A Dynamic Theory of Learning and Relationship Lending

Yunzhi Hu and Felipe Varas*

February 15, 2019

Abstract

We introduce learning into a banking model to study the dynamics of relationship lending. In our model, an entrepreneur chooses between bank and market financing. Bank lending facilitates learning *over time*, but it subjects the borrower to the downside of hold-up cost. We construct an equilibrium in which the entrepreneur starts with bank financing and subsequently switch to the market and find conditions under which this equilibrium is unique. Our model generates several novel results: 1) Endogenous zombie lending, i.e. the bank is willing to roll over loans known to be bad for the prospect of future loan sales. 2) Short maturity could encourage zombie lending and deteriorate credit quality; and 3) the hold-up cost may decrease with the length of the lending relationship.

Keywords: private learning, experimentation, relationship bank, hold-up cost, debt rollover, zombie lending, adverse selection, dynamic games

*Hu: Kenan-Flagler Business School, UNC Chapel Hill; Yunzhi_Hu@kenan-flagler.unc.edu. Varas: Fuqua School of Business, Duke University; felipe.varas@duke.edu.

1 Introduction

How do lending relationships evolve over time? How do firms choose *dynamically* between bank and market financing? Why do banks sometimes roll over loans that are known to be insolvent? To answer these questions, we introduce a dynamic framework in the context of relationship lending. By doing so, we also examine how the magnitude of hold-up cost changes as lending relationship continues.

It has been widely documented that bank loans contain important information about borrowers that is not available to market-based lenders (Addoum and Murfin, 2017; James, 1987; Gustafson et al., 2017). Moreover, as suggested by Lummer and McConnell (1989), such information is not produced upon a bank's first contact with a borrower, but, instead, through repeated interactions during prolonged lending relationships which involve substantive screening and monitoring. On the other hand, as shown in Rajan (1992), learning provides information advantage to the relationship bank and thus increases the hold-up cost so that ultimately, the borrower may switch to lenders in the financial market. When should borrowers switch from relationship lending to market financing? How do entrepreneurs balance the tradeoff between learning and hold-up cost? How does loan maturity affect these decisions?

To answer these questions, we introduce private learning into a dynamic model of relationship lending. Specifically, we model an entrepreneur investing in a long-term, illiquid project whose quality is either good or bad. Only a good project has positive net present value (NPV) and should be financed. A bad project should be liquidated immediately. The liquidation value is a constant and independent of the project's quality. Initially, the quality of the project is unknown to anyone, including the entrepreneur herself. She can raise funding from either the competitive financial market or a bank that will develop into a relationship. Market financing takes the form of arm's-length debt so that lenders only need to break even given their beliefs on the project's quality. Under market financing, no information is ever produced and therefore, the maturity of the market debt is irrelevant. In contrast, if the entrepreneur borrows from a bank, screening and monitoring will produce "news" about the project's quality. We model news arrival as a Poisson event and assume this news is only observed by the entrepreneur and the bank once the relationship starts. In other words, the bank and the entrepreneur learn *privately* about the project's quality as time goes by. Meanwhile, all agents, including lenders in the financial market can observe the time since the initialization of the project, which will turn out to be the state variable.

Given the structure of learning, the bank and the borrower possess one of the three types of private information after time 0: 1) news has arrived and implies the project is good – the informed-good type g; 2) news has arrived and implies the project is bad – the informed-bad type b; and 3) no news has arrived yet – the uninformed type u. Upon the maturity of the bank loan, the bank and the entrepreneur jointly determine whether to roll it over, to liquidate the project, or to switch to market-based financing. The entrepreneur may also self-finance the entire project. In the case of rollover, the price of the loan is determined by Nash Bargaining between the bank and the entrepreneur.

By solving the model in closed form, we characterize the equilibrium with two thresholds $\{t_g, t_b\}$ in the time since project initialization. Consequently, the equilibrium is characterized into three stages. If the bank loan matures between 0 and t_b, an informed-bad type's project will be liquidated. All other types' matured loans will be rolled over. During this period, the average quality of borrowers who remained with banks drifts up because the informed-bad types get liquidated and exit funding. These liquidation decisions are socially efficient and therefore we name this stage after *efficient liquidation*. If the bank loan matures between t_b and t_g, however, it will be rolled over irrespective of the quality of the project. In particular, the relationship bank will roll over the loan matured between t_b and t_g even if bad news has arrived. Clearly, this rollover decision is inefficient. This result on banks' rolling over bad loans can be interpreted as *zombie lending*. Finally, after time passes t_g, all entrepreneurs will switch to market financing upon their bank loans maturing – the *market financing* stage.

The intuitions for these results can be best explained backwards in time. When time elapsed gets sufficiently long, all entrepreneurs will ultimately switch to market financing, driven by the assumption that market-based lenders are competitive and have lower costs of capital. This effect is captured by the threshold t_g. Now, imagine a scenario that bad news arrives shortly before t_g, the relationship-bank could liquidate the project, in which case it receives a fixed payoff. Alternatively, it can roll over the loan and pretend as if no bad news has arrived yet. Essentially, by hiding losses *today*, the bank helps the borrower accumulate reputation so that the loan could be sold to the market in the *future*. Such "extending and pretending" incur relatively low costs since shortly afterwards, these bad loans will be sold to the lenders in the market, and the part of the loss will be shared. On the other hand, if negative news arrives early on, "extending and pretending" are much more costly, due to both large time discounting and the high probability that before t_g, the project may mature and the loss will be entirely born by the relationship bank. In this case, liquidating the project is the more profitable option. The threshold t_b captures the time at which an informed-bad type is indifferent between liquidating and rolling over. Note that there is a significant gap between t_b and t_g so that the zombie lending stage lasts for a significant period. During this period, the average quality of borrowers stays unchanged. However, this period is necessary to force informed-bad types to liquidate and exit before t_b, which leads to an improvement in the average quality during the efficient liquidation stage.

We show that short-term loan leads to a longer-period of zombie lending and reduces the credit quality that is ultimately financed with by the market. This result is in contrast with previous studies, which show debt with shorter maturities can better align incentives across different parties (Diamond, 1991a). Intuitively, under shorter maturity the loan can be sold faster once the market financing stage arrives. As a result, the benefits to "extend and pretend" get higher so that fewer of the informed-bad types liquidate their projects before t_b, deteriorating the credit quality.

We show the magnitude of hold-up cost, proxied by the continuation payoff of the entrepreneur, can be non-monotonic in the length of relationship. This pattern is especially prominent for the informed-bad type if 1) loan maturity is short, 2) the entrepreneur's bargaining power is high, and 3) the project's liquidation value is low. Intuitively, two effects are at work here. First, as time approaches the market financing stage, the value of a bad project increases and so is the surplus from rolling over a bad loan. Ceteris paribus, the entrepreneur's continuation payoff should increase. However, there is a second, counter-veiling effect. In the zombie lending stage, the bank's outside option in Nash Bargaining is to liquidate the project which only generates a (relatively) low value. In this case, the entrepreneur is essentially "holding up" the bank. By contrast, during the market financing stage the bank will be very likely to recover the full value of the loan. The ability for the entrepreneur to hold-up the bank then gets more and more limited as the time gets closer and closer to the market financing stage. Ceteris paribus, the entrepreneur's continuation payoff should decrease. The overall pattern therefore depends on the relatively magnitude of these two effects.

Related Literature

Our paper is extends the literature to study the dynamics of relationship lending (Diamond, 1991b; Rajan, 1992). In Diamond (1991b), the lender's decision is mypoic because borrowers' projects mature after one period. Therefore, a lender would never want to engage in zombie lending. Rajan (1992) studies the tradeoff between relationship-based lending and arm's length debt. It implies that the hold-up problem increases over time as the bank gets more and more informative. However, if the relationship-bank keeps rolling over, it is good news to the market, which will lead highly-reputable borrowers switch to market finance. Our paper explicitly studies such a switch and examine how the hold-up cost varies as the lending relationship continues. Our paper is also related to Parlour and Plantin (2008), which study the efficiency of a secondary market for loan sales.

Existing explanations on zombie lending largely rely on either loan officers' career concerns (Rajan, 1994) or additional regulatory capital triggered by writing off bad loans (Caballero et al., 2008; Peek and Rosengren, 2005). We offer a dynamic explanation based on the prospect of future loan sales. This result is also related to Kremer and Skrzypacz (2007) and Fuchs and Skrzypacz (2015) which study how suspension and delaying trading can promote efficiency in markets plagued by adverse selection.

Finally, our modeling approach builds on the emerging literature on private and public learning (Che and Hörner, 2017; Akcigit and Liu, 2015; Kremer et al., 2014). Our paper is one of the first paper in this literature that studies dynamic bank lendings.

2 Model

We consider a continuous-time model with an infinite horizon. An entrepreneur (she) invests in a long-term project whose quality is unknown at $t = 0$. She can borrow from either a bank or a competitive financial market. The bank has a superior monitoring technology than the market which allows the bank to privately learn about the quality of the long-term project before maturity. However, the bank financing is relatively expensive as the bank has a higher cost of capital, which in the model is captured by a higher discount rate. Thus, compared to market financing, bank financing has the advantage of producing valuable information but with the downside of a higher cost of capital and also the possibility that the entrepreneur becomes held-up by the bank. Next, we describe the model in detail.

2.1 Project

We consider a long-term project that generates a constant stream of interim cash flows cdt. The project matures at a random time τ_ϕ, which arrives at an exponential time with intensity $\phi > 0$. Upon maturity, the project may produce some random final cash flows

$\tilde{R}$, depending on its type. A good (g) project produces cash flows $\tilde{R} = R$ with certainty, whereas a bad (b) project produces $\tilde{R} = R$ with probability θ. With probability $1 - \theta$, however, a matured bad project fails to produce anything, $\tilde{R} = 0$. At any time before the project matures, it can be terminated with a liquidation value $L > 0$. The liquidation value is independent of the project's quality so we can assume that it corresponds to a liquidation of the physical asset used in production. Let $r > 0$ be the entrepreneur's discount rate so the fundamental value of the project is given by the discounted value of its future cash flows:

$$NPV^g = \frac{c + \phi R}{r + \phi} \tag{1a}$$

$$NPV^b = \frac{c + \phi \theta R}{r + \phi} \tag{1b}$$

$$NPV^u = q_0 NPV^g + (1 - q_0) NPV^b. \tag{1c}$$

2.2 Agents, Debt Financing and Rollover

The borrower has deep pocket and therefore can self-finance the project if needed. Under this assumption, the borrower should probably be understood as a manager of a more matured firm. However, in the extension, we show the deep-pocket assumption is only made for convenience so that the borrower can also be understood as a manager of a start-up venture. As in the traditional trade-off theory the benefit of debt financing is given by the tax shield advantage. Let γ be the tax rate. We consider two types of debt available to the entrepreneur: bank financing and market financing. First, she can take out a loan from a banker (he), who has the same discount rate r. Following Leland (1998), we assume a bank loan lasts for a random period and matures at a random time τ_m, upon the arrival of an independent Poisson event with intensity $m > 0$. As a result, the expected remaining maturity of the loan is always $\frac{1}{m}$.

The second type of debt is provided by the market and thus can be thought as public bond. In particular, we introduce a competitive public market with discount rate δ satisfying $\delta \in (0, r)$. As a result, market finance is cheaper than bank finance so that eventually, entrepreneurs will switch to market finance. The assumption $\delta < r$ captures the realistic feature that banks have higher cost of capital than the market, which can be justified by either regulatory requirements or the skin in the game needed to monitor borrowers (see Holmstrom and Tirole (1997) for example). [1] As it will be clear shortly, the maturity of the public debt has no effect on the model's outcome and for simplicity, we assume it only matures with the project.

We assume at any time t, the entrepreneur can only take one type of debt. Both types of debt share the same exogenously-specified face value: $F = \frac{c}{r} \in (0, R)$. Our paper intends to study the tradeoff between bank loan and public debt, rather than the optimal leverage. For the same reason, we assume the bank loan carries coupon payments $rFdt$ over the period $(t, t + dt)$, whereas the public debt has coupon payments $\delta F dt$. At $t = 0$, the entrepreneur chooses between a loan and public debt. Once the bank loan matures at τ_m, she can replace it with public debt. Alternatively, she may also roll it over with the same bank who may have information monopoly over the project's quality. In the case that she chooses to roll over with the bank, we follow Rajan (1992) and model the rollover event as a Nash Bargaining game with $(\beta, 1 - \beta)$ being the entrepreneur's and the bank's bargaining power.

