Fisherian Debt-Deflation Zero Lower Bound

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Abstract

In this paper, we build a nonlinear two-sector DSGE model with capital accumulation, in which the Zero Lower Bound (ZLB) of interest rate and the collateral constraint are occasionally binding. We show the interaction of ZLB and the deleveraging cycle triggered by a binding collateral constraint can be a powerful mechanism in exacerbating the financial crisis as well as generating the prolonged liquidity trap and stagnation after the crisis. In particular, a binding ZLB can be triggered by capital over-accumulation, and when ZLB is binding, output is decreasing in capital stock. We also find an equilibrium does not exist when the capital stock is too high, while the existence of equilibrium can be restored by adding the adjustment cost of capital into the model. In our numerical results, we find the amplification effect of the collateral constraint is modest when the ZLB is not binding, but is quantitatively large when the ZLB is binding. In addition, with collateral constraint and ZLB, the recovery of the economy is slow since it takes longer for the borrowers to restore their net worth, and due to insufficient demand, the duration of the liquidity trap is longer. Lastly, in a society with better access to the credit market, the borrowers use higher leverage ex ante, and the average duration of ZLB is longer once the economy is hit by adverse shocks.

1 Introduction

After the financial crisis in 2008, the economy of the United States entered a staggering liquidity trap lasting for seven years. As the nominal interest rate essentially hits the zero lower bound (ZLB), the liquidity trap poses a real challenge to the monetary policymakers who attempts to boost the employment and price level, and the ZLB accounts for a significant role in exacerbating the recession and the slow recovery after the crisis (e.g., Gust et al., 2017).

Many scholars, including Eggertsson and Krugman (2012) and Korinek and Simsek (2016), emphasize leverage as a major driving force of the liquidity trap. During the crisis, the households' and firms' ability to borrow are restricted, which lead them to spend less and save more. Thus on the aggregate level, both the output and the price level drop. Since it is impossible for the central bank to drop the nominal interest rate below zero, once the ZLB is hit and in the presence of price rigidity, the real interest rate is higher than the natural interest rate if the ZLB were absent. A higher real interest rate further reduces consumption and investment, and the recession is exacerbated due to the insufficient demand. In addition, a higher real interest rate aggravates the real debt burden of the borrowers, whose net worth and borrowing capacity are further reduced. Since the borrowers in general have higher propensity of spending, the drop in their net worth further reduces the aggregate demand. In this line of argument, the over-indebtedness problem and the liquidity trap problem reinforce each other, make the crisis more severe and form a major challenge for the economic recovery after the crisis.[1]

In this paper, we construct a nonlinear DSGE model with capital accumulation, in which the ZLB and the collateral constraint are occasionally binding. There are two sectors in the economy, the entrepreneurs and the households, who are both of measure one and infinitely lived. The entrepreneurs accumulate capital, produce the intermediate good using capital and labor hired from the households. In the meantime, the entrepreneurs can borrow from the households subject to a collateral constraint. There is also a sector of monopolistically competitive retailers who produce the final good using the intermediate good. For simplicity, following Korinek and Simsek (2016), we assume complete price rigidity in our model, i.e., the price of the final good is fixed. As a result, in the presence of ZLB, both the nominal interest rate and the real interest rate are bounded below by zero. We assume that the monetary policy is to set the real interest rate to match the natural rate unless the ZLB is binding.

Our model generates several innovative results which are consistent with the observations during the crisis and the liquidity trap. To illustrate the key mechanism, we first solve a deterministic two-period model analytically, and find that the ZLB is binding when the capital stock is high. Besides, when the ZLB is binding, output is decreasing in capital stock, which is consistent with the "paradox of toil" proposed by Eggertsson and

[1] In his seminal paper on debt-deflation, Fisher (1933) wrote: "I have, at present, a strong conviction that these two economic maladies, the debt disease and the price-level disease (or dollar disease), are, in the great booms and depressions, more important causes than all others put together."

Krugman (2012) when the problem during a liquidity trap is insufficient demand instead of supply. Since the entrepreneurs are willing to hold capital only if the return to capital is weakly larger than the interest rate, a binding ZLB naturally implies an upper bound for capital holding with decreasing marginal return to capital. Actually, a higher capital stock reduces investment when the entrepreneurs can keep the un-depreciated part of capital stock for future production. Since the consumption are also reduced in the presence of ZLB, a higher initial capital has to be accompanied by a reduction in output to clear the market. Actually, we show that there is a threshold of capital stock above which there is no equilibrium. In addition, we show that certain degree of capital adjustment cost is necessary to restore the equilibrium. In particular, if we add irreversibility of investment into the model, i.e., investment cannot be negative, then the equilibrium always exists.

We then extend the two-period model into a infinite-horizon DSGE model with Markov shock to the growth rate of TFP. All the results from the two-period model still hold qualitatively in the infinite-horizon model. In particular, the ZLB tends to bind when the leverage is high, capital stock is high and when the growth rate of TFP is low. We show that the interaction between the ZLB and the collateral constraint is important, and the performance of the economy around the ZLB is highly nonlinear. When the ZLB is not binding, the amplification effect of the collateral constraint is modest. However, when both ZLB and the collateral constraint are binding, the amplification effect is very large. This is helpful in explaining the severity of the financial crisis in 2008. The model also shed light on the stagnation and the prolonged duration of the liquidity trap after the financial crisis. Since the entrepreneurs are financially constrained, it takes long time for them to restore their net worth, and consequently the recovery of the aggregate demand is also slow.

Lastly, we quantitatively show that in a society with better access to credit market, the probability of a binding collateral constraint is lower, but the duration of ZLB is higher. The intuition here is similar to the phenomenon of *volatility paradox* proposed by Brunnermeier and Sannikov (2014). With looser borrowing constraint, the entrepreneurs tend to borrow more. Once the economy is hit by an adverse shock, the deleveraging cycle is more severe, and it would take longer time for the entrepreneurs to restore the wealth level, and for the economy to recover from the liquidity trap.

The rest of this paper is organized as follows. Section 2 gives a simple two-period model to analytically show how the ZLB arises in a model with capital accumulation. Section 3 add investment irreversibility into the two-period model and shows that in this case, an equilibrium always exists. Section 4 extends the model with investment irreversibility into an infinite-horizon model with shocks to the growth rate of the TFP.

Section 5 calibrates the infinite-horizon model and discusses the numerical results.

