mathbb{D} satisfying*

$$\left\{ \frac{1}{2\pi} \sup_{0 \leq r < 1} \int_{-\pi}^{\pi} |g(re^{i\omega})|^2 d\omega \right\}^{1/2} < \infty.$$

It can be verified that the expression on the preceding inequality defines a norm on $\mathbf{H}^2(\mathbb{D})$, denoted as $\|g\|_{\mathbf{H}^2}$. The Hardy space can also be viewed as a certain closed vector subspace of the complex L^2 space for the unit circle \mathbb{T} . This connection is provided by the fact that the radial limit

$$\tilde{g}(e^{i\omega}) = \lim_{r \uparrow 1} g(re^{i\omega})$$

exists for almost all $\omega \in [-\pi, \pi]$. The function \tilde{g} belongs to the space $L^2(\mathbb{T})$ of functions $f : \mathbb{T} \rightarrow \mathbb{C}$ with the inner product

$$\langle f_1, f_2 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(e^{i\omega}) \overline{f_2(e^{i\omega})} d\omega, \quad f_1, f_2 \in L^2(\mathbb{T}).$$

Then we have

$$\|g\|_{\mathbf{H}^2} = \|\tilde{g}\|_{L^2} = \lim_{r \uparrow 1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{i\omega})|^2 d\omega \right\}^{1/2} < \infty.$$

Denote by $\mathbf{H}^2(\mathbb{T})$ the vector subspace of $L^2(\mathbb{T})$ consisting of all limit functions \tilde{g} , when g varies in $\mathbf{H}^2(\mathbb{D})$.

Theorem 4 (Katznelson 1976) *$f \in \mathbf{H}^2(\mathbb{T})$ if and only if $f \in L^2(\mathbb{T})$ and $\hat{f}_n = 0$ for all $n < 0$, where \hat{f}_n is the Fourier coefficient of a function f integrable on the unit circle,*

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\omega}) e^{-i\omega n} d\omega, \quad n = 0, \pm 1, \pm 2, \dots$$

Suppose that $\tilde{g} \in \mathbf{H}^2(\mathbb{T})$ and \tilde{g} has Fourier coefficients $\{a_n\}$ with $a_n = 0$ for all $n < 0$. We define

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1.$$

The following theorem ensures $g \in \mathbf{H}^2(\mathbb{D})$. Thus we have a bijection between $\mathbf{H}^2(\mathbb{D})$ and $\mathbf{H}^2(\mathbb{T})$.

Theorem 5 *If $f(z)$ is an analytic function in \mathbb{D} and its Laurent expansion is*

$$f(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then $f \in \mathbf{H}^2(\mathbb{D})$ if and only if $\{b_n\}_{n=0}^{\infty}$ is square summable, i.e., $\sum_{n=0}^{\infty} |b_n|^2 < \infty$. When this condition is satisfied

$$\sum_{n=0}^{\infty} |b_n|^2 = \|f\|_{\mathbf{H}^2}.$$

We call the map from the sequence $\{b_n\}_{n=0}^{\infty}$ to $f(z)$ a z -transform. Theorem 5 also allows us to give an equivalent definition of the Hardy space $\mathbf{H}^2(\mathbb{D})$ as the class of analytical functions $f : \mathbb{D} \rightarrow \mathbb{C}$, which are the z -transforms of some square summable sequences. Thus solving for $\{M_j\}_{j=0}^{\infty}$ in (E.1) is equivalent to solving for a function $M(z)$ in the hardy space $\mathbf{H}^2(\mathbb{D})$. In particular, we can write $y_t = M(L) \epsilon_{at}$, where $M(z) \in \mathbf{H}^2(\mathbb{D})$ is the object we will solve for. We can use Theorem 5 to compute the variance of y_t easily because

$$\text{Var}(y_t) = \sigma_a^2 \sum_{j=0}^{\infty} M_j^2 = \sigma_a^2 \|M(z)\|_{\mathbf{H}^2}.$$

Finally, a rational function $f(z) \in \mathbf{H}^2(\mathbb{D})$ if and only if $f(z)$ is analytic in the closed unit disk. In particular, poles are not allowed on the unit circle.

F Computing Expectations in the Frequency Domain

We present our approach in a general framework. Suppose that the signal is an ℓ -dimensional variable X_t , defined in terms of infinite-order moving average processes.¹⁰ Let \mathbb{C} denote the complex plane, \mathbb{T} denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, and \mathbb{D} denote the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

Definition 2 (*signal representation*) *The ℓ -dimensional real-valued signal process $\{X_t\}$ is linearly regular and admits representation*

$$X_t = \underset{\ell \times 1}{H(L)} \underset{\ell \times k}{\eta_t} \underset{k \times 1}{}, \quad \ell \leq k,$$

where L denotes the lag operator, $\{\eta_t\}$ represents structural Gaussian innovations with mean zero and covariance matrix Σ_η , and $H(z)$ is an $\ell \times k$ matrix analytic function defined on the open unit disk \mathbb{D} in the matrix-valued Hardy space $\mathbf{H}^2(\mathbb{D})$.¹¹

We call $H(\cdot)$ the signal matrix or the transfer function as in the mathematics literature. To simplify the signal extraction problem, it is useful to assume a maximal rank condition for the signal process so that no redundant information is contained in X_t .