During lending relationship, we denote the continuation value of the entrepreneur and the bank by E_t and B_t, and denote the price of bank and market debt by P_t and D_t. To be

[1] The entire model can be written as one where $r = \delta$ but there is cost associated with rolling over bank debt.

succinct, we will also refer to E_t and B_t as equity and bank value, respectively. By definition,

$$E_{t-} = \mathbb{E}_{t-}\Bigg\{ \int_t^\tau e^{-r(s-t)} r\gamma F ds + e^{-r(\tau-t)} \Bigg[\mathbb{1}_{\tau=\tau_\phi} (\tilde{R} - F)^+ + \mathbb{1}_{\tau=\tau_m} \Big(\mathbb{1}_{\text{rollover}} (E_{\tau_m} + P_{\tau_m} - F) + \mathbb{1}_{\text{market}} (E_{\tau_m} + D_{\tau_m} - F) \Big) \Bigg] \Bigg\}$$

$$B_{t-} = \mathbb{E}_{t-}\Bigg\{ \int_t^\tau e^{-r(s-t)} r F ds + e^{-r(\tau-t)} \Bigg[\mathbb{1}_{\tau=\tau_\phi} \min(\tilde{R}, F) + \mathbb{1}_{\tau=\tau_m} \Big(\mathbb{1}_{\text{rollover}} (B_{\tau_m} + F - P_{\tau_m}) + \mathbb{1}_{\text{market}} F + \mathbb{1}_{\text{liquidation}} L \Big) \Bigg] \Bigg\},$$

where $(\tilde{R} - F)^+ = \max \left\{ \tilde{R} - F, 0 \right\}$, and the indicators variables indicate whether the loan is rolled over, the project is liquidated, or the entrepreneur obtains market financing. The expectation operator $\mathbb{E}_{t-}$ is defined as the expectation across all information that is available to the entrepreneur and the bank as time s satisfies $\lim s \uparrow t$, to be specified in the next subsection.

2.3 Learning and Information Structure

The quality of the project is initially unknown. Let $q_0 \in (0, 1)$ be the exogenous belief at $t = 0$ that the project is good. This belief is based on public information such as credit ratings and is commonly shared by all agents in the economy. If the entrepreneur finances with the bank, i.e., if she takes out a loan, the entrepreneur-bank pair can privately learn the true quality of the project through "news". News arrives at a random time τ_λ, modeled as an independent Poisson event with intensity $\lambda > 0$. Upon arrival, the news fully reveals the true type of the project. In practice, one can think of the news process as information learned during bank screening and monitoring, which includes due diligence and covenant violations. We assume that such news can only be observed by the two parties and there is no committable mechanism to share it with third parties such as credit bureaus and market participants. In this sense, the news can be understood as soft information on project quality (Petersen, 2004).

Although the public market participants do not observe the news, they can observe t – the project's time since initialization and therefore make inference about the project's quality. In the benchmark model, we assume the realization of each rollover event τ_m is unobservable to market participants. We denote the type of the bank/entrepreneur by $i \in \{u, g, b\}$, where u, g, and b refer to the uninformed, informed-good and informed-bad types, respectively. We assume that that any failure to rollover the debt is publicly observable because in this case the project would either be liquidated or the entrepreneur would seek financing from the market. In other words, the market cannot observe when the bank debt has been refinanced but can observe whether the firm still owns bank debt. We also assume the entrepreneur will never self-finance the project during the roll over date. Self finance will only affect the entrepreneur's outside option in the Nash Bargaining problem between the bank and the entrepreneur and therefore, the results are similar to those when we allow the entrepreneur to be financially constrained.

Given the unique feature of Poisson learning, the *private* belief process, i.e., the belief held by the bank and the entrepreneur, is straightforward. If news hasn't arrived yet, the belief remains at q_0. In this case, no news is simply no news. Upon news arrival at t_λ, the private belief jumps to 1 in the case of good news and 0 if bad. To characterize the public belief process, we introduce a belief system $\left\{\pi_t^u, \pi_t^g, \pi_t^b\right\}$, where π_t^u is the public's belief at time t that news hasn't arrived yet, and π_t^g (π_t^b) is the public belief that the news has arrived and is good (bad). In any equilibrium where the belief is rational, π_t^i is consistent with the actual probability that the bank and the entrepreneur are of type $i \in \{u, g, b\}$. Given $\left\{\pi_t^u, \pi_t^g, \pi_t^b\right\}$, the public belief that the project is good is given by

$$q_t = \pi_t^u q_0 + \pi_t^g. \tag{2}$$

To simplify notation, we will abuse notation and use $\{\pi_t^i, q_t\}$ to denote $\left\{\pi_{t-}^i, q_{t-}\right\}$. We will state them differently whenever they cause confusions.

2.4 Strategies and Equilibrium

The public history $\mathcal{H}_t$ consists of the entrepreneur's and the bank's actions up to t. Specifically, it includes the entrepreneur's decision at $t = 0$ and at any $s \leq t$, whether the project has been liquidated and whether the entrepreneur has sought finance from the market. The strategy of the public market is therefore summarized by the price of market debt D_t. Given that the market is competitive, the price of debt at which it breaks even satisfies

$$D_t = \frac{\delta F + \phi\left[q_t + (1 - q_t)\,\theta\right] F}{\delta + \phi}. \tag{3}$$

The private history h_t consists of the public history $\mathcal{H}_t$, whether the rollover has occurred, as well as the Poisson event on news arrival and the news content. Let $V_t^i = E_t^i + B_t^i$ be the joint value of the pair. A strategy of the entrepreneur is a stopping time determining the time at which to switch to market financing. The strategy of the bank specifies whether to roll over debt at each rollover date τ_m. Given the Nash Bargaining assumption at each rollover date, we can treat the bank and the entrepreneur as one entity and the problem for the entity is to choose whether to roll over the loan once it matures in order to maximize the surplus of the coalition.

Let $\bar{V}_{\tau_m}^i$ be the continuation value when the entrepreneur finances with the market at time τ_m:

$$\bar{V}_{\tau_m}^i = D_{\tau_m} + \frac{\left[r - \delta\,(1 - \gamma)\right] F}{r + \phi} + \frac{\phi q_{\tau_m}^i\,(R - F)}{r + \phi}, \tag{4}$$

where D_{τ_m} is consistent with (3). The second term $\frac{[r-\delta(1-\gamma)]F}{r+\phi}$ is the remaining discounted coupon payments received over time, and the last term $\frac{\phi q_{\tau_m}(R-F)}{r+\phi}$ captures the final payoff upon project finally matures, where $q_{\tau_m}^i$ is the entrepreneur's and relationship bank's belief that the project is good.

The expected payoff V_t^i before replacing bank debt with public debt satisfies the following Bellman equation:

$$V_t^u = \mathbb{E}_t\left\{ \int_t^\tau e^{-r(s-t)} rF(1+\gamma)\, ds + e^{-r(\tau-t)} \left[\mathbb{1}_{\tau=\tau_\phi} \left[q_0 + (1-q_0)\theta\right] R \right.\right. \tag{5a}$$

$$\left.\left. + \mathbb{1}_{\tau=\tau_\lambda} \left[q_0 V_\tau^g + (1-q_0) V_\tau^b \right] + \mathbb{1}_{\tau=\tau_m} \max\left\{ V_\tau^u, L, \bar{V}_\tau^u \right\} \right] \right\}$$

$$V_t^g = \mathbb{E}_t\left\{ \int_t^\tau e^{-r(s-t)} rF(1+\gamma)\, ds + e^{-r(\tau-t)} \left[\mathbb{1}_{\tau=\tau_\phi} R + \mathbb{1}_{\tau=\tau_m} \max\left\{ V_\tau^g, L, \bar{V}_\tau^g \right\} \right] \right\} \tag{5b}$$

$$V_t^b = \mathbb{E}_t\left\{ \int_t^\tau e^{-r(s-t)} rF(1+\gamma)\, ds + e^{-r(\tau-t)} \left[\mathbb{1}_{\tau=\tau_\phi} \theta R + \mathbb{1}_{\tau=\tau_m} \max\left\{ V_\tau^b, L, \bar{V}_\tau^b \right\} \right] \right\}. \tag{5c}$$

With some abuse of notation, in the first equation $\tau = \min\{\tau_\phi, \tau_\lambda, \tau_m\}$ while in the last two equations $\tau = \min\{\tau_\phi, \tau_m\}$. The first term, $rF(1+\gamma)\, ds$ is the value of tax shields from over time $(s, s+ds)$, where γ is the tax rate. The project matures and pays off $q_t R$ if $\tau = \tau_\phi$. If $\tau = \tau_m$, the bank debt matures and the pair chooses among rolling over, liquidation, or switch to market financing to maximize their value. In the case that the pair is uninformed, news arrives at random time τ_λ, after which the party gets informed.

We denote the price of the bank loan at the rollover time given a type i by P_t^i and the joint continuation value when the entity pursues the outside option by O_t^i. The outside option corresponds to the maximum of liquidating the project or replacing bank debt with the public debt. If the project gets liquidated at a rollover time τ_m, then the outside option is given by $O_{\tau_m}^i = L$, which will be recouped by the bank. If the borrower replaces the maturing debt with public debt, then the outside option is $O_{\tau_m}^i = \bar{V}_{\tau_m}^i$.

The bank loan will be rolled over at time τ_m if and only if

$$V_{\tau_m}^i \equiv E_{\tau_m}^i + B_{\tau_m}^i > O_{\tau_m}^i. \tag{6}$$

In this case, the price of the loan is the solution to the Nash Bargaining problem. If the bank does not roll over the loan and the entrepreneur defaults, the bank gets a payoff L. On the other hand, if the bank does not rollover and the entrepreneur obtains market financing, the bank gets a payoff of F. Accordingly, the bank outside option is $O^i_{Bt} = \mathbb{1}_{\text{liquidation}} L + \mathbb{1}_{\text{market}} F$. Finally, if the loan is rolled over with the same bank, its price ($P^i_{\tau_m}$) is determined by Nash Bargaining, and it is given by

$$B^i_{\tau_m} + F - P^i_{\tau_m} = O^i_{B\tau_m} + (1-\beta)(V^i_{\tau_m} - O^i_{\tau_m}). \tag{7}$$

We look for a perfect Bayesian equilibrium of this game.

Definition 1. *An equilibrium of the game satisfies*

1. *Optimality: the roll over decisions maximize all three types' continuation value* (5a), (5b), *and* (5c)*, given the beliefs* $\{\pi^i_t, q_t\}$*. The price of bank loan at roll over dates satisfy* (3).

2. *Belief Consistency: For any history on the equilibrium path, the belief process* $\left\{\pi^u_t, \pi^g_t, \pi^b_t\right\}$ *is consistent with Baye's rule.*

3. *Market Breakeven: the price of the market debt satisfies* (3).

4. *No (unrealized) Deals: for any* $t > 0$ *and* $i \in \{u, g, b\}$, $V^i_t \geq \mathbb{E}\left[\bar{V} \middle| \mathcal{H}_t\right]$.

The first three conditions are standard. The *No Deals* condition follows Daley and Green (2012), reflecting the requirement that the market cannot profitably deviate by making an offer that the entrepreneur and the bank will accept.

As standard in the literature Daley and Green (2012), we use a refinement to rule out equilibrium that arise only due to unreasonable beliefs off the equilibrium path. Specifically, we restrict the belief on the off-equilibrium to be non-decreasing

Definition 2. *Belief monotonicity is satisfied if q_t – the public's belief that the project is good – is non-decreasing in t. An equilibrium that satisfies belief monotonicity is referred to as a monotonic equilibrium.*

We will show there is one unique monotone equilibrium. On the other hand, we show that the equilibrium is unique – that is, that any perfect Bayesian equilibrium is monotone – if the maturity of the loan is sufficiently long. In that case, the requirement on monotone equilibrium is not needed.