2 Simple Two-Period Model

Two-period economy $t = 0, 1$ with a measure one of identical households and measure one of identical entrepreneurs. We assume no-uncertainty, perfectly sticky prices.

2.1 Economic Environment

Households The households maximize the discounted utility

$$\log c_0' - L_0' + \beta \left(\log c_1' - L_1' \right) \tag{1}$$

subject to sequential budget constraint

$$P_t c_t' + \frac{b_t'}{R_t} \leq b_{t-1}' + w_t L_t' + \int_0^1 \Xi_t(z) dz \tag{2}$$

for $t = 0, 1$, where R_t is the nominal interest rate, and $\int_0^1 \Xi_t(z) dz$ is the aggregate profit from intermediate good retailers.

Final Good Producers and Retailers Following Iacoviello (2005), we assume the retailers, indexed by $z \in [0, 1]$, purchase the intermediate good from the entrepreneurs at the wholesale price P_t^e, differentiate it a no cost and sell the differentiated goods to a representative final good producer at price $p_t(z)$. The final good producer combines the differentiated goods using a CES production technology

$$Y_t = \left(\int_0^1 (y_t(z))^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}},$$

where $\epsilon > 1$ is the elasticity of substitution between retailers' goods. It is standard to show that the price of the final good is given by

$$P_t = \left(\int_0^1 p_t(z)^{1-\epsilon} dz \right)^{\frac{1}{1-\epsilon}},$$

and each retailer z face an iso-elastic demand curve

$$y_t(z) = \left(\frac{p_t(z)}{P_t} \right)^{-\epsilon} Y_t,$$

for its intermediate good. Denote the markup of the retailers' sector as $X_t = \frac{P_t}{P_t^e}$.

Entrepreneurs The entrepreneurs are identical and maximize the discounted utility

$$\log c_0 + \gamma \log c_1 \tag{3}$$

subject to the sequential budget constraint:

$$P_t c_t + P_t k_t + \frac{b_t}{R_t} \leq b_{t-1} + P_t (1-\delta) k_{t-1} + P_t^e Y_t^e - w_t L_t \tag{4}$$

The entrepreneur produces using the production technology:

$$Y_t^e = A_t k_{t-1}^{\alpha} L_t^{1-\alpha} \tag{5}$$

where

$$A_t = aG_t$$

and

$$\frac{G_t}{G_{t-1}} = 1 + g.$$

Given the production function (5) and a competitive factor market, the wage w_t and total return from capital holding R_t^K can be expressed as

$$w_t = P_t^e (1-\alpha) A_t \left(\frac{k_{t-1}}{L_t}\right)^{\alpha},$$

$$R_t^K = P_t (1-\delta) + P_t^e \alpha A_t \left(\frac{k_{t-1}}{L_t}\right)^{\alpha-1}.$$

The entrepreneurs are also subject to the following borrowing constraint:

$$b_t + mR_{t+1}^K k_t \geq 0, \tag{6}$$

where $m \in [0,1]$.

Let ω_t denote the *normalized financial wealth* of the entrepreneurs:

$$\omega_t = \frac{R_t^K k_{t-1} + b_{t-1}}{R_t^K k_{t-1}}, \tag{7}$$

and ω_t' denote the normalized financial wealth of the households:

$$\omega_t' = \frac{b_{t-1}'}{R_t^K k_{t-1}}.$$

By the bond market clearing condition, $\omega_t' = 1 - \omega_t$ in any competitive equilibrium.

Price-Stickiness and Monetary Policy If the retailers have no constraint on their prices then they choose price and quantity to maximize profit:

$$\Xi_t(z) = \max_{0 \le p_t(z), \tilde{y}_t(z)} \tilde{y}_t(z) \left(p_t(z) - P_t^e\right)$$

subject to

$$\tilde{y}_t(z) \le y_t(z) = \left(\frac{p_t(z)}{P_t}\right)^{-\epsilon} Y_t,$$

which implies

$$p_t(z) = \frac{\epsilon}{\epsilon - 1} P_t^e, \tag{8}$$

and

$$P_t = \frac{\epsilon}{\epsilon - 1} P_t^e.$$

When P_t is normalized to 1, we obtain

$$P_t^e = \frac{\epsilon - 1}{\epsilon},$$
$$X_t = \frac{\epsilon}{\epsilon - 1}.$$

But in this simple example, we assume that the retailers' price $p_t(z)$ are perfectly sticky

$$p_t(z) \equiv 1,$$

which implies

$$P_t = 1, \forall t.$$

The retailers only decide on the quantity supplied to the market:

$$\Xi_t(z) = \max_{0 \le \tilde{y}_t(z) \le y_t(z)} \tilde{y}_t(z) \left(1 - P_t^e\right).$$

So

$$y = y_t(z)$$

if $P_t^e < 1$.

However, at $t = 0$, the central bank chooses R_0 to replicate the equilibrium under perfect price-stickiness in which

$$p_0(z) = \frac{\epsilon}{\epsilon - 1} P_0^e = 1,$$

unless R_0 is constrained by the ZLB. When the ZLB binds, P_0^e endogenously decreases to clear markets and $X_0 > \frac{\epsilon}{\epsilon-1}$.

At $t = 1$, any level of $P_1^e \leq 1$, corresponds to an equilibrium. In order to simplify the notations, we focus on the flexible price equilibrium:

$$P_1^e = \frac{\epsilon - 1}{\epsilon},$$
$$X_1 = \frac{\epsilon}{\epsilon - 1}.$$

Equilibrium Definition We use the following standard equilibrium definition.

Definition 1. Sequence of prices $\{P_0^e, P_1^e\}$ and interest rate R_0, such that markets clear

$$c_t + c_t' + (k_t - (1 - \delta)k_{t-1}) = Y_t, \tag{9}$$
$$b_t + b_t' = 0, \tag{10}$$
$$L_t = L_t', \tag{11}$$
$$Y_t = Y_t^e, \tag{12}$$

and the households, entrepreneurs, retailers, final good producers maximize their objectives subject to their constraints.