¹⁰We can extend the definition to contain information about future innovations (e.g. Bachetta and Wincoop, 2008).

¹¹See Appendix E for the definition of the Hardy space.

Assumption 4 *The ℓ -dimensional signal process X_t has maximal rank, i.e. the rank of its associated spectral density $f_x(\omega)$ equals its dimension:*

$$\text{rank}(f_x(\omega)) = \ell$$

for almost all $\omega \in [-\pi, \pi]$.

An important methodological contribution of our paper is that we study a non-square signal representation in that $\ell < k$. The existing literature focuses on the case of square signal representations with $\ell = k$ (e.g., Kasa, Walker, and Whiteman (2014), and Rondina and Walker (2015)). To use the Wiener-Hopf prediction formula, we need the Wold fundamental representation for the signal process. For the case of non-square signal representation, finding the Wold representation is non-trivial. We use spectral factorization techniques to solve this problem.

F.1 A Two-Step Spectral Factorization Method

Our goal is to find a Wold representation for $\{X_t\}$. We are looking for an analytic matrix function $\Gamma(\cdot)$ in the Hardy space $\mathbf{H}^2(\mathbb{D})$ such that

$$X_t = \begin{matrix} \Gamma(L) \\ \ell \times 1 \end{matrix} \begin{matrix} e_t \\ \ell \times \ell \end{matrix}, \quad f_x(\omega) = \Gamma(e^{-i\omega}) \Gamma^*(e^{-i\omega}), \quad \omega \in [-\pi, \pi], \quad (\text{F.1})$$

where asterisk denotes the conjugate transpose, $\{e_t\}$ is some mutually uncorrelated Wold (fundamental) innovation process with mean zero and an identity covariance matrix, f_x is the spectral density, and $\Gamma(\cdot)$ is an analytic function.¹²

For the square signal case with $\ell = k$, we can directly apply the Beurling-Blaschke factorization method to derive the Wold representation as in Kasa, Walker, and Whiteman (2014) and Rondina and Walker (2015). However, this method does not apply to the non-square case with $\ell < k$. We propose a two-step spectral factorization procedure. In step 1 we apply the convolution theorem to find the spectral density $f_x(\omega)$ of the signal process $\{X_t\}$. Then we use the Rozanov (1967) theorem to find a lower triangular decomposition of $f_x(\omega)$. In step 2 we apply the Beurling-Blaschke factorization method to the lower triangular matrix.

We start with the following result.

Lemma 4 *Suppose that X_t is the vector of signals defined in Definition 2 and Assumption 4 holds. Moreover, the transfer function $H(z)$ is a non-square matrix function with dimension $k > \ell$. Then the spectral density $f_x(\omega)$ is an $\ell \times \ell$ matrix function defined on $[-\pi, \pi]$ and*

$$f_x(\omega) = H(e^{-i\omega}) \Sigma_\eta H^*(e^{-i\omega}) = H(z) \Sigma_\eta H(z^{-1})^\top, \quad z = e^{-i\omega},$$

¹²Note that the Wold fundamental innovations can have non-diagonal, non-normalized covariance matrices. Using the unitary eigen-decomposition of the covariance matrix, we can obtain the orthonormal Wold representations with an identity covariance matrix.

where the superscript \top denotes the transpose of a matrix. Furthermore, $f_x(\omega)$ is a Hermitian normal matrix that is non-negative definite for almost all $\omega \in [-\pi, \pi]$. If we extend the definition of z to the entire complex plane \mathbb{C} , then the autocovariance generating function is given by $S_x(z) = H(z)\Sigma_\eta H(z^{-1})^\top$, but without the Hermitian non-negativeness property for general $z \in \mathbb{C}$.

Lemma 4 allows us to transform the non-square signal transfer matrix function into the square spectral density matrix $f_x(\omega)$. Based on this lemma, the first step of the spectral factorization method is to decompose $f_x(\omega)$ into triangular matrix functions using Rozanov's (1967) analytical method.