2.5 Parametric Assumptions

Through out the paper, we assume that it is socially optimal to liquidate a bad project rather than continue financing it until the maturity date, τ_ϕ. This requires that the liquidation value is high enough. On the other hand, liquidation is costly so the liquidation payoff must be lower than the value of continuing a good project until maturity. Thus, we assume that the liquidation value satisfies the following condition:

Assumption 1 (Liquidation value)**.**

$$L \in \left(\frac{c + \phi \left(q_0 + (1 - q_0)\, \theta \right) R}{r + \phi}, \frac{c + \phi R}{r + \phi} \right) \tag{8}$$

According to Assumption 1, the NPV of a good project is above its liquidation value, which is in turn above the NPV of an unknown project.

Assumption 2 (Risky debt)**.**

$$F > \max \left\{ \theta R, L \right\}. \tag{9}$$

Assumption 2 assumes the face value of the debt is above both the liquidation value and the expected repayment; otherwise both the bank loan and the market-based debt can be riskless.

Assumption 3 (Tax shields)**.**

$$L > \frac{rF(1+\gamma) + \phi\theta R}{r+\phi} \tag{10}$$

Assumption 3 imposes that the size of the tax shields cannot be too large; otherwise, an informed-bad type will not liquidate the project just due to the large benefits coming out of tax shields.

3 Equilibrium

In this section, we solve the equilibrium in two steps. We treat the bank and the entrepreneur as one entity and solve the optimal choice of timing when they liquidate the project and when they switch to market finance. Given the assumption of Nash Bargaining, their incentives are aligned when facing the choice of whether to roll over the loan. Clearly, at each rollover event, the project is liquidated if the joint surplus falls below the liquidation value L. By contrast, the project is sold to the market if the joint surplus falls below $\bar{V}^i_{\tau_m}$, the continuation value if the borrower switches to market financing by accepting the price of the market debt at D_t. In subsection 3.2, we explicitly solve the Nash Bargaining problem between the bank and the entrepreneur. By doing so, we are able to examine how the hold-up cost evolves over time.

The economy is characterized by state variables in private and public beliefs $\{q_t, \pi^u_t, \pi^g_t, \pi^b_t\}$. It turns out all of them are deterministic function of time elapsed. Therefore, we use the elapsed time t as the state variable to describe the economy. Specifically, the equilibrium that we construct can be characterized by two thresholds $\{t_b, t_g\}$, as illustrated by Figure 1. If $t \in [0, t_b]$, the bank and the entrepreneur will liquidate the project upon debt maturity after bad news has arrived – efficient liquidation region. If $t \in [t_b, t_g]$, the pair will roll over the loan even if bad news has arrived – zombie lending region. Finally, if $t \in [t_g, \infty)$, the

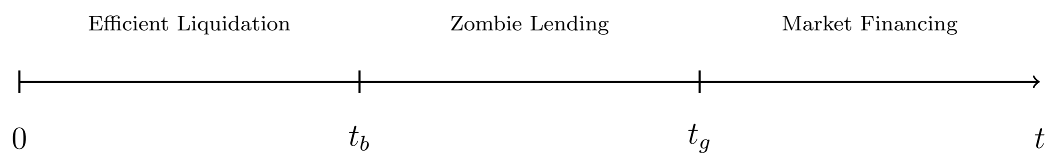


Figure 1: Equilibrium regions

pair will always sell the project to the market upon debt maturity – market financing. Below we first describe each region in details. Later, we will prove that this equilibrium is unique if the maturity of loans is long enough, that is, if m is low enough.

The first step in the analysis is to derive the evolution of beliefs given the equilibrium conjecture. The market beliefs depend crucially on type-b's roll over decisions. Given our conjectured equilibrium, the evolution of beliefs is given by the following lemma.

Lemma 1. *Consider a monotone equilibrium with threshold $\{t_b, t_g\}$. In the absence of liquidation or market financing, the public beliefs $(\pi_t^u, \pi_t^g, \pi_t^b)$ satisfy the following differential equation:*

- *For any $t \in (0, t_b)$,*

$$\dot{\pi}_t^u = -\lambda \pi_t^u + m \pi_t^u \pi_t^b \tag{11a}$$

$$\dot{\pi}_t^g = \lambda \pi_t^u q_0 + m \pi_t^g \pi_t^b \tag{11b}$$

$$\dot{\pi}_t^b = \lambda \pi_t^u (1 - q_0) - m \pi_t^b \left(1 - \pi_t^b\right). \tag{11c}$$

- *For any $t > t_b$,*

$$\dot{\pi}_t^u = -\lambda \pi_t^u \tag{12a}$$

$$\dot{\pi}_t^g = \lambda \pi_t^u q_0 \tag{12b}$$

$$\dot{\pi}_t^b = \lambda \pi_t^u (1 - q_0). \tag{12c}$$

- *With initial condition* $\pi_0^u = 1$, $\pi_0^g = \pi_0^b = 0$, *and* q_0.

Figure 2 provides a graphical illustration to the public belief systems for $t < t_g$. In all three panels, the dashed red line (vertical) marks the position of t_b, which equals 5.4782 under given parameter values. The top panel shows π^u, which decreases monotonically due to the arrival of news over time. In contrast, π^g keeps increasing, since the informed good type gets discovered over time and keeps rolling over. Finally, π^b evolves non-monotonically. During $[0, t_b]$, it increases initially as bad types get revealed (note that that don't exit immediately due to the finite maturity of the loan). Ultimately, it starts to decline as more and more of the informed bad types get liquidated and exit funding. After t passes t_b, becomes no bad type will ever liquidate their projects and more and more uninformed types learn through news, π_t^b starts to increase again.

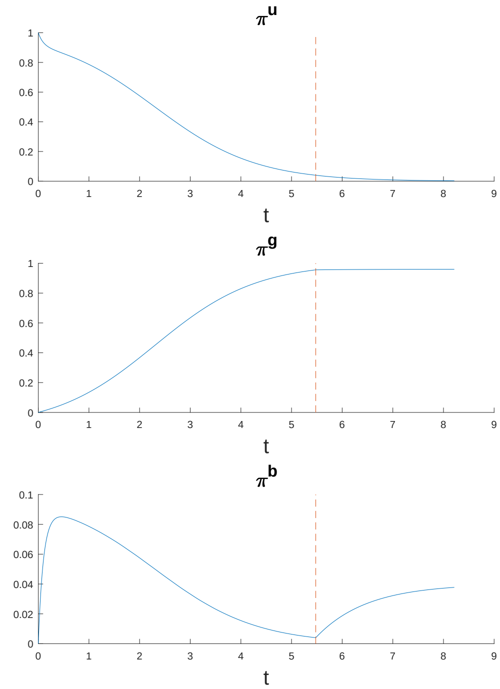


Figure 2: Public Beliefs when $t < t_g$

This figure plots the public beliefs process with the following parameter values: $r = 0.1$, $\delta = 0.05$, $m = 10$, $F = 1$, $\phi = 1$, $R = 2$, $\theta = 0.1$, $L = 1.2 \times NPV^b$, $\lambda = 1$, $q_0 = 1$, $\beta = 0.5$, $\gamma = 0.2$

3.1 Rollover Decision

In this section we analyze the rollover decision. First, we derive the HJB equation and the optimality conditions that determine the thresholds t_b and t_g. We conclude with the complete characterization of the equilibrium. The continuation value if the entrepreneur finances with the market at time t, $\bar{V}_t^i$, is given by (4).

By considering the changes in valuation V_t^i, $i \in \{u, g, b\}$ over a small interval $[t, t + dt]$, we are able to derive the following Hamilton-Jacobi-Bellman (HJB) equation system:

$$(r+\phi)\,V_t^u = \dot{V}_t^u + rF\,(1+\gamma) + \phi\,[q_0 + (1-q_0)\,\theta]\,R + \lambda\left[q_0 V_t^g + (1-q_0)\,V_t^b - V_t^u\right] + m\mathcal{R}(V_t^u, \bar{V}_t^u) \tag{13a}$$

$$(r+\phi)\,V_t^g = \dot{V}_t^g + rF\,(1+\gamma) + \phi R + m\mathcal{R}(V_t^g, \bar{V}_t^g) \tag{13b}$$

$$(r+\phi)\,V_t^b = \dot{V}_t^b + rF\,(1+\gamma) + \phi\theta R + m\mathcal{R}(V_t^b, \bar{V}_t^b), \tag{13c}$$

where

$$\mathcal{R}(V_t^i, \bar{V}_t^i) \equiv \max\left\{0, \bar{V}_t^i - V_t^i, O^i - V_t^i\right\} \tag{14}$$

$$O^i = L \cdot \mathbb{1}_{i\in\{u,b\}} + NPV^g \cdot \mathbb{1}_{i=g}$$

In Equation (13a), the first term on the right hand side $\dot{V}_t^u$ is the change in valuation as a function of belief evolution, which is determined by time elapsed t. The second term captures the benefits of interim cash flow and resulting tax shields $rF\,(1+\gamma)\,dt$ over the short-time horizon $(t, t+dt)$. The third term describes the scenario that the project matures, which occurs with (approximately) probability ϕdt. In this case, the bank and the entrepreneur receive final payoff R with an average probability $q_0 + (1-q_0)\,\theta$. The fourth term stands for the event of news arrival at rate λ, after which the bank and the entrepreneur get informed and become either informed-good or informed-bad. Finally, upon debt maturity which happens with intensity m, the bank and the entrepreneur choose between rolling over the debt (0 in Equation (14), repaying the bank debt and in turn switch to the market ($\bar{V}_t^i - V_t^i$ in (14)), liquidating the project ($O^i - V_t^i$ in (14) where $O^i = L$), or switch to all-equity financing (O^i in (14) where $O^i = NPV^g$). Equation (13b) and (13c) can be interpreted in the similar vein.

Below, we specify the HJB equation in each of the equilibrium regions. To better explain the economic intuition, we describe the equilibrium backwards in the time elapsed.

3.1.1 Market Financing: $\left[t_g, \infty\right)$

As time goes by long enough, all types will replace with loan with the public debt because market finance is cheaper, for two reasons: First, market participants have lower discount rates, $\delta < r$. Second, and more importantly, market participants are willing to pay the price that reflects the average quality of the project which exceeds the initial quality q_0. The average quality of the project increases over time because, in equilibrium some bad type would have liquidated the project earlier at a rollover date in the interval $[0, t_b)$, whereas only good and uninformed banks would have always rolled the loan. Once all types decide to go into the market at the rollover date, we get that

$$\dot{q}_t = \dot{\pi}_t^g + q_0 \dot{\pi}_t^u = 0$$

so the average quality of the firm remains constant from the market perspective. This means that the continuation value of getting market financing remains constant after time t_g, and is given by $\bar{V}^i$. Hence, in the interval (t_g, ∞), the HJB equation reduces to

$$\begin{aligned}
(r + \phi + \lambda + m) V_t^u &= rF(1+\gamma) + \phi\left[q_0 + (1-q_0)\theta\right] R \qquad (15a)\\
&\quad + \lambda\left[q_0 V_t^g + (1-q_0) V_t^b\right] + m\bar{V}^u \\
(r + \phi + m) V_t^g &= rF(1+\gamma) + \phi R + m\bar{V}^g \qquad (15b)\\
(r + \phi + m) V_t^b &= rF(1+\gamma) + \phi\theta R + m\bar{V}^b \qquad (15c)
\end{aligned}$$

Note that compared to equations (13a)-(13c), the term $\dot{V}_t^i,\ i \in \{u, g, b\}$ is dropped because the belief q_t stays unchanged. The last term $\mathcal{R}(V_t^i, \bar{V}_t^i)$ is maximized by letting it equals $\bar{V}_t^i - V_t^i$.