2.2 Equilibrium Properties

Using the market clearing condition of $Y_t = Y_t^e$, the total profit from the retailers' sector is

$$\int_0^1 \Xi_t(z)dz = \left(1 - \frac{1}{X_t}\right) Y_t.$$

Using (7), the budget constraints (2) and (4) can be written as

$$c_t' + \frac{b_t'}{R_t} \leq R_t^K k_{t-1} (1 - \omega_t) + w_t L_t' + \left(1 - \frac{1}{X_t}\right) Y_t, \tag{13}$$

$$c_t + k_t + \frac{b_t}{R_t} \leq R_t^K k_{t-1} \omega_t, \tag{14}$$

and the expressions of w_t and R_t^K can be written as

$$w_t = \frac{1-\alpha}{X_t} A_t \left(\frac{k_{t-1}}{L_t}\right)^{\alpha}, \tag{15}$$

$$R_t^K = 1 - \delta + \frac{\alpha}{X_t} A_t \left(\frac{k_{t-1}}{L_t}\right)^{\alpha-1}. \tag{16}$$

The feasibility constraint is

$$c_t + c_t' + k_t = (1-\delta)\, k_{t-1} + Y_t. \tag{17}$$

Denote the Lagrangian multiplier on the collateral constraint as $\frac{\mu_0}{c_0}$. The F.O.C.s of the entrepreneurs are:

$$-\frac{1 - mR_1^K \mu_0}{c_0} + \gamma \frac{R_1^K}{c_1} = 0, \tag{18}$$

$$\mu_0 \left(b_0 + mR_1^K k_0\right) = 0, \tag{19}$$

$$-\frac{1 - R_0 \mu_0}{c_0} + \gamma \frac{R_0}{c_1} = 0. \tag{20}$$

The F.O.C.s for the households are

$$\frac{w_0}{c_0'} = 1, \tag{21}$$

$$\frac{w_1}{c_1'} = 1, \tag{22}$$

$$\frac{1}{c_0'} = \beta R_0 \frac{1}{c_1'}. \tag{23}$$

Lastly, the ZLB constraint implies

$$(R_0 - 1)\left(X_0 - \frac{\epsilon}{\epsilon - 1}\right) = 0 \tag{24}$$

2.3 Last Period

In the last period 1, there are no borrowing or lending, and thus we have $b_1 = 0$. The entrepreneurs makes no further investment either, i.e., $k_1 = 0$. We assume the markup takes its steady state value, $X_1 = \frac{\epsilon}{\epsilon-1}$. In the last period, the entrepreneurs and households consume all their wealth. From equations (9),(10), and (13), (14) we have

$$c_1 = R_1^K k_0 \omega_1, \tag{25}$$

$$c_1' = R_1^K k_0 (1 - \omega_1) + \left(1 - \frac{\alpha}{X_1}\right) Y_1. \tag{26}$$

Given ω_1 and k_0, for a labor supply L_1, we can solve w_1 and R_1^K from (15) and (16), and c_1 and c_1' from (25) and (26). Lastly, from (21), we solve L_1 by the following equation:

$$\frac{1-\alpha}{X_1} L_1^{-\alpha} - \left(1 - \frac{\alpha}{X_1}\omega_1\right) L_1^{1-\alpha} = (1-\omega_1)(1-\delta)\frac{1}{A_1} k_0^{1-\alpha}. \tag{27}$$

We see that L_1 is decreasing in both k_0 and ω_1.

2.4 Model with Natural Borrowing Limit, $m = 1$

We first consider the model with $m = 1$. In this case, The collateral constraint (6) corresponds to the natural borrowing limit and should not be binding in equilibrium. Otherwise, c_1 becomes zero from (7) and (25).

In this case, we must have

$$R_1^K = R_0.$$

Suppose this relationship does not hold. If $R_1^K > R_0$, the return of investment is higher than the interest rate, and the entrepreneurs can make infinite profit by borrowing from the households and invest in capital. Otherwise, if $R_1^K < R_0$, no entrepreneurs are willing to hold capital, i.e., $k_0 = 0$, which implies zero consumptions in the last period. Both situations cannot arise in the equilibrium. Therefore, we have $R_1^K = R_0$ when the collateral constraint is not binding.

From (20) and (23), we have

$$R_0 = \frac{c_1}{\gamma c_0} = \frac{c_1'}{\beta c_0'}.$$

Combining the results above with (25), the entrepreneurs' budget (14) becomes

$$c_0 + \frac{c_1}{R_0} = b_{-1} + R_0^K k_{-1}.$$

and we get the entrepreneurs' consumptions as

$$c_0 = \frac{1}{1+\gamma} R_0^K k_{-1} \omega_0, \tag{28}$$

$$c_1 = \frac{\gamma R_0}{1+\gamma} R_0^K k_{-1} \omega_0. \tag{29}$$

which suggests that the entrepreneurs consume $\frac{1}{1+\gamma}$ fraction of their lifetime wealth.

We can express all the other variables as functions of R_0 and X_0. From (16), (15) and (21), given k_0, in the last period we have

$$\frac{k_0}{L_1} = \left(\frac{X_1}{\alpha}\frac{R_0 - 1 + \delta}{A_1}\right)^{\frac{1}{\alpha-1}},$$

$$c_1' = \frac{1-\alpha}{X_1} A_1 \left(\frac{X_1}{\alpha}\frac{R_0 - 1 + \delta}{A_1}\right)^{\frac{\alpha}{\alpha-1}}, \tag{30}$$

$$Y_1 = \frac{X_1}{\alpha}\left(R_0 - 1 + \delta\right) k_0. \tag{31}$$

From (23), (16) and (15), in the first period we have

$$c_0' = \frac{1}{\beta R_0}\frac{1-\alpha}{X_1} A_1 \left(\frac{X_1}{\alpha}\frac{R_0 - 1 + \delta}{A_1}\right)^{\frac{\alpha}{\alpha-1}}, \tag{32}$$

$$L_0 = \left(\frac{G}{\beta R_0}\frac{X_0}{X_1}\right)^{-\frac{1}{\alpha}} \left(\frac{X_1}{\alpha}\frac{R_0 - 1 + \delta}{A_1}\right)^{\frac{1}{1-\alpha}} k_{-1}, \tag{33}$$

$$R_0^k = 1 - \delta + \left(\beta R_0\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{G X_0}{X_1}\right)^{-\frac{1}{\alpha}} \left(R_0 - 1 + \delta\right).$$

Notice that the households' consumptions, c_0' and c_1' are purely determined by R_0. In other words, given R_0, c_0' and c_1' are independent of k_{-1}, ω_0 and X_0.