Proposition 1 *Given an $\ell \times \ell$ spectral density matrix $f_x(\omega)$ with full rank almost everywhere, there exists an $\ell \times \ell$ lower triangular matrix function $\tilde{\Gamma}(e^{-i\omega})$ such that*

$$f_x(\omega) = \tilde{\Gamma}(e^{-i\omega})\tilde{\Gamma}^*(e^{-i\omega}),$$

where

$$\tilde{\Gamma}(z) = \begin{bmatrix} \tilde{\Gamma}_{11}(z) & 0 & \dots & 0 \\ \tilde{\Gamma}_{21}(z) & \tilde{\Gamma}_{22}(z) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\Gamma}_{\ell 1}(z) & \tilde{\Gamma}_{\ell 2}(z) & \dots & \tilde{\Gamma}_{\ell \ell}(z) \end{bmatrix}.$$

If $f_x(\omega)$ is rational, then all elements of the matrix function are rational and analytic in the closed unit disk $\mathbb{T} \cup \mathbb{D}$ and hence in the $\mathbf{H}^2(\mathbb{D})$ space. Moreover, $\tilde{\Gamma}(e^{-i\omega})$ has full rank in \mathbb{D} except for at most a finite number of points.

If the determinant of the analytic matrix $\tilde{\Gamma}(z)$ vanishes at finitely many points inside the unit disk, it is not a Wold spectral factor. Without loss of generality, let $\{z_1, z_2, \dots, z_n\}$ be the finite set of distinct points such that $\det(\tilde{\Gamma}(z_j)) = 0$, $|z_j| < 1$, $j \in \{1, 2, \dots, n\}$. Let \bar{z}_j denote the conjugate of z_j . We assume that all zeros are of order 1 (this property is generic).

The second step of our spectral factorization method employs a multivariate version of the Beurling-Blaschke factorization theorem to remove any zeros inside the unit disk.

Proposition 2 *The Wold spectral factor $\Gamma(z)$ is given by the factorization for Hardy space functions*

$$\Gamma(z) = \tilde{\Gamma}(z) \prod_{j=1}^n V_j^{-1} B_j(z),$$

where the $\ell \times \ell$ Blaschke matrices $B_j(z)$ are (inverse) inner matrix functions of the form

$$B_j(z) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1-\bar{z}_j z}{z-z_j} \end{bmatrix},$$

and the constant unitary matrix V_j is given by the singular value decomposition of $\tilde{\Gamma}(z)$ evaluated at the zeros

$$\tilde{\Gamma}(z_j) = U_j D V_j,$$

where D is a diagonal matrix containing the singular values.

The constant unitary matrices V_j remove the unwelcome poles brought in by the Blaschke factors. There are different ways of computing these matrices, and we use the eigen-decomposition method. In particular, the orthonormal column vectors of V_j can be directly picked from normalized linear independent eigenvectors of the Hermitian matrix $G_j(z_j) = \tilde{\Gamma}^*(z_j) \tilde{\Gamma}(z_j)$, which are automatically pairwise-orthogonal for distinct eigenvalues. For more complicated systems, the eigenvectors can be found easily using symbolic toolboxes in Matlab or Mathematica.

F.2 Wiener-Hopf Prediction Formula

Using the Wold representation for the signal process, we can compute the conditional expectations given the history of signals. Since in our model agents need to perform optimal linear filtering to estimate unobserved shocks, we use the Wiener-Hopf prediction formula, a generalization of the Wiener-Kolmogorov forecasting formula.

Consider any random vector Θ_t satisfying $\Theta_t = G(L)\eta_t$, where $G(z)$ is a matrix analytic function in some matrix-valued Hardy space, we wish to compute the conditional expectation $\mathbb{E}[L^m \Theta_t | \{X_{t-n}\}_{n=0}^\infty]$ given the history of signals $\{X_{t-n}\}_{n=0}^\infty$, where m is any integer. The Wiener-Hopf prediction formula gives

$$\mathbb{E}[L^m \Theta_t | \{X_{t-n}\}_{n=0}^\infty] = \Xi(L) X_t, \quad (\text{F.2})$$

where the analytic matrix function $\Xi(z)$ is given by

$$\Xi(z) = [z^m S_{\Theta_x}(z) (\Gamma^{-1}(z^{-1}))^\top]_+ \Gamma^{-1}(z). \quad (\text{F.3})$$

Here $\Gamma(z)$ is the Wold spectral factor derived in the previous subsection and $S_{\Theta_x}(z) = G(z) \Sigma_\eta H (1/z)^\top$ is the covariance generating function. The annihilation operator $[\cdot]_+$ is linear and is used to remove the principal part of the Laurent series expansion of the analytic functions around a common region of convergence.¹³ This formula reduces to the Wiener-Kolmogorov formula when $\Theta_t = X_t$ so that $\Xi(z) = [z^m \Gamma(z)]_+ \Gamma^{-1}(z)$. If the forecast objects follow geometrically discounted processes, the formula reduces to the Hansen-Sargent optimal prediction formula.

¹³See Kailath, Sayed, and Hassibi (2000) for a textbook proof of the Wiener-Hopf prediction formula. Hansen and Sargent (1980) provide a practical method of computing the annihilation operator using elementary complex analysis.