3.1.2 Zombie Lending: $\left[t_b, t_g\right)$

Next, we consider the region $\left[t_b, t_g\right)$ over which the bank rolls over bad loans. This is the region with zombie lending. In this region, the bad type does not need to wait very long until t_g, when they can replace the matured loan with public debt, in which case the market will share the loss. Therefore, when bad news arrives, both the entrepreneur and the bank would rather hide it by rolling over the loan and pretending as if no bad news has occurred yet. The following HJB equations summarize the value functions in this region

$$(r+\phi+\lambda) V_t^u = \dot{V}_t^u + rF(1+\gamma) + \phi\left[q_0 + (1-q_0)\theta\right] R + \lambda\left[q_0 V_t^g + (1-q_0) V_t^b\right] \tag{16a}$$

$$(r+\phi) V_t^g = \dot{V}_t^g + rF(1+\gamma) + \phi R \tag{16b}$$

$$(r+\phi) V_t^b = \dot{V}_t^b + rF(1+\gamma) + \phi\theta R. \tag{16c}$$

The interpretation of these equations are similar to (15a), (15b), and (15c), except that on the right-hand side, there is a new term $\dot{V}_t^s$, standing for the change in the continuation value. In addition, $\mathcal{R}(V_t^i, \bar{V}_t^i)$ is maximized by letting it equals 0.

Given all types rolling over the loan, the evolution of the public belief does not rely on loan maturity. The average quality of the pool also stays unchanged. Equilibrium in this region is clearly inefficient. A bad project should be liquidated but instead, the bank and the entrepreneur roll it over in the hope of sharing the loss with the market lenders after t_g. By not liquidating between 0 and t_b, they have accumulated "good" reputation and as a result, this type of "zombie lending" can show up in equilibrium.

3.1.3 Efficient Liquidation: $\left[0, t_b\right)$

Next, we look at the initial region $\left[0, t_b\right)$, where bad loans are not rolled over and the project is liquidated. When the time that has passed since the beginning of the lending relationship is relatively short, only the uninformed and informed-good types will roll over their matured loans. Intuitively, the joint continuation payoff of the bank and the entrepreneur cannot decrease relative to that at $t = 0$ because no bad news has arrived. Therefore without any bad news, there is no reason to switch to market finance or liquidate the project, conditional on a bank loan has been taken at $t = 0$. By contrast, type b can be worse off after they have learned the bad news. Assumption 1 guarantees that liquidation possesses a higher payoff than continuing the project until the final date t_ϕ. In equilibrium, it is going to be the case that type g will sell the project to the market after the elapsed time gets long enough. By continuity, liquidation still has a higher payoff if type b needs to wait quite a long time to sell. Therefore, they would rather liquidate the project instead. The following HJB equations summarize the value functions in this region:

$$(r+\phi+\lambda)\,V_t^u = \dot{V}_t^u + rF\,(1+\gamma) + \phi\,[q_0 + (1-q_0)\,\theta]\,R + \lambda\left[q_0 V_t^g + (1-q_0)\,V_t^b\right] \tag{17a}$$

$$(r+\phi)\,V_t^g = \dot{V}_t^g + rF\,(1+\gamma) + \phi R \tag{17b}$$

$$(r+\phi+m)\,V_t^b = \dot{V}_t^b + rF\,(1+\gamma) + \phi\theta R + mL. \tag{17c}$$

The equations are similar to (16a), (16b), and (16c), except that in Equation (17c), the informed-bad types liquidate the project if bank debt matures and therefore receive payoff L instead. In other words, $\mathcal{R}(V_t^b, \bar{V}_t^b)$ is maximized by letting it equals $L - V_t^b$.

Given the strategies, the average quality of the pool gets better over time: a bad project is liquidated with some positive probability, whereas others remain. The equilibrium is socially efficient in this region.

Boundary Conditions

The final step in the construction of the equilibrium is to specify the boundary conditions that the HJB equation must satisfy at the thresholds. Between rollover dates, the continuation payoff is a continuous function of time, which means that HJB equation must satisfy value matching conditions at $\{t_b, t_g\}$. However, the value matching conditions do not allow to uniquely pin-down the threshold $\{t_b, t_g\}$. The next two conditions allow to pin-down $\{t_b, t_g\}$

$$V^b(t_b) = L \tag{18a}$$

$$V'^g(t_g) = 0. \tag{18b}$$

The first condition is the indifference condition for the optimality of the liquidation policy of the bad type. This is the traditional value matching condition in optimal stopping problem, which states that a bad type whose bank debt matures at time t_b is indifferent between liquidating and rolling over. In this case, rolling over brings exactly the same payoff L and thus by continuity and monotonicity, she prefers liquidating when $t_m < t_b$ and rolling over when $t_m > t_b$. The second condition, smooth pasting, comes from the *No-Deals* condition. We show in Lemma 3 of Appendix A.1.2 that if this conditions fails then the type g will have strictly higher incentives to sell the loan before t_g, which constitutes an arbitrage opportunity for market participants. In that case, the *No Deals* condition fails to hold. Given those boundary conditions, we can uniquely pin-down $\{t_b, t_g\}$, which is given by the following proposition.

Proposition 1. *There exists a unique m^* such that the constructed equilibrium exists for any $m > m^*$. In the unique monotone equilibrium, the rollover thresholds t_b and t_g are given by*

$$t_g = t_b + \frac{1}{r+\phi} \log\left(\frac{(r+\phi)V^b_{t_g} - (rF(1+\gamma) + \phi\theta R)}{(r+\phi)L - (rF(1+\gamma) + \phi\theta R)}\right). \tag{19}$$

and

$$t_b = \min\{t : q_t = \bar{q}\}, \tag{20}$$

where

$$\bar{q} = \frac{1}{(1-\theta)} \left\{ \frac{1}{\phi} \left[\frac{(\delta+\phi)\left[r\gamma\left(1+\frac{r+\phi}{m}\right)+\phi+\delta(1-\gamma)\right]}{r+\phi} - \delta \right] - \theta \right\} \tag{21}$$

$$V_{t_g}^b = \frac{rF(1+\gamma)+\phi R}{r+\phi} - \frac{\phi R(1-\theta)+m\frac{\phi(R-F)(1-\theta)}{r+\phi}}{r+\phi+m}, \tag{22}$$

and, for any $t < t_b$,

$$q_t = \frac{q_0\left(1-q_0+q_0 e^{\lambda t}\right)^{\frac{1}{\lambda}-1} e^{mt}}{1+m\int_0^t \left(1-q_0+q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds}. \tag{23}$$

Proposition 1 shows that such an equilibrium exists if the maturity of the loan gets short enough $m > m^*$. We defer to subsection 3.1.4 to discuss the equilibrium when loans have long maturity.

Figure 3 plots the value function of all three types. In this example, the equilibrium $t_b = 5.48$ and $t_g = 8.22$. In all three panels, the blue solid lines stand for the value function, whereas the red dashed line shows the levels of L. Clearly, all three value functions stay constant after t passes t_g. In fact, as shown in Lemma 4 of Appendix A.1.2, V_g stays a constant throughout the entire range. In other words, the informed-good types always expect the same continuation value. By contrast, the value of informed-bad types (bottom panel) exceeds L only after t passes t_b and then increases sharply until $t = t_g$.

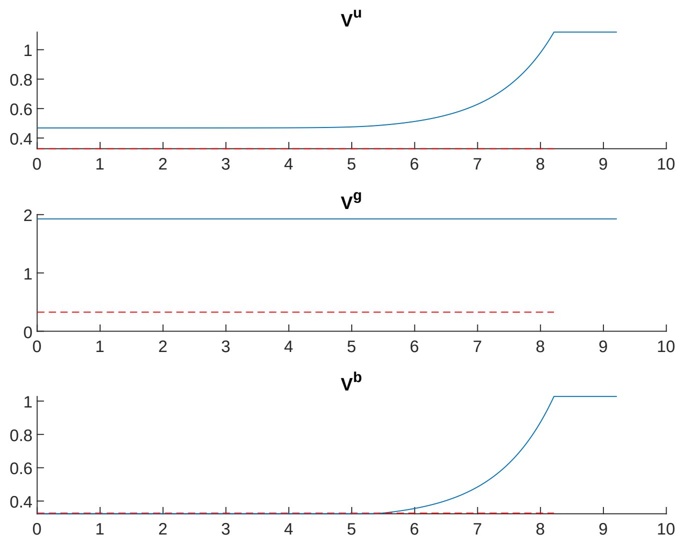


Figure 3: Value Functions

This figure plots the value function with the following parameter values: $r = 0.1$, $\delta = 0.05$, $m = 10$, $F = 1$, $\phi = 1$, $R = 2$, $\theta = 0.1$, $L = 1.2 \times NPV^b$, $\lambda = 1$, $q_0 = 1$, $\beta = 0.5$, $\gamma = 0.2$

We conclude this subsection with a result on equilibrium uniqueness.

Proposition 2. *There exists $\bar{m}$ such that, for any $m < \bar{m}$, the monotone equilibrium in Proposition 1 is the unique equilibrium.*

3.1.4 Debt Maturity, Credit Quality, and Zombie Lending

In this subsection, we study how (expected) debt maturity $\frac{1}{m}$ affects credit quality, zombie lending, and firm valuation. The first result draws upon Proposition 1.

Corollary 1. *The credit quality financed by market lenders $\bar{q}$ increases with expected debt maturity $\frac{1}{m}$.*

This result implies that long-term debt is associated with higher credit quality, which is in contrast with existing literature that has focused on liquidity risk (Diamond, 1991a). In previous studies, short-term debt gives more sensitivity to new information and therefore provides more incentives through the refinance decision. In our study, however, short-term debt means the relationship bank and the entrepreneur can replace bank debt with market-based debt without much delaying. Since delay is effectively a punishment towards hiding bad information, less delay imposes lower penalty and therefore induces lower credit quality.

We offer a heuristic proof below. Note that the informed-good type g can always roll over the loan with the same bank, in which case the pair receive $\underline{V}^g = \frac{rF(1+\gamma)+\phi R}{r+\phi}$, including discounted flow of coupon, tax shields as well as the final payoff. The *No Deals* condition implies this will be their final payoff. On the other hand, if the entity refinance with the market at the next roll-over date, the payoff becomes $\frac{rF(1+\gamma)+\phi R+m\bar{V}_g}{r+\phi+m}$. Clearly, $\bar{V}_g > \underline{V}_g$, otherwise market financing is never used. Short-term debt (high m) therefore decreases the continuation value $\bar{V}_g$ that good types receive by switching to market financing by decreasing the price of the market debt $\bar{D}$. Since market lenders break even, this implies the credit quality goes down.

Next, we study how debt maturity affects the length of zombie lending period $t_g - t_b$.

Corollary 2. *The length of the zombie lending period $t_g - t_b$ decreases with expected debt maturity $\frac{1}{m}$.*

This result implies the concern for zombie lending is bigger for short-term debt, and the intuition again depends on delaying. Intuitively, $t_g - t_b$ is the length of the period that a type-b borrower needs to wait and pretend as others. If bank debt has short maturity (say instantly maturing), that necessarily implies the informed-bad type can refinance their debt immediately after t_g. In that case, to discourage them from imitating, the length of the "black-out" period needs to be even longer. Both Corollary 1 and 2 rely on the intuition of delay and discourage type b imitating. Meanwhile, if the bank and the borrower can secretly renegotiate the maturity, the results no longer hold.

Finally, we examine how the initial value of the project V_0^u varies with the expected debt maturity $\frac{1}{m}$. Figure 4 shows the results under given parameter values. Clearly, V_0^u decreases with expected maturity $\frac{1}{m}$, implying that if the firm could choice its maturity structure at the initial date, it would prefer loans with short maturity. Intuitively, short-term loans enable the informed-bad types to liquidate their projects early to mitigate the loss and therefore increase the initial valuation.