The aggregate supply (AS) curve is

$$Y_0^S = \frac{X_1}{\alpha G}\left(\frac{G_1}{\beta R_0}\frac{X_0}{X_1}\right)^{\frac{\alpha-1}{\alpha}} \left(R_0 - 1 + \delta\right) k_{-1}. \tag{34}$$

We see that the AS curve is increasing in R_0 and k_{-1}, but is not affected by the wealth distribution ω_0 directly.

From (28) and (20), we can solve c_0 and c_1 as:

$$c_0 = \frac{1}{1+\gamma}\omega_0 \left[1-\delta+(\beta R_0)^{\frac{1-\alpha}{\alpha}}\left(\frac{GX_0}{X_1}\right)^{-\frac{1}{\alpha}}(R_0-1+\delta)\right]k_{-1}, \tag{35}$$

$$c_1 = \frac{\gamma R_0}{1+\gamma}\omega_0 \left[1-\delta+(\beta R_0)^{\frac{1-\alpha}{\alpha}}\left(\frac{GX_0}{X_1}\right)^{-\frac{1}{\alpha}}(R_0-1+\delta)\right]k_{-1}. \tag{36}$$

Then from the feasibility (17) in period 1, (36) and (30), we get

$$k_0 = \frac{c_1+c_1'}{1-\delta+\frac{X_1}{\alpha}(R_0-1+\delta)}. \tag{37}$$

By the feasibility condition (17), the aggregate demand (AD) curve is

$$Y_0^D = c_0 + c_0' + \left[k_0 - (1-\delta)k_{-1}\right]. \tag{38}$$

The variables on the right-hand side are computed using (32), (35) and (37). The term $k_0-(1-\delta)k_{-1}$ is investment.

Lastly, by the feasibility condition (17) in period 0, we have a unique equation to pin down R_0:(or X_0 depending on whether the ZLB is binding or not.)

$$\begin{aligned}\frac{1}{k_{-1}} =& \frac{(1-\delta)^2\left(1-\frac{1}{1+\gamma}\omega_0\right)\left(1-\frac{X_1}{\alpha}\right)+(1-\delta)\left(\frac{X_1}{\alpha}-\frac{1}{1+\gamma}\omega_0\left(1+\frac{X_1}{\alpha}\right)\right)R_0}{\frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha}\frac{R_0-1+\delta}{A_1}\right)^{\frac{\alpha}{\alpha-1}}\left[1+\frac{1}{\beta R_0}(1-\delta)\left(1-\frac{X_1}{\alpha}\right)+\frac{X_1}{\alpha\beta}\right]} \\ &+\frac{\left[(1-\delta)\left(1-\frac{X_1}{\alpha}\right)\left(\frac{X_0}{\alpha}-\frac{1}{1+\gamma}\omega_0\right)+\left(\frac{X_0X_1}{\alpha^2}-\frac{\frac{X_1}{\alpha}+\gamma}{1+\gamma}\omega_0\right)R_0\right](\beta R_0)^{\frac{1-\alpha}{\alpha}}\left(\frac{GX_0}{X_1}\right)^{-\frac{1}{\alpha}}(R_0-1+\delta)}{\frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha}\frac{R_0-1+\delta}{A_1}\right)^{\frac{\alpha}{\alpha-1}}\left[1+\frac{1}{\beta R_0}(1-\delta)\left(1-\frac{X_1}{\alpha}\right)+\frac{X_1}{\alpha\beta}\right]}.\end{aligned} \tag{39}$$

2.4.1 Equilibrium when ZLB is not Binding

Proposition 1. *With $m=1$, R_0 is decreasing in k_{-1} and decreasing in ω_0. Given ω_0, ZLB binds if and only if $k_{-1} \geq \hat{k}_{-1}$ given in Appendix A.*

Proof. Appendix A □

Proposition 1 shows the ZLB binds when k_{-1} is high enough, or when ω_0 is low enough. The intuition can be analyzed by plotting the aggregate supply curve (AS) and the aggregate demand curve (AD) in Figure 1.

We see that in Figure 1, the AS curve is positively sloped. The reason is the following. From (35), a higher R_0 reduces c_0' in two ways. First, it increases the opportunity cost of consumption and reduces c_0' through the substitution effect channel. Second, with

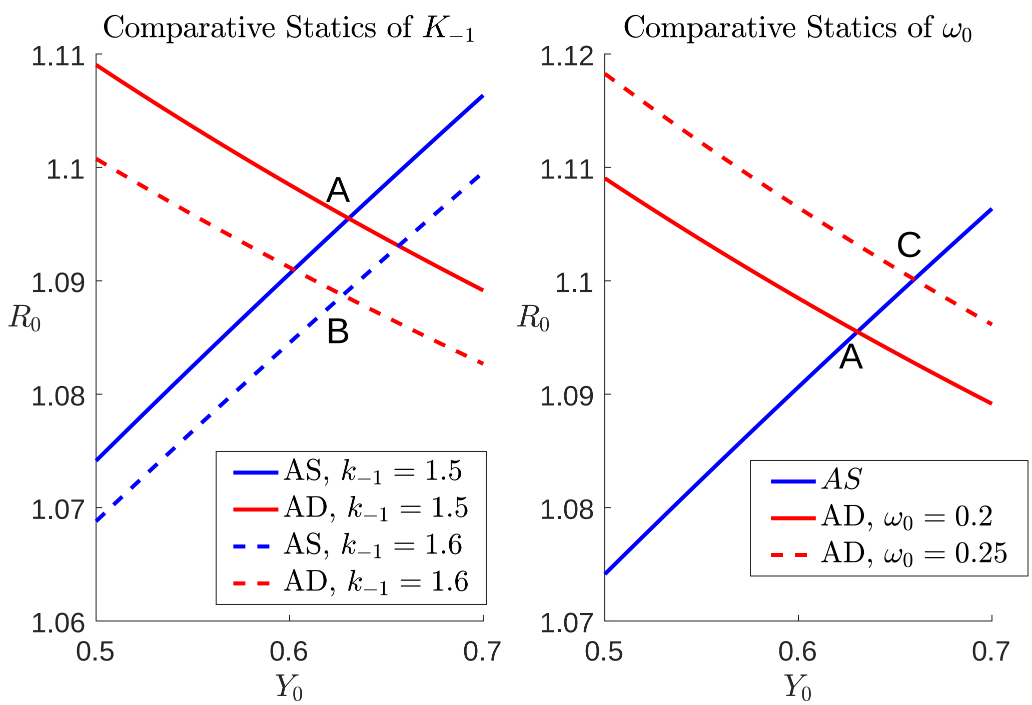


Figure 1: AS-AD Curves when $R_0 > 1$

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $g = 0$ and $\epsilon = 21$. We choose $k_{-1} = 1.5$ and $\omega_0 = 0.2$ as the baseline case.

$R_0 = R_1^k$, a higher R_0 implies a lower capital-labor ratio in the last period by (16), and thus lower w_1 and c_1' by (15) and (21), which reduces c_0' through the income effect channel. Lastly, labor cost becomes cheaper by the labor supply equation (21), which boosts the aggregate supply.

The AD curve is negatively sloped. With higher interest rate, the entrepreneurs and households choose to save more into the future and consume less. On the other hand, with bond and capital being perfect substitutes, the entrepreneurs choose to invest less when the cost of borrowing is higher.

Now consider an exogenous increase of the initial capital, k_{-1}. In (34), given R_0, the output increases proportionally to k_{-1}, and the AS curve shifts to the right. This result is natural since larger k_{-1} implies greater production capacity. The AD curve, on the contrary, shifts to the left. We have shown that given R_0, c_0' is fixed as in (32). Although c_0 and k_0 are increasing in k_{-1} as in (35) and (37), they increase by a smaller amount compared to the increase in $(1-\delta)k_{-1}$ as long as R_0 is not too large.[2] Thus the aggregate demand is decreasing in k_{-1}. As a result, R_0 is lower in equilibrium. This is illustrated in

[2]With our calibrated parameters and $G_1 = 1$ and $\omega_0 = 1$, the AD curve is negatively sloped when $R_0 < 1.3$.

the left graph of Figure 1.

For an exogenous increase in the wealth distribution ω_0, the AS curve is not affected. For the AD curve, since the entrepreneurs now have more wealth, on the one hand, c_0 in (35) increases. On the other hand, the entrepreneurs also increase their consumption in period 1, c_1, which requires higher capital holding k_0 by (37). Since given R_0, c_0' is not affected by ω_0 as in (32), the AD curve shifts to the right. In equilibrium, both R_0 and output Y_0 are higher. This is illustrated in the right graph of Figure 1.

2.4.2 Equilibrium when ZLB is Binding

When ZLB binds, we have $R_0 = R_1^K = 1$. From (32) and (30), the consumptions of the households, c_0' and c_1' are constants:

$$c_1' = \frac{1-\alpha}{X_1} A_1 \left(\frac{\delta X_1}{A_1 \alpha}\right)^{\frac{\alpha}{\alpha-1}}, \tag{40}$$

$$c_0' = \frac{1}{\beta}\frac{1-\alpha}{X_1} A_1 \left(\frac{\delta X_1}{A_1 \alpha}\right)^{\frac{\alpha}{\alpha-1}}. \tag{41}$$

The other variables can be solved as

$$c_0 = \frac{1}{1+\gamma}\omega_0 \left[1-\delta+\delta\beta^{\frac{1-\alpha}{\alpha}}\left(\frac{GX_0}{X_1}\right)^{-\frac{1}{\alpha}}\right] k_{-1}, \tag{42}$$

$$c_1 = \frac{\gamma}{1+\gamma}\omega_0 \left[1-\delta+\delta\beta^{\frac{1-\alpha}{\alpha}}\left(\frac{GX_0}{X_1}\right)^{-\frac{1}{\alpha}}\right] k_{-1}, \tag{43}$$

$$k_0 = \frac{\frac{\gamma}{1+\gamma}\omega_0 \left[1-\delta+\delta\beta^{\frac{1-\alpha}{\alpha}}\left(\frac{GX_0}{X_1}\right)^{-\frac{1}{\alpha}}\right] k_{-1} + \frac{1-\alpha}{X_1}A_1\left(\frac{\delta X_1}{A_1\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{1-\delta+\delta\frac{X_1}{\alpha}}, \tag{44}$$

$$L_0 = \left(\frac{G_1}{\beta}\frac{X_0}{X_1}\right)^{-\frac{1}{\alpha}} \left(\frac{\delta X_1}{A_1\alpha}\right)^{\frac{1}{1-\alpha}} k_{-1}. \tag{45}$$

From (34), the AS curve with binding ZLB is

$$Y_0^S = \left(\frac{G_1}{\beta}\frac{X_0}{X_1}\right)^{\frac{\alpha-1}{\alpha}} \frac{\delta X_1}{\alpha G_1} k_{-1}, \tag{46}$$

in which the output Y_0 increases proportionally with k_{-1}, and decreases in X_0. When the markup X_0 is higher, the price of intermediate good produced by the entrepreneurs,

$P_0^e = \frac{1}{X_0}$ gets lower, which discourages production.

The AD curve is still

$$Y_0^D = c_0 + c_0' + \left[k_0 - (1-\delta) k_{-1}\right], \tag{47}$$

in which the expressions of c_0, c_0' and k_0 are given by (41), (43), and (44). In the AD curve, Y_0 is increasing in ω_0, and decreasing with k_{-1} with common parameter values.

By setting $R_0 = 1$ in (39) and after some calculation, we have the following equation to pin down X_0:

$$\begin{aligned}&\left[(1+\beta) c_0' - \left(1 - \frac{\alpha}{X_1}\right)\frac{\delta X_1}{\alpha}\Gamma_0\right]\frac{1}{k_{-1}} \\ =&\Gamma_1\left(1-\frac{\alpha}{X_0}\right)X_0^{\frac{\alpha-1}{\alpha}} + \left(1-\frac{\alpha}{X_1}\right)\frac{\delta X_1}{\alpha}\frac{\left(1-\delta+\Gamma_1 X_0^{\frac{\alpha-1}{\alpha}}\right)}{\frac{1-\delta+\frac{\delta X_1}{\alpha}}{\gamma}+1} + \left[1-\delta+\delta\beta^{\frac{1-\alpha}{\alpha}}\left(\frac{GX_0}{X_1}\right)^{-\frac{1}{\alpha}}\right](1-\omega_0).\end{aligned} \tag{48}$$

in which c_0' is constant and given in (41), and Γ_0 and Γ_1 are given in Proposition 1.