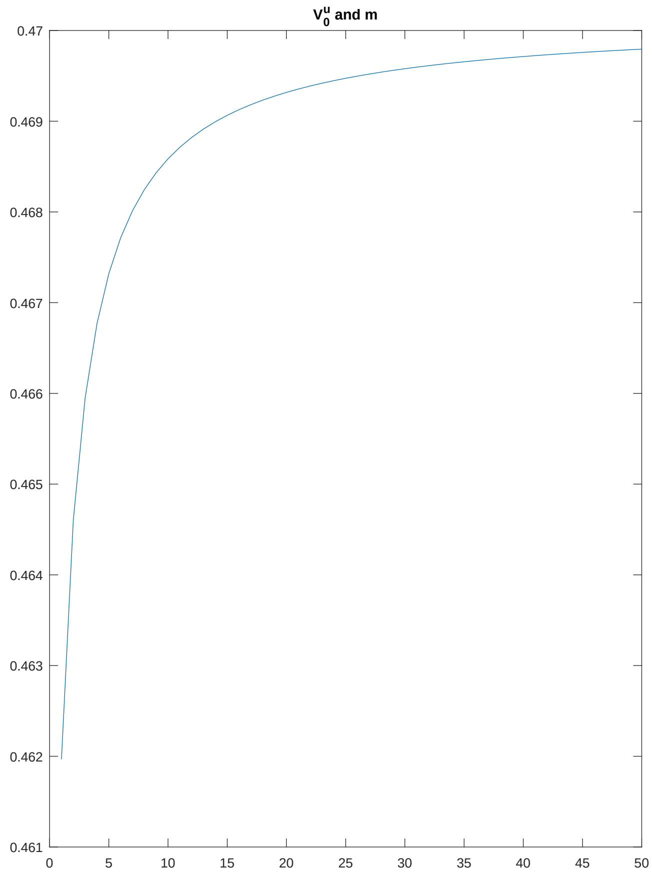


Figure 4: V_0^u and m

This figure plots the value function with the following parameter values: $r = 0.1$, $\delta = 0.05$, $m = 10$, $F = 1$, $\phi = 1$, $R = 2$, $\theta = 0.1$, $L = 1.2 \times NPV^b$, $\lambda = 1$, $q_0 = 1$, $\beta = 0.5$, $\gamma = 0.2$

We conclude this subsection be discussing the equilibrium when $m < m^*$. In this case, the maturity of the loan gets so long that the period of zombie lending is no longer needed, and the equilibrium is simply characterized by one single time cutoff $\bar{t}$. From t to $\bar{t}$, there is efficient liquidation, whereas market financing occurs right after $\bar{t}$. The boundary condition in this case is captured by the value matching condition $V_{\bar{t}}^{b} = L$.[2] Intuitively, this period is necessary to incentivize the bad types not to mimic good types during the efficient liquidation stage. When the maturity of the loan gets long enough, even if the market financing stage has arrived – t passes t_g, the informed bad types still need to wait for a period until the loan matures and sell it. For a lower m, the expected length of this period gets longer, so that it is more likely the project may actually mature before the loan matures. In the most extreme case when the loan never matures, the informed bad types can never sell the loan to the market so that it is never incentive-compatible to "pretend and extend".

3.2 Bank and Entrepreneur's Value

In the previous section, we study the value of the firm as the joint surplus between the entrepreneur and the bank. In this section, we examine how this surplus is distributed between the bank and the entrepreneur. To do so, we need to derive the continuation value of the bank and the entrepreneur, which will directly determine the price of bank debt at each rollover event. For simplicity, we will only offer the equations that determine the entrepreneur's continuation value and leave the relevant expressions for the bank to Appendix A.2.

[2] The smooth-pasting condition no longer holds. In general, it would be the case that $\frac{dV_{t_g}^{g}}{dt} \geq 0$. See the proof for Lemma 3 in Appendix A.1.2 for details.

In the *Market Financing* region (t_g, ∞), the results are straightforward:

$$E_t^u = \frac{\phi\left[q_0 + (1-q_0)\,\theta\right](R-F)}{r+\phi+\lambda+m} + \frac{\lambda\left[q_0 E_t^g + (1-q_0)\,E_t^b\right]}{r+\phi+\lambda+m} \tag{24a}$$

$$+ \frac{r\gamma F + m\left(\bar{D} - F\right) + m\frac{(r-\delta)F}{r+\phi} + m\frac{\phi[q_0+(1-q_0)\theta](R-F)}{r+\phi} + m\frac{\delta F\gamma}{r+\phi}}{r+\phi+\lambda+m} \tag{24b}$$

$$E_t^g = \frac{\phi\,(R-F) + r\gamma F + m\left(\bar{D} - F\right) + m\frac{(r-\delta)F}{r+\phi} + m\frac{\phi(R-F)}{r+\phi} + m\frac{\delta F\gamma}{r+\phi}}{r+\phi+m} \tag{24c}$$

$$E_t^b = \frac{\phi\theta\,(R-F) + r\gamma F + m\left(\bar{D} - F\right) + m\frac{(r-\delta)F}{r+\phi} + m\frac{\phi\theta(R-F)}{r+\phi} + m\frac{\delta F\gamma}{r+\phi}}{r+\phi+m} \tag{24d}$$

In equation (24a), the firs term represents the case that a project matures, which leads to the repayment of $R-F$ with expected probability $q_0 + (1-q_0)\,\theta$. The second term describes the event of news arrival, whereas the third term shows the event when the loan matures and gets replaced by market-based bonds with coupon δF. Equation (24c) and (24d) can be analyzed similarly.

In the other two regions, the HJBs for the uninformed and the informed-good types are clear. At each rollover event, the loan is rolled over and the entrepreneur (bank) receives rollover gains (losses). That is, when $t \in (0, t_g)$,

$$(r+\phi)\,E_t^u = \dot{E}_t^u + rF\gamma + \phi\left[q_0 + (1-q_0)\,\theta\right](R-F) + m\,(P_t^u - F) \tag{25a}$$

$$+ \lambda\left[q_0 E_t^g + (1-q_0)\,E_t^b - E_t^u\right]$$

$$(r+\phi)\,E_t^g = \dot{E}_t^g + rF\gamma + \phi\,(R-F) + m\,(P_t^g - F)\,, \tag{25b}$$

where

$$\begin{aligned} F - P_t^u &= L + (1-\beta)\,(V_t^u - L) - B_t^u = -\left[\beta\,(V_t^u - L) - E_t^u\right] \\ F - P_t^g &= F + (1-\beta)\,(V_t^g - NPV^g) - B_t^g = -\left[(NPV^g - F) + \beta\,(V_t^g - NPV^g) - E_t^g\right]. \end{aligned}$$

The rollover gains above directly follow from the Nash Bargaining assumption in (7). Note that according to Assumption (1), an uninformed entrepreneur will prefer liquidation to all-equity financing, whereas an informed-good type entrepreneur has the opposite preference. Therefore, for the uninformed type, the disagreement point in Nash Bargaining is $(0, L)$, whereas for type u, the disagreement point becomes $(NPV^g - F, F)$.

In contrast, the value function for a bad-type entrepreneur differs between two regions. When $t \in (0, t_b)$

$$(r + \phi + m)\, E_t^b = \dot{E}_t^b + rF\gamma + \phi\theta\,(R - F) \quad \forall t \in (0, t_b) \tag{26a}$$

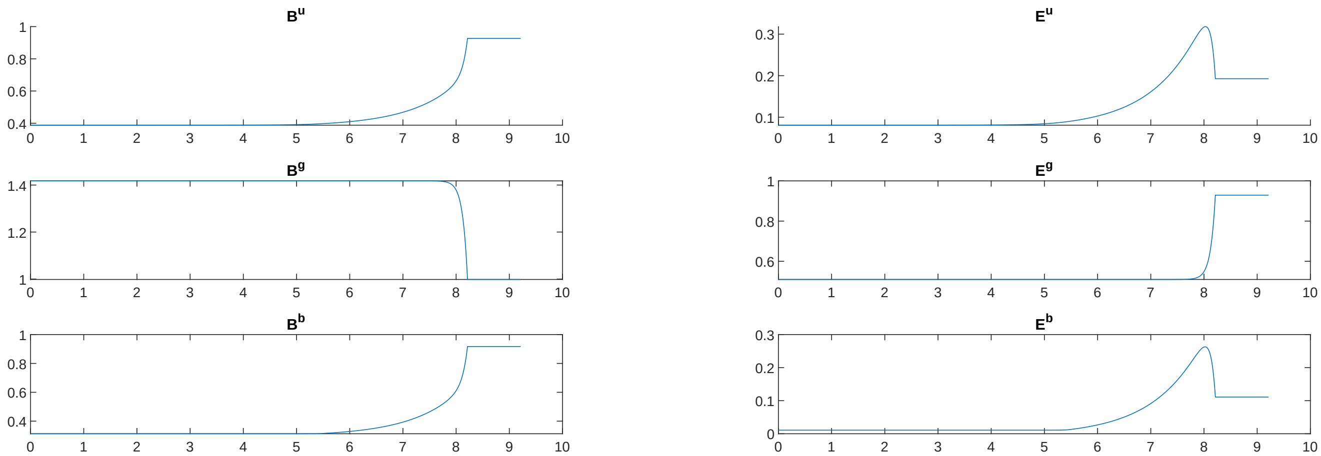
$$(r + \phi)\, E_t^b = \dot{E}_t^b + rF\gamma + \phi\theta\,(R - F) + m\left(P_t^b - F\right) \quad \forall t \in (t_b, t_g)\,, \tag{26b}$$

where

$$F - P_t^b = L + (1 - \beta)\,(V_t^b - L) - B_t^b = -\left[\beta(V_t^b - L) - E_t^b\right].$$

Intuitively, in the *efficient liquidation* region, a bad project gets liquidated when the loan matures, whereas in the *zombie lending* region, the same loan will get rolled over. Figure 5 respectively plot the value functions.

A prominent feature of is the non-monotonicity of the entrepreneurs' value function E_t^u and E_t^b. Intuitively, there are two forces at work here. First, the value of a bad project is increasing, as time gets closer and closer to the market financing stage. As a result, the surplus of rolling over the bad loan $V_t^b - L$ gets larger. Ceteris paribus, the entrepreneur's value function also increases. However, there is a second, counterveiling force. During the market financing stage, the disagreement point in the Nash bargaining game is $(0, L)$: if the bargaining does not reach an agreement, the bank only receives the liquidation value L. During the market financing stage, however, the bank will always gets fully repaid and thus receives F. As time t gets closer to the market financing stage, the entrepreneur's ability to hold up the bank gets more limited because it is increasingly likely that the next roll-



(a) This figure plots the banks' value function with the following parameter values: $r = 0.1$, $\delta = 0.05$, $m = 10$, $F = 1$, $\phi = 1$, $R = 2$, $\theta = 0.1$, $L = 1.2 \times NPV^b$, $\lambda = 1$, $q_0 = 1$, $\beta = 0.5$, $\gamma = 0.2$

(b) This figure plots the entrepreneurs' value function with the following parameter values: $r = 0.1$, $\delta = 0.05$, $m = 10$, $F = 1$, $\phi = 1$, $R = 2$, $\theta = 0.1$, $L = 1.2 \times NPV^b$, $\lambda = 1$, $q_0 = 1$, $\beta = 0.5$, $\gamma = 0.2$

Figure 5: Bank and Entrepreneur Value Functions

over event will occur during the market financing stage. Therefore, the entrepreneur's value function decreases. Given the opposite effects of these two forces, the overall effect can be non-monotonic. As we show in lemma 6 in the Appendix, the sign of E_t^b can only change sign at most once though. In subsection 3.3, we show that in the case with instantly-maturing debt, the first force will lead to the entrepreneur's value function to increase until t reaches t_g^-. At t_g, it experiences a discontinuous downwards jump due to the second effect.

3.3 Special Case: Instantly-Maturing Debt

In this subsection, we show the solution to a special case of our model – instantly-maturing debt. Specifically, we take the maturity intensity of the debt m to infinity and study how the solution depends on primitives. We will state the main results, with details supplemented in Appendix A.3.

Proposition 3. *When bank loans mature instantly,*

$$q_t = \frac{q_0}{q_0 + (1 - q_0)e^{-\lambda t}} \quad \forall t < t_b \tag{27a}$$

$$t_b = \frac{1}{\lambda}\left[\log\left(\frac{\bar{q}}{1-\bar{q}}\right) - \log\left(\frac{q_0}{1-q_0}\right)\right] \tag{27b}$$

$$t_g = t_b + \frac{1}{r+\phi}\log\left(\frac{\phi(1-\theta)F}{(r+\phi)L - rF(1+\gamma) - \phi\theta R}\right) \tag{27c}$$

$$\bar{q} = \frac{1}{(1-\theta)}\left\{\frac{1}{\phi}\left[\frac{(\delta+\phi)\left[r\gamma + \phi + \delta\left(1-\gamma\right)\right]}{r+\phi} - \delta\right] - \theta\right\}. \tag{27d}$$

When bank loans mature instantly, a project is immediately liquidated once it is known as bad before t_b. In this case, the length of this efficient liquidation region only depends on the speed of learning λ and the ultimate credit quality $\bar{q}$.

Entrepreneur's value function

Next, we study the valuation of entrepreneur before t reaches t_g. We will focus on the informed-bad case and leave the other cases in the appendix. With instant maturing loans, the contract is renegotiated continuously. Therefore, for $t < t_b$, it is immediately clear that $E_t^b = 0$ and $B_t^b = L$. For $t \in (t_b, t_g)$, let Gdt be the rollover gains that entrepreneur receive during $[t, t + dt]$. In this case, the HJB of entrepreneur becomes

$$\left(r + \phi\right)E_t^b = \dot{E}_t^b + \phi\theta\left(R - F\right) + rF\gamma + G_t^b. \tag{28}$$

Lemma 2. *For the instantly-maturing debt, when $t \in (t_b, t_g)$, the entrepreneur receives rollover gains $G^i dt$ where*

$$G^b = \beta\left[(r + \phi\theta)F - \left(r + \phi\right)L\right] - (1 - \beta)\left[\phi\theta\left(R - F\right) + rF\gamma\right] \tag{29}$$

The proof follows Proposition 1 in Moscarini (2005), which is also in the Appendix.

Clearly, $G_t^b > 0$ for large enough β. When the entrepreneur has more bargaining power, she receives gains by constantly renegotiating the loan contract.

The non-monotonicity in entrepreneur's value function can be easily seen in the case of instantly-maturing debt. Lemma 2 implies $\dot{E}_t^b = (r+\phi)\left(E_t^b + \beta L\right) - \beta\left[\phi\theta R + rF\left(1+\gamma\right)\right]$. Also according to Lemma 6 implies in the Appendix, E_t^b is non-monotonic on $[t_b, t_g]$ if and only if $\dot{E}_{t\to t_g^-}^b < 0$, which will be the case if the liquidation value is low enough.

Proposition 4. *In the case of instantly-maturing debt, E_t^b is non-monotonic on $[t_b, t_g]$ if and only if*

$$L < F - \frac{1-\beta}{\beta}\frac{\phi\theta(R-F) + rF\gamma}{r+\phi}. \tag{30}$$

Note that (30) holds when $\beta = 0$ but can never hold when $\beta = 1$. That is, the non-monotonic pattern is more prominent when the entrepreneur has more bargaining power. Intuitively, the non-monotonicity happens because the outside option of the bargaining experience a discontinuous jump at $t = t_g$. Prior to that, the bank can liquidate the project for a value of L, whereas the entrepreneur receives nothing. For low levels of L, such non-monotonicity pattern is more prominent. When the entrepreneur has more bargaining power, she can extract more of the surplus early on before t reaches t_g. If her bargaining power gets high enough, the entrepreneur is essentially holding up the bank by extracting a large fraction of the surplus. In the case of $\beta = 1$, she is only offering the bank a break-even price $P_{\tau_m} = L$ at each roll over event until t reaches t_g. This extraction can improve the entrepreneur's value so much that it even exceeds the future value after financing with the market.

4 Conclusion

In this paper, we introduce private learning into a banking model and study the dynamic tradeoffs of relationship-based lending. Compared to market financing, bank financing enables learning about the quality of the project being financed, but is also subject to the downside of information monopoly and hold-up cost. We construct an equilibrium in which an entrepreneur starts with bank financing and subsequently switch to market financing. We characterize the timing of such a switch and study how it is affected by factors such as debt maturity, project illiquidity, credit rating and learning. Our model can endogenously generate zombie lending, i.e., a bank is willing to roll over its debt even after learning the project's quality is bad.

One interesting extension is to introduce learning as banks' endogenize choices, which is left as on-going work.

References

Addoum, J. M. and J. Murfin (2017). Hidden in plain sight: Equity price discovery with informed private debt.

Akcigit, U. and Q. Liu (2015). The role of information in innovation and competition. *Journal of the European Economic Association 14*(4), 828–870.

Caballero, R. J., T. Hoshi, and A. K. Kashyap (2008). Zombie lending and depressed restructuring in japan. *American Economic Review 98*(5), 1943–77.

Che, Y.-K. and J. Hörner (2017). Recommender systems as mechanisms for social learning. *The Quarterly Journal of Economics 133*(2), 871–925.

Daley, B. and B. Green (2012). Waiting for news in the market for lemons. *Econometrica 80*(4), 1433–1504.

Diamond, D. W. (1991a). Debt maturity structure and liquidity risk. *The Quarterly Journal of Economics 106*(3), 709–737.

Diamond, D. W. (1991b). Monitoring and reputation: The choice between bank loans and directly placed debt. *Journal of political Economy 99*(4), 689–721.

Fuchs, W. and A. Skrzypacz (2015). Government interventions in a dynamic market with adverse selection. *Journal of Economic Theory 158*, 371–406.

Gustafson, M., I. Ivanov, and R. R. Meisenzahl (2017). Bank monitoring: Evidence from syndicated loans.

Holmstrom, B. and J. Tirole (1997). Financial intermediation, loanable funds, and the real sector. *the Quarterly Journal of economics 112*(3), 663–691.

James, C. (1987). Some evidence on the uniqueness of bank loans. *Journal of financial economics 19*(2), 217–235.

Kremer, I., Y. Mansour, and M. Perry (2014). Implementing the wisdom of the crowd. *Journal of Political Economy 122*(5), 988–1012.

Kremer, I. and A. Skrzypacz (2007). Dynamic signaling and market breakdown. *Journal of Economic Theory 133*(1), 58–82.

Leland, H. E. (1998). Agency costs, risk management, and capital structure. *The Journal of Finance 53*(4), 1213–1243.

Lummer, S. L. and J. J. McConnell (1989). Further evidence on the bank lending process and the capital-market response to bank loan agreements. *Journal of Financial Economics 25*(1), 99–122.

Moscarini, G. (2005). Job matching and the wage distribution. *Econometrica 73*(2), 481–516.

Parlour, C. A. and G. Plantin (2008). Loan sales and relationship banking. *The Journal of Finance 63*(3), 1291–1314.

Peek, J. and E. S. Rosengren (2005). Unnatural selection: Perverse incentives and the misallocation of credit in japan. *American Economic Review 95*(4), 1144–1166.

Petersen, M. A. (2004). Information: Hard and soft.

Rajan, R. G. (1992). Insiders and outsiders: The choice between informed and arm's-length debt. *The Journal of finance 47*(4), 1367–1400.

Rajan, R. G. (1994). Why bank credit policies fluctuate: A theory and some evidence. *The Quarterly Journal of Economics 109*(2), 399–441.

A Appendix

A.1 Proofs

A.1.1 Proof of Lemma 1

Proof. The proof relies on the filtering formula for counting processes in Lipster and Shiryaev (Chapter 19). Let x_t^i be the probability that a type $i \in \{g, b, u\}$ firm looks for external financing at time t and let ℓ_t^i be the probability that a type i firm liquidates at time t. Let L_t be the counting process associated to the liquidation time and and M_t be the counting process associated with going to the market. If we denote the type of the firm at time t by $i(t)$ then L_t has intensity $m\ell_t^{i(t)}$ while M_t has intensity $mx_t^{i(t)}$. The process $i(t)$ has transitions governed by the infinitesimal generator

$$\Lambda \equiv \begin{pmatrix} -\lambda & \lambda q_0 & \lambda(1-q_0) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using Theorem 19.6 (and following similar calculations to the ones in Examples 2 and 3 therein) we get that

$$\begin{aligned} d\pi_t^u = -\lambda\pi_t^u dt &+ \pi_t^u \left(\frac{(\ell_t^u - \ell_t^b)(1-\pi_t^u) - (\ell_t^g - \ell_t^b)\pi_t^g}{\pi_t^u\ell_t^u + \pi_t^g\ell_t^g + \pi_t^b\ell_t^b} \right) \cdot [dL_t - m(\pi_t^u\ell_t^u + \pi_t^g\ell_t^g + \pi_t^b\ell_t^b)dt] \\ &+ \pi_t^u \left(\frac{(x_t^u - x_t^b)(1-\pi_t^u) - (x_t^g - x_t^b)\pi_t^g}{\pi_t^u x_t^u + \pi_t^g x_t^g + \pi_t^b x_t^b} \right) \cdot [dM_t - m(\pi_t^u x_t^u + \pi_t^g x_t^g + \pi_t^b x_t^b)dt] \end{aligned}$$

From here, we get that in absence of liquidation and market financing beliefs are given by

$$\dot{\pi}_t^u = -\lambda\pi_t^u - m\pi_t^u \left((\ell_t^u + x_t^u - \ell_t^b - x_t^b)(1-\pi_t^u) - (\ell_t^g + x_t^g - \ell_t^b - x_t^b)\pi_t^g \right)$$

Suppose that $\ell_t^b = 1$ and $\ell_t^u = \ell_t^g = x_t^u = x_t^g = x_t^b = 0$, then we have

$$\dot{\pi}_t^u = -\lambda \pi_t^u + m\pi_t^u \pi_t^b$$

Similarly, we get

$$\dot{\pi}_t^g = \lambda q_0 \pi_t^u - m\pi_t^g \left[(\ell_t^g + x_t^g - \ell_t^b - x_t^b)(1 - \pi_t^g) - (\ell_t^u + x_t^u - \ell_t^b - x_t^b)\pi_t^u \right]$$
$$\dot{\pi}_t^b = \lambda(1 - q_0)\pi_t^u - m\pi_t^b \left[(\ell_t^b + x_t^b - \ell_t^g - x_t^g)(1 - \pi_t^b) - (\ell_t^u + x_t^u - \ell_t^g - x_t^g)\pi_t^u \right]$$

so in the particular case that $\ell_t^b = 1$ and $\ell_t^u = \ell_t^g = x_t^u = x_t^g = x_t^b = 0$, then we get

$$\dot{\pi}_t^g = \lambda q_0 \pi_t^u + m\pi_t^g \pi_t^b$$
$$\dot{\pi}_t^b = \lambda(1 - q_0)\pi_t^u - m\pi_t^b(1 - \pi_t^b)$$

□

A.1.2 Value function and boundary condition

Lemma 3. *The No Deals condition implies the good type's value function must satisfy smooth-pasting at $t = t_g$. That is*

$$V'^g(t_g) = 0.$$

Proof. We prove by contradiction. Suppose $\frac{dV^g_{\underline{t_g}}}{dt} < 0$, then Equation (16b) implies $V_t^g < \frac{rF(1+\gamma)+\phi R}{(r+\phi)}$. However, this is impossible because $\frac{rF(1+\gamma)+\phi R}{(r+\phi)}$ is the continuation value of the good types if they never finance with the market.

Next, let us assume $\frac{dV^g_{\underline{t_g}}}{dt} > 0$. Under the constructed equilibrium, $\dot{q}_t = 0$ for any $t > t_b$. As a result, $\bar{V}_t^g$ – the continuation payoff when the good type financed with the market at time t also stays at a constant after t_b. Let is be $\bar{V}^g$. If $\frac{dV^g_{t_g^-}}{dt} > 0$, that implies that for ε sufficiently small, $V^g_{t_g-\varepsilon} < \bar{V}^g$ so that the *No Deals condition* fails. Note that this step relies on the fact that $\bar{V}^g_t$ stays a constant for $t \in [t_b, t_g]$. In the equilibrium without the zombie lending stage ($m < m^*$), this condition no longer holds so that in general, $\frac{dV^g_{t_g}}{dt} \geq 0$.