Proposition 2. *When $m = 1$ and the ZLB is binding, as k_{-1} increases, X_0 increases, and the output Y_0 decreases.*

Proof. From (46), the term on the right-hand side is decreasing in X_0. Since the left-hand side is constant, as k_{-1} increases, X_0 must increase to equate (46). Since the right-hand side is decreasing in ω_0, when ω_0 increases, X_0 decreases to equate (46). For the output, insert the expression of Γ_0 and Y_0 from (46) into (48), we have

$$\begin{aligned}&(1+\beta) c_0' - \left(1-\frac{\alpha}{X_1}\right)\frac{\delta X_1}{\alpha}\Gamma_0 \\ =&\left(1-\omega_0\frac{\alpha}{X_0}\right)Y_0 + \left(1-\frac{\alpha}{X_1}\right)\frac{\frac{\delta X_1}{\alpha}}{\frac{1-\delta+\frac{\delta X_1}{\alpha}}{\gamma}+1}\left[(1-\delta)k_{-1}+Y_0\right] \\ &+(1-\delta)(1-\omega_0).\end{aligned} \tag{49}$$

Since the left-hand side of (49) is constant, and X_0 increases in k_{-1}, with larger k_{-1}, Y_0 must decrease to equate (49). Thus, we have Y_0 decreases in k_{-1} when the ZLB binds. Since X_0 decreases in ω_0, as ω_0 increases, Y_0 must increase to keep (49) hold. Thus Y_0 is increasing in ω_0.

□

The surprising result in Proposition 2 that output decreases in k_{-1} is consistent with the *paradox of toil* proposition from Eggertsson and Krugman (2012) when the economy is

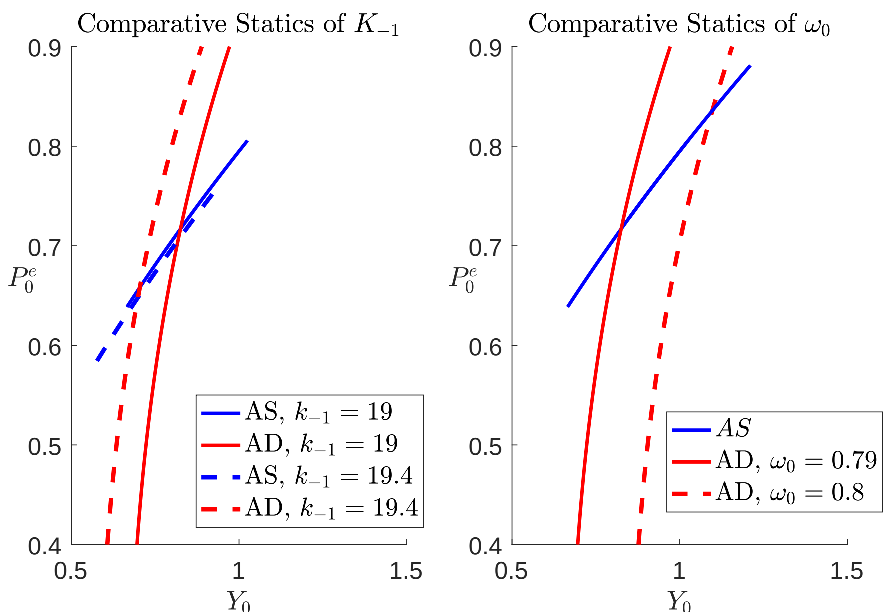


Figure 2: AS-AD Curves when ZLB Binds

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $g = 0$ and $\epsilon = 21$. We choose $k_{-1} = 19$ and $\omega_0 = 0.79$ as the baseline case.

in the liquidity trap.[3] We illustrate the intuition of Proposition 2 by plotting the AS curve in (46) and the AD curve in (47) in Figure 2. Output is plotted on the x-axis, while the price of the intermediate good, $P_0^e = \frac{1}{X_0}$ is plotted on the y-axis.

We see that the AS curve takes the normal shape and is upward sloping. When the price level is higher, the entrepreneurs will produce more. However, the AD curve here is also upward sloping.[4] When the ZLB is binding, consumptions of the households are constants as in (40) and (41). When X_0 is higher, P_0^e is smaller, and the entrepreneurs' total wealth $\omega_0 R_0^k k_{-1}$ is smaller. Their consumption and investment are thus reduced. Adding up all the three components, the aggregate demand decreases in P_0^e.

When k_{-1} increases, we see the AS curve shifts to the right and the AD curve shifts to the left, resulting in lower price level and lower output. On the one hand, while a higher initial capital increases the production capacity of the economy, the aggregate demand is constrained by the ZLB. Thus a higher potential output will simply depress the price level P_0^e further, leading to lower real output. This is the intuition of the "paradox of toil" discussed in Eggertsson and Krugman (2012). In addition, there is more to it. In

[3]A quote from Eggertsson (2010): "the paradox reveals some fundamental weaknesses in New Keynesian theory. While the standard New Keynesian model studied here is very specific in many respects, I conjecture that the same paradox occurs in a broad class of models where nominal spending determines aggregate output. If one wants to interpret the paradox from this perspective it poses a relatively broad challenge to this class of models."

[4]The AD curve is steeper than the AS curve which is consistent with Eggertsson and Krugman (2012).

our dynamic model with investment, a higher initial capital also depresses the aggregate demand directly. When the ZLB binds, the entrepreneurs' investment is constrained because too much investment would depress the return to capital, R_1^k below the ZLB. Thus the investment in period 0, $k_0 - (1-\delta)\,k_{-1}$ decreases in k_{-1}, so as the aggregate demand.

When ω_0 increases, the AS curve is not affected. For the aggregate demand, since the entrepreneurs' total wealth increases with ω_0, and their consumption and investment become larger. When ZLB binds, the consumption of the households is constant as in (41). Thus the AD curve will shift to the right, resulting in higher output and price level.

Proposition 3. *Given ω_0, there is an upper bound of initial capital $\bar{k}_{-1}$, such that an equilibrium does not exist when $k_{-1} > \bar{k}_{-1}$. $\bar{k}_{-1}$ is increasing in ω_0 and the technology growth rate G_1.*

Proof. By setting $X_0 \to \infty$ in (48), we have the expression for $\bar{k}_{-1}$:

$$\bar{k}_{-1} = \frac{\frac{1-\alpha}{X_1} A_1 \left(\frac{\delta X_1}{\alpha A_1}\right)^{\frac{\alpha}{\alpha-1}} \left[1 + \frac{1}{\beta}(1-\delta)\left(1 - \frac{X_1}{\alpha}\right) + \frac{X_1}{\alpha\beta}\right]}{(1-\delta)^2 \left(1 - \frac{X_1}{\alpha}\right) + \frac{X_1}{\alpha} - \delta(1-\delta)\left(\frac{X_1}{\alpha} - 1\right)\frac{1}{1+\gamma}\omega_0} \tag{50}$$

By Proposition 2, X_0 increases with k_{-1}. Since at $k_{-1} = \bar{k}_{-1}$, X_0 cannot be increased further, and equilibrium does not exist when $k_{-1} > \bar{k}_{-1}$. We can easily see from (50) that $\bar{k}_{-1}$ is increasing in ω_0, and increasing in A_1 and thus G_1 when A_0 is fixed.