□

Lemma 4. *V^g_t stays at a constant in any equilibrium that is constructed under t_b and t_g.*[3]

Proof. This directly follows after plugging (15b) into (16b) and (17b). □

A.1.3 Proof of Proposition 1

Proof. By applying the smooth pasting condition

$$V^g_{t_g} = \frac{rF(1+\gamma)+\phi R}{r+\phi} = \frac{rF+\phi R+m\bar{V}^g}{r+\phi+m},$$

we get

$$\bar{q} = \frac{1}{(1-\theta)}\left\{\frac{1}{\phi}\left[\frac{(\delta+\phi)\left[r\gamma\left(1+\frac{r+\phi}{m}\right)+\phi+\delta(1-\gamma)\right]}{r+\phi}-\delta\right]-\theta\right\}$$

after some derivation.

Clearly, the equation system in the last region shows

$$V^g_{t_g} - V^b_{t_g} = \frac{\phi R(1-\theta)+m\left(\bar{V}^g-\bar{V}^g\right)}{r+\phi+m} = \frac{\phi R(1-\theta)+m\frac{\phi(R-F)(1-\theta)}{r+\phi}}{r+\phi+m}.$$

In that case, using the same smooth pasting condition, we get

$$V^b_{t_g} = \frac{rF(1+\gamma)+\phi R}{r+\phi} - \frac{\phi R(1-\theta)+m\frac{\phi(R-F)(1-\theta)}{r+\phi}}{r+\phi+m}.$$

[3] This is true under any equilibrium that we construct, which consists of thresholds $\{t_b, t_g\}$. However, it may not hold under any arbitrary equilibrium, which could exist when m gets very large.

Given that, let us solve for $t_g - t_b$ using the ODE system in region 2. In particular, for any $t \in [t_b, t_g]$,

$$V_t^b = e^{(r+\phi)(t-t_g)} V_{t_g}^b + \frac{rF(1+\gamma) + \phi\theta R}{r+\phi} \left[1 - e^{(r+\phi)(t-t_g)}\right].$$

Using the boundary condition $V_{t_b}^b = L$, we can get

$$t_g - t_b = -\frac{1}{(r+\phi)} \log \left(\frac{L - \frac{rF(1+\gamma)+\phi\theta R}{r+\phi}}{V_{t_g}^b - \frac{rF(1+\gamma)+\phi\theta R}{r+\phi}} \right).$$

The threshold t_b is determine by the condition

$$t_b = \min\{t : q_t = \bar{q}\}.$$

The final step is to find the solution for q_t in the interval $[0, t_b]$. For simplicity, let $\{u_t, g_t, b_t\}$ respectively be $\{\pi_t^u, \pi_t^g, \pi_t^g\}$. The ODE system in region 1 becomes

$$\begin{aligned}
\dot{u}_t &= -\lambda u_t + m u_t b_t \\
\dot{g}_t &= \lambda u_t q_0 + m g_t b_t \\
\dot{b}_t &= \lambda u_t (1 - q_0) - m b_t (1 - b_t).
\end{aligned}$$

Let us define $z_t = \frac{g_t}{u_t}$, then,

$$\begin{aligned}
\dot{z}_t &= \frac{\dot{g}_t u_t - g_t \dot{u}_t}{u_t^2} = \frac{\dot{g}_t}{u_t} - z_t \frac{\dot{u}_t}{u_t} \\
&= \lambda q_0 + m z_t (1 - g_t - u_t) - z_t (-\lambda + m (1 - g_t - u_t)) \\
&= \lambda (q_0 + z_t).
\end{aligned}$$

Therefore, we have the solution

$$z_t = \frac{g_t}{u_t} \exp(\lambda t - 1) \Rightarrow g_t = q_0 \left(e^{\lambda t} - 1\right) u_t.$$

Since $u_t + g_t + b_t = 1$, we also have

$$b_t = 1 - \left(q_0 e^{\lambda t} + 1 - q_0\right) u_t.$$

Plugging both back to the ODE system, we have a second-order ODE for u_t

$$\dot{u}_t = (m - \lambda) u_t - m \left(q_0 e^{\lambda t} + 1 - q_0\right) (u_t)^2.$$

This equation is a continuous-time Riccati equation. We can transform it into a second-order ODE. Let $v_t = -m \left(q_0 e^{\lambda t} + 1 - q_0\right) u_t$ and $R_t = q_0 e^{\lambda t} + 1 - q_0$,

$$\dot{v}_t = v_t^2 + \frac{v_t}{R_t} \left[q_0 e^{\lambda t} \lambda + R_t (m - \lambda)\right].$$

Further, we let $v_t = -\frac{\dot{y}_t}{y_t} \Rightarrow \dot{v}_t = -\frac{\ddot{y}_t}{y_t} + (v_t)^2$. The ODE above is transformed to a second-order ODE

$$\ddot{y}_t = \frac{\dot{y}_t}{R_t} \left[q_0 e^{\lambda t} \lambda + R_t (m - \lambda)\right]$$

From here, we get that

$$\dot{y}_t = \dot{y}(0) e^{\int_0^t \lambda \frac{1}{1 + \frac{1 - q_0}{q_0} e^{-\lambda s}} ds + (m - \lambda) t}$$

Moreover,

$$\int_0^t \frac{1}{1 + \frac{1 - q_0}{q_0} e^{-\lambda s}} ds = \frac{1}{\lambda} \log\left(1 - q_0 + q_0 e^{\lambda t}\right)$$

so

$$\dot{y}_t = \dot{y}(0) \left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda}} e^{(m - \lambda) t}$$

Integrating one more time, we get

$$y_t = y(0) + \dot{y}(0) \int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds.$$

Using the definition of v_t and y_t, we have

$$\dot{y}_0 = -v_0 y(0) = m y(0)$$

so

$$y_t = y(0) \left(1 + m \int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds\right).$$

Using the definition of v_t we get

$$v_t = -\frac{m \left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)t}}{1 + m \int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds}$$

so

$$u_t = \frac{\left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda}-1} e^{(m-\lambda)t}}{1 + m \int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds}$$

Thus, using the definition for q_t, we get

$$\begin{aligned} q_t &= g_t + q_0 u_t \\ &= q_0 \left(e^{\lambda t} - 1\right) u_t + q_0 u_t \\ &= q_0 e^{\lambda t} u_t \\ &= \frac{q_0 \left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda}-1} e^{mt}}{1 + m \int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds} \end{aligned}$$

Finally, because q_t is monotone, the solution for t_b and t_g is unique.

Finally, we examine $t_g - t_b$, which equals

$$\frac{1}{r+\phi}\log\left(\frac{(r+\phi)V^b_{t_g} - (rF(1+\gamma)+\phi\theta R)}{(r+\phi)L - (rF(1+\gamma)+\phi\theta R)}\right).$$

A necessary condition for the equilibrium to be true is $t_g - t_b > 0$. However, if $m = 0$, this is clearly violated because Assumption 3 guarantees $V^b_{t_g} < L$. If $m \to \infty$,

$$V^b_{t_g} \to \frac{rF(1+\gamma)+\phi R}{r+\phi} - \frac{\phi(R-F)(1-\theta)}{r+\phi}$$

so that it exceeds L. Finally, a quick comparative static analysis shows that $\frac{dV^b_{t_g}}{dm} > 0$.Therefore, there exists a unique m^* so that such an equilibrium exists if and only if $m > m^*$. □

A.1.4 Proof of Proposition 2

Proof. We have already shown that the unique monotone equilibrium is the one in Proposition 1. It is only left to show that any equilibrium must be monotone if m is low enough. The proof for monotonicity follows the traditional skimming property in bargaining models. In our case, the skimming property is satisfied only if m is low enough. In particular, we show that

$$V^g_t - \bar{V}^g > V^u_t - \bar{V}^u > V^b_t - \bar{V}^b. \tag{31}$$

Let $x^i_t \in \{0,1\}$ and $\ell^u_t \in \{0,1\}$ be the rollover and liquidation decision, respectively. The expected payoff, given strategy (x^i, ℓ^i) is

$$V_t^u = \mathbb{E}_t\Bigg\{ \int_t^\tau e^{-r(s-t)} rF\,(1+\gamma)\,ds + e^{-r(\tau-t)}\Bigg[\mathbb{1}_{\tau=\tau_\phi}\left[q_0 + (1-q_0)\,\theta\right] R$$

$$+ \mathbb{1}_{\tau=\tau_\lambda}\left[q_0 V_\tau^g + (1-q_0)\,V_\tau^b\right] + \mathbb{1}_{\tau=\tau_m}\left[x_\tau^u V_\tau^u + \ell_\tau^u L + (1 - x_\tau^u - \ell_\tau^u)\bar{V}^u\right]\Bigg]\Bigg\}$$

$$V_t^g = \mathbb{E}_t\Bigg\{ \int_t^\tau e^{-r(s-t)} rF\,(1+\gamma)\,ds + e^{-r(\tau-t)}\Bigg[\mathbb{1}_{\tau=\tau_\phi} R + \mathbb{1}_{\tau=\tau_m}\left[x_\tau^g V_\tau^g + \ell_\tau^g L + (1 - x_\tau^g - \ell_\tau^g)\bar{V}^g\right]\Bigg]\Bigg\}$$

$$V_t^b = \mathbb{E}_t\Bigg\{ \int_t^\tau e^{-r(s-t)} rF\,(1+\gamma)\,ds + e^{-r(\tau-t)}\Bigg[\mathbb{1}_{\tau=\tau_\phi} \theta R + \mathbb{1}_{\tau=\tau_m}\left[x_\tau^b V_\tau^b + \ell_\tau^b L + (1 - x_\tau^b - \ell_\tau^b)\bar{V}^b\right]\Bigg]\Bigg\}.$$

A good type can always mimic the strategy of a low type, hence the continuation payoff of a good type must at least as high as the payoff of mimicking the strategy of the bad type.

$$V_t^g \geq \mathbb{E}_t\Bigg\{ \int_t^\tau e^{-r(s-t)} rF\,(1+\gamma)\,ds + e^{-r(\tau-t)}\Bigg[\mathbb{1}_{\tau=\tau_\phi} R$$

$$+ \mathbb{1}_{\tau=\tau_m}\left[x_\tau^b V_\tau^g + \ell_\tau^b L + (1 - x_\tau^b - \ell_\tau^b)\bar{V}^g\right]\Bigg]\Bigg\}.$$

Hence, for all $t < \tau$, we have

$$V_t^g - V_t^b \geq \mathbb{E}_t\Bigg\{ e^{-r(\tau-t)}\Bigg[\mathbb{1}_{\tau=\tau_\phi}(1-\theta)R + \mathbb{1}_{\tau=\tau_m}\left[x_\tau^b(V_\tau^g - V_\tau^b) + (1 - x_\tau^b - \ell_\tau^b)(\bar{V}^g - \bar{V}^b)\right]\Bigg]\Bigg\}.$$

Because the time $\tau = \min\{\tau_\phi, \tau_m\}$ is exponentially distributed with mean arrival rate $m + \phi$, we we can write the previous expression as

$$V_t^g - V_t^b \geq \frac{\phi(1-\theta)R}{r+\phi+m} + \int_t^\infty e^{-(r+m+\phi)(s-t)} m\left[x_\tau^b(V_s^g - V_s^b) + (1 - x_s^b - \ell_s^b)(\bar{V}^g - \bar{V}^b)\right] ds,$$

$$\bar{V}^g - \bar{V}^b = \frac{\phi(1-\theta)(R-F)}{r+\phi} > 0.$$

Letting $\Delta_t \equiv (V_t^g - \bar{V}^g) - (V_t^b - \bar{V}^b)$, we can write the previous inequality as

$$\Delta_t = A + \int_t^\infty e^{-(r+m+\phi)(s-t)} m \left[x_s^b \Delta_\tau + (1 - \ell_s^b)(\bar{V}^g - \bar{V}^b) \right] ds,$$

where

$$A \equiv \frac{\phi(1-\theta)}{(r+\phi+m)(r+\phi)} \left[(r+\phi)F - m(R-F) \right]$$

Differentiating Δ_t, we get the differential equation

$$\dot{\Delta}_t = \left(r + m(1 - x_t^b) + \phi \right) \Delta_t + m(1 - \ell_t^b)(\bar{V}^g - \bar{V}^b) - (r + m + \phi)A.$$

The solution to this equation is given by

$$\Delta_t = \int_t^\infty e^{-(r+\phi)(s-t)+\Psi(s,t)} \left[\phi(1-\theta) \left(F - \frac{m}{r+\phi}(R-F) \right) + \frac{m}{m+\phi}(1-\ell_s^b)(\bar{V}^g - \bar{V}^b) \right] ds$$

where

$$\Psi(s,t) \equiv \int_t^s m(1 - x_u^b) du$$

From here we get that if

$$m < (r+\phi) \frac{F}{R-F}$$

then $\Delta_t > 0$ for any policy ℓ_τ^b. Hence, in any equilibrium we have that

$$V_t^g - \bar{V}^g > V_t^b - \bar{V}^b,$$

which means that if there is a time $\tilde{t}_g$ at which the good type chooses market financing with positive probability, then the bad type chooses market financing for sure. Repeating the same calculations for the pairs $\{g, u\}$ and $\{u, b\}$, we can conclude that the skimming property (31) holds.