□

We emphasize here that the existence of $\bar{k}_{-1}$ does not result from our setup of the two-sector model. When $\omega_0 = 0$, the model is isomorphic to a representative agent model with the household sector only who maximizes (1) by choosing consumption, bond, investment and labor. In that setup, when ZLB binds, their consumption and capital holding become constant from (41) and (44):

$$c_0' = \frac{1}{\beta}\frac{1-\alpha}{X_1} A_1 \left(\frac{\delta X_1}{A_1 \alpha}\right)^{\frac{\alpha}{\alpha-1}},$$

$$k_0 = \frac{\frac{1-\alpha}{X_1} A_1 \left(\frac{\delta X_1}{A_1 \alpha}\right)^{\frac{\alpha}{\alpha-1}}}{1 - \delta + \delta \frac{X_1}{\alpha}}.$$

In the feasibility condition $c_0' + k_0 = (1-\delta)\,k_{-1} + Y_0$, on the left-hand side, the aggregate demand is constant. as k_{-1} increases, Y_0 must decrease by the same amount to make the equation true. To reduce Y_0, $P_0^e = \frac{1}{X_0}$ decrease to discourage production. However, there is a limit. When $k_{-1} = \bar{k}_{-1} = \frac{c_0' + k_0}{1-\delta}$, X_0 goes to infinity, Y_0 drops to zero, and there is no equilibrium when $k_{-1} > \bar{k}_{-1}$. As A_1 (or G_1) is higher, with higher wealth, the total demand is higher , and $\bar{k}_{-1}$ increases.

When ω_0 is positive, the demand of the entrepreneurs is increasing in k_{-1}. On the one hand, their consumption c_0 is increasing in k_{-1} as in (42). On the other hand, their capital holding k_0 also increases in k_{-1} as in (44), since the entrepreneurs have more wealth and choose to invest more into the future. However, c_0 and k_0 increases by smaller amount compared to the non-depreciated part of the initial capital stock, $(1-\delta)\,k_{-1}$, and the aggregate demand decreases in k_{-1} and at certain value of k_{-1}, output still drops to zero.

2.5 Model with $m < 1$

Now we consider the case with $m < 1$ in the collateral constraint (6), which is tighter than the natural borrowing limit.

Proposition 4. *When $m < 1$, given ω_0, there is a threshold value of initial capital, k_{-1}^{CC}, such that the collateral constraint is binding if and only if $k_{-1} < k_{-1}^{CC}$. k_{-1}^{CC} decreases with ω_0, and its expression is given in Appendix B. When $k_{-1} \geq k_{-1}^{CC}$, the equilibrium is the same as the one with natural borrowing limit as discussed in Propositions 1 to 3. In particular, given ω_0,*

(i) when $k_{-1} \in \left[k_{-1}^{CC}, \hat{k}_{-1}\right]$, R_0 decreases with k_{-1} and increases with ω_0;

(ii) when $k_{-1} \in \left[\hat{k}_{-1}, \bar{k}_{-1}\right]$, ZLB binds, and X_0 increases with k_{-1}, while Y_0 decreases with k_{-1};

(iii) when $k_{-1} > \bar{k}_{-1}$, there is no equilibrium.

Proof. *The proof is given in Appendix B, in which we derive the expression of $k_{-1}^{CC}(\omega_0)$. It is natural that the equilibrium is the same as the one with natural borrowing limit when $k_{-1} \geq k_{-1}^{CC}$. If the collateral constraint is not binding, it will not affect the equilibrium outcomes.*

□

In the following, we analyze the situation when $k_{-1} < k_{-1}^{CC}$ and the collateral constraint is binding. In this case, we have $R_1^K > R_0$ which encourages the entrepreneurs to borrow to the limit. By (6) and (7), we have

$$b_0 = -mR_1^K k_0, \tag{51}$$

$$\omega_1 = 1 - m. \tag{52}$$

Next, we express all the other variables as functions of R_1^K and R_0. In the last period, from (27) and others, we have

$$L_1 = \frac{\frac{1-\alpha}{X_1}}{\frac{m\alpha(1-\delta)}{X_1\left(R_1^K-1+\delta\right)} + 1 - \frac{\alpha}{X_1}(1-m)},$$

$$k_0 = \frac{\frac{1-\alpha}{X_1}\left(\frac{X_1}{\alpha}\frac{R_1^K-1+\delta}{A_1}\right)^{\frac{1}{\alpha-1}}}{\frac{m\alpha(1-\delta)}{X_1\left(R_1^K-1+\delta\right)} + 1 - \frac{\alpha}{X_1}(1-m)}, \tag{53}$$

$$c_1 = \frac{(1-m)\frac{1-\alpha}{X_1}\left(\frac{X_1}{\alpha}\frac{R_1^K-1+\delta}{A_1}\right)^{\frac{1}{\alpha-1}} R_1^K}{\frac{m\alpha(1-\delta)}{X_1\left(R_1^K-1+\delta\right)} + 1 - \frac{\alpha}{X_1}(1-m)}, \tag{54}$$

$$c_1' = \frac{1-\alpha}{X_1} A_1 \left(\frac{X_1}{\alpha}\frac{R_1^K-1+\delta}{A_1}\right)^{\frac{\alpha}{\alpha-1}},$$

$$Y_1 = \frac{\frac{1-\alpha}{\alpha}\left(\frac{X_1}{\alpha A_1}\right)^{\frac{1}{\alpha-1}}\left(R_1^K-1+\delta\right)^{\frac{\alpha}{\alpha-1}}}{\frac{m\alpha(1-\delta)}{X_1\left(R_1^K-1+\delta\right)} + 1 - \frac{\alpha}{X_1}(1-m)}.$$