Using the definition $q_t = q_0 \pi_t^u + \pi_t^g$ and the evolution of beliefs in the proof of Lemma 1, we get that the evolution of q_t is given by the following differential equation.

$$\dot{q}_t = mq_t\left[(\ell_t^b + x_t^b)(1-\pi_t^u-\pi_t^g) - (\ell_t^u + x_t^u)(1-\pi_t^u) + (\ell_t^g + x_t^g)\pi_t^g\right] \\ + m\pi_t^g\left[(\ell_t^u + x_t^u) - (\ell_t^g + x_t^g)\right]$$

If $\ell_t^g + x_t^g = \ell_t^u + x_t^u = 0$ and $\ell_t^b + x_t^b = 1$ then $\dot{q}_t = mq_t(1-\pi_t^u-\pi_t^g) > 0$. On the other hand, by the skimming property, we have that:

1. If $\ell_t^g + x_t^g = 1$, then $\ell_t^u + x_t^u = \ell_t^b + x_t^b = 1$ so $\dot{q}_t = 0$.
2. If $\ell_t^b + x_t^b = 0$, then $\ell_t^g + x_t^g = \ell_t^u + x_t^u = 0$ and also $\dot{q}_t = 0$.
3. If $\ell_t^g + x_t^g = 0$ and $\ell_t^u + x_t^u = 1$, then $\ell_t^b + x_t^b = 1$ so $\dot{q}_t = m\pi_t^g(1-q_t) > 0$.

Hence, in any equilibrium, the trajectory of q_t must be non-decreasing in time so an equilibrium must be monotone.

□

A.1.5 Rollover gains with instantly-maturing debt

Let us write the full version of Lemma 2, including the rollover gains to other types

Lemma 5. *For the instantly-maturing debt, when $t \in (t_b, t_g)$, the entrepreneur receives rollover gains $G^i dt$ where*

$$G^b = \beta\left[(r+\phi\theta)F - (r+\phi)L\right] - (1-\beta)\left[\phi\theta(R-F) + rF\gamma\right] \tag{32a}$$

$$G^g = -(1-\beta)rF\gamma \tag{32b}$$

$$G^u = \beta\left[\big(r+\phi(q_0+(1-q_0)\theta)\big)F - (r+\phi)L\right] - (1-\beta)\left[\phi(q_0+(1-q_0)\theta)(R-F) + rF\gamma\right]. \tag{32c}$$

Proof. Nash Bargaining implies $\beta\left(B_t^b - L\right) = (1-\beta)\, E_t^b$, which further implies $\beta \dot{B}_t^b = (1-\beta)\, \dot{E}_t^b$. Multiplying Equation (36) by $(1-\beta)$ (28) by β and take their difference:

$$\begin{aligned} -\beta\,(r+\phi)\,L &= (1-\beta)\left[\phi\theta\,(R-F) + rF\gamma + G^b\right] - \beta\left[rF + \phi\theta F - G^b\right] \\ \Rightarrow G^b &= \beta\left[rF + \phi\theta F - (r+\phi)\,L\right] - (1-\beta)\left[\phi\theta\,(R-F) + rF\gamma\right]. \end{aligned}$$

Repeating the same calculations for uninformed we get

$$G^u = \beta\left[rF + \phi\,(q_0 + (1-q_0)\theta)\,F - (r+\phi)\,L\right] - (1-\beta)\left[\phi\,(q_0 + (1-q_0)\theta)\,(R-F) + rF\gamma\right]$$

The only difference in the case of the good type is that the outside option is different now. In this case, Nash bargaining implies that $\beta\,(B_t^g - F) = (1-\beta)\,(E_t^g - NPV^g + F)$ so we get

$$G^g = (1-\beta)\left((r+\phi)NPV^g - r(1+\gamma)F - \phi R\right) = -(1-\beta)r\gamma F.$$

□

A.2 Bank and Entrepreneur Value Function

In this subsection, we supplement the details in subsection 3.2. Specifically, in the market financing region, we have

$$\begin{aligned} B_t^u &= \frac{rF + \phi\left[q_0 + (1-q_0)\,\theta\right]F + \lambda\left[q_0 B_t^g + (1-q_0)\,B_t^b\right] + mF}{r+\phi+\lambda+m} \\ B_t^g &= F \\ B_t^b &= \frac{rF + \phi\theta F + mF}{r+\phi+m}. \end{aligned}$$

When $t \in (0, t_b)$,

$$(r + \phi + m)\, B_t^b = \dot{B}_t^b + rF + \phi\theta F + mL,$$

whereas when $t \in (t_b, t_g)$,

$$(r + \phi)\, B_t^b = \dot{B}_t^b + rF + \phi\theta F + m\left(F - P_t^b\right).$$

Lemma 6 proves that in the region of (t_b, t_g), $\dot{E}_t^b$ will change sign at most once. Therefore, the value of E_t^b is either monotonically increasing, or first increases and then decreases.

Lemma 6. *$\ddot{E}_t^b > 0$ for $t \in (t_b, t_g)$.*

Proof. Take derivative to both sides of equation (26b), we can get

$$\ddot{E}_t^b = (r + \phi + m)\, \dot{E}_t^b - m\beta \dot{V}_t^b.$$

This implies any local extrema of E_t^b (which satisfies $\dot{E}_t^b = 0$) is a local maximum. if $\dot{V}_t^b > 0$. Therefore, if $\dot{V}_t^b > 0$ for any $t \in (t_b, t_g)$, E_t^b cannot change sign more than once over $t \in (t_b, t_g)$. To show this, let us take derivative to both sides of equation (16c)

$$\ddot{V}_t^b = (r + \phi)\, \dot{V}_t^b.$$

At $t = t_b$, $\dot{V}_{t_b}^b = (r + \phi)\, L - rF\,(1 + \gamma) - \phi\theta R > 0$ following Assumption 3. Therefore, since $sgn\left(\dot{V}_t^b\right) = sgn\left(\ddot{V}_t^b\right)$ for any $t \in (t_b, t_g)$, that implies $\dot{V}_t^b > 0$ in this region as well.

□

A.3 Instantly-Maturing Debt

When debt is rolled over in a continuous basis, at any time $t < t_b$ and as long as the project has not been liquidated, the market knows that the bank has not received bad news. In this case, the solution for q_t simplifies significantly, and the solution in (23) converges to

$$q_t = \frac{q_0}{q_0 + (1 - q_0)e^{-\lambda t}}.$$

Thus, we can solve for the threshold t_b

$$t_b = \frac{1}{\lambda}\left[\log\left(\frac{\bar{q}}{1-\bar{q}}\right) - \log\left(\frac{q_0}{1-q_0}\right)\right] \tag{33a}$$

$$t_g = t_b + \frac{1}{r+\phi}\log\left(\frac{\phi(1-\theta)F}{(r+\phi)L - rF(1+\gamma) - \phi\theta R}\right). \tag{33b}$$

We can also solve for the firm value in closed form. For $t \leq t_b$, the HJB equation reduces to

$$(r + \phi + \lambda)\, V_t^u = \dot{V}_t^u + rF\,(1+\gamma) + \phi\,[q_0 + (1-q_0)\,\theta]\, R + \lambda\,[q_0 V_t^g + (1-q_0)\, L] \tag{34a}$$

$$(r+\phi)\, V_t^g = \dot{V}_t^g + rF\,(1+\gamma) + \phi R \tag{34b}$$

$$V_t^b = L. \tag{34c}$$

For $t \in (t_b, t_g)$, the equations are unchanged, except that the informed-bad type's becomes

$$(r+\phi)\, V_t^b = \dot{V}_t^b + rF\,(1+\gamma) + \phi\theta R. \tag{35}$$

With instant maturing debt, the entrepreneur refinances with market-based lenders right upon time t reaches t_g, in which case the valuation equations are identical to (4). This equations can be solved in closed form. For $t < t_b$, the value of the good and uninformed firm are

$$
\begin{aligned}
V_t^u &= \bar{V}^u e^{-(r+\phi+\lambda)(t_g-t)} + q_0 \bar{V}^g \left(1 - e^{-\lambda(t_g-t)}\right) e^{-(r+\phi)(t_g-t)} \\
&\quad + \left[\frac{\lambda}{(r+\phi)(r+\phi+\lambda)} + \frac{e^{-(r+\phi+\lambda)(t_g-t)}}{r+\phi+\lambda} - \frac{e^{-(r+\phi)(t_g-t)}}{r+\phi}\right] q_0 \big(rF\,(1+\gamma) + \phi R\big) \\
&\quad + \frac{1 - e^{-(r+\phi+\lambda)(t_g-t)}}{r+\phi+\lambda} \big(rF\,(1+\gamma) + \phi\,[q_0 + (1-q_0)\,\theta]\,R + \lambda(1-q_0)L\big) \\
V_t^g &= \frac{1 - e^{-(r+\phi)(t_g-t)}}{r+\phi} (rF\,(1+\gamma) + \phi R) + e^{-(r+\phi)(t_g-t)} \bar{V}^g
\end{aligned}
$$

Next, we supplement the HJBs of bank when $t \in (t_b, t_g)$.

$$
(r+\phi)\, B_t^b = \dot{B}_t^b + rF + \phi\theta F - G_t^b. \tag{36}
$$

A.4 An Alternative Model

In this subsection, we present an alternative model where the entrepreneur is no longer deep-pocketed. For that reason, we may also drop the assumption of tax shields which offer reasons for her to raise funding from the bank or the market. Instead, the entrepreneur has no wealth to begin with and does not receive any wealth before the project finally matures. In fact, as we will see shortly, she will simply consume all the wealth even if she receive any interim cash flows, since this will enhance the value of her outside option.

Following the alternative assumption, the entrepreneur cannot incur any loss during the rollover date. For all types $i \in \{u, g, b\}$, it must be the case that $P_{\tau_m}^i \geq F$. Under this result, a type-g entrepreneur can no longer be held-up by the bank.

The HJBs for the value function $\{V_t^i,\ i \in \{u, g, b\}\}$ are unchanged. Again, we can use two thresholds $\{t_b, t_g\}$ to characterize the equilibrium solutions. However, the boundary conditions can differ, due to the assumption that the ability for the bank to hold up the entrepreneur is limited. Specifically, let us define

$$B^b_{\max}(\Delta) = \int_{t_b}^{t_g} e^{-(r+\phi)(s-t_b)}\phi\theta F ds + e^{-(r+\phi)(t_g-t_b)}\bar{B}^b.$$

Apparently, $B^b_{\max}(\Delta)$ is the maximum value that a type-b bank may attain at time t_b, given that the rollover gain for the entrepreneur is exactly 0: $P_{\tau_m} = F$ for any $\tau_m \in (t_b, t_g)$. Here, $\Delta = t_g - t_b$ is the length of the zombie-lending period. $\bar{B}^b = \frac{m}{r+\phi+m}F + \frac{\phi\theta F}{r+\phi+m}$ will be the value of the bank after t_g. Given any t_g, if we use the same boundary condition $V^b_{t_b} = L$, then it must be that $B^b_{\max}(\Delta) \geq L$. If this condition is violated, however, then the boundary condition $V^b_{t_b} = L$ no longer holds and as a result, t_b needs to be higher. The boundary condition is instead replaced by $B^b_{\max}(\Delta) = L$. This condition essentially defines $\Delta = t_g - t_b$ not being too large.

We will work out the value of bank and equity subsequently. (TBA)