In the first period, we have

$$c_0' = \frac{1}{\beta R_0}\frac{1-\alpha}{X_1} A_1 \left(\frac{X_1}{\alpha}\frac{R_1^K-1+\delta}{A_1}\right)^{\frac{\alpha}{\alpha-1}}, \tag{55}$$

$$L_0 = \left(\beta R_0 \frac{X_1}{G X_0}\right)^{\frac{1}{\alpha}} \left(\frac{X_1}{\alpha}\frac{R_1^K-1+\delta}{A_1}\right)^{\frac{1}{1-\alpha}} k_{-1},$$

$$R_0^K = 1-\delta + (\beta R_0)^{\frac{1-\alpha}{\alpha}} \left(\frac{X_1}{G X_0}\right)^{\frac{1}{\alpha}} \left(R_1^K - 1 + \delta\right), \tag{56}$$

$$Y_0 = \frac{X_1}{\alpha G}\left(R_1^K - 1 + \delta\right)\left(\beta R_0 \frac{X_1}{G X_0}\right)^{\frac{1-\alpha}{\alpha}} k_{-1}. \tag{57}$$

From (28),

$$c_0 = \frac{\omega_0}{1+\gamma}\left[1-\delta + (\beta R_0)^{\frac{1-\alpha}{\alpha}} \left(\frac{X_1}{G X_0}\right)^{\frac{1}{\alpha}} \left(R_1^K - 1 + \delta\right)\right] k_{-1}. \tag{58}$$

Given $R_0 > 1$, we have two unknowns: R_0 and R_1^k, and the other variables can be expressed as functions of these two. We use the following two equations to derive and R_0 and R_1^k.

The first one is derived by the feasibility condition (17) in period 0:

$$k_0 = \left[(1-\delta)\left(1 - \frac{\omega_0}{1+\gamma}\right) + \left(\frac{X_0}{\alpha} - \frac{\omega_0}{1+\gamma}\right)\left(\frac{X_1}{GX_0}\right)^{\frac{1}{\alpha}} \left[\frac{1}{\beta R_0}\left(R_1^K - 1 + \delta\right)^{\frac{\alpha}{\alpha-1}}\right]^{\frac{\alpha-1}{\alpha}} \right] k_{-1} - \frac{1-\alpha}{X_1} A_1^{\frac{1}{1-\alpha}} \left(\frac{X_1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \left[\frac{1}{\beta R_0}\left(R_1^K - 1 + \delta\right)^{\frac{\alpha}{\alpha-1}}\right]. \tag{59}$$

in which k_0 is a function of R_1^K given by (53). Denote (59) as *Condition 1*. If we increase R_1^K in (59), R_0 decreases.

The second equation is derived by the entrepreneurs' consumption and investment choices. Since the collateral constraint is binding, $b_0 = -\frac{mR_1^K}{R_0} k_0$. From (28), the entrepreneurs consume $\frac{1}{1+\gamma}$ fraction of their total wealth, and the remaining part are spent on capital:

$$\left(1 - \frac{mR_1^K}{R_0}\right) k_0 = \frac{\gamma}{1+\gamma} R_0^k \omega_0 k_{-1}, \tag{60}$$

in which $1 - \frac{mR_1^K}{R_0}$ is the down payment ratio of capital, and k_0 and R_0^k are functions of R_1^K and R_0 given in (53) and (56). Denote (60) as *Condition 2*. The relationship between R_1^K and R_0 in (60) is positive.

Lemma 1. *When the collateral constraint is binding and $R_0 > 1$, R_1^k decreases in k_{-1}.*

Proof. As k_{-1} increases, both Condition 1 and Condition 2 shift to the left, resulting in a lower R_1^k. See Figure 3 as an example.

□

The result of Lemma 1 is useful in determining the different regions regarding the collateral constraint and the ZLB. Since the collateral constraint is binding if and only if $R_1^k > R_0$, and R_0 is bounded below by one, as k_{-1} increases, either the ZLB binds, or k_{-1} gets larger than k_{-1}^{CC}, $R_1^k = R_0$ and the collateral constraint is not binding.

Notice that as k_1 increases, R_0 might decrease or increase, as indicated in Figure 3. On the one hand, with larger k_{-1}, the households have more wealth in the first period and would like to save more into the future, which depresses R_0. On the other hand, higher k_{-1} generates higher initial wealth for the entrepreneurs as well. As in (60), their optimal choice is to spend $\frac{\gamma}{1+\gamma}$ of their wealth on buying new capital and borrowing to the limit, which tends to increase R_0. Thus whether R_0 increases or decreases in k_{-1} becomes a quantitative question. In Figure 4, we given an example with binding collateral constraint in which R_0 increases in k_{-1} in some region.

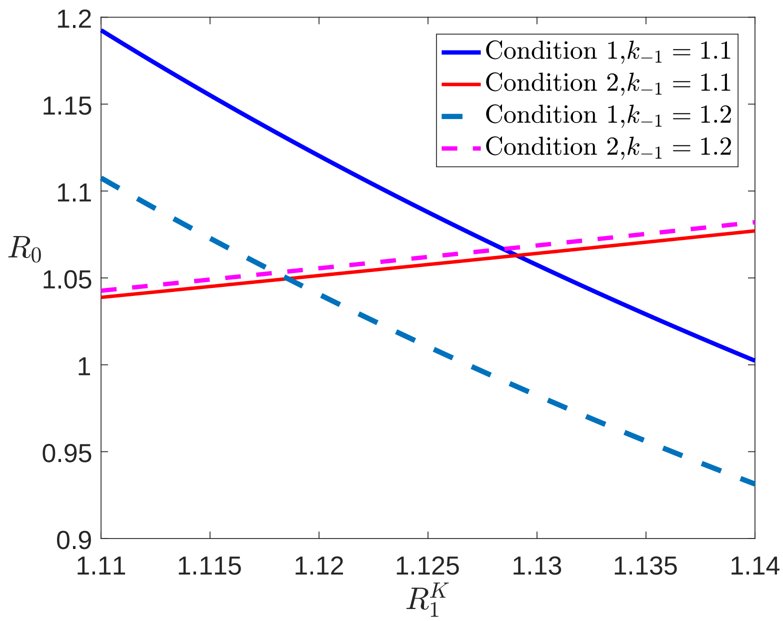


Figure 3: Comparative Statics when Collateral Constraint Binds and $R_0 > 1$

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $g = 0$ and $\epsilon = 21$. We choose $\omega_0 = 0.05$.

Proposition 5. *Given ω_0, there is an upper bound of initial capital $\bar{k}_{-1}$, such that an equilibrium does not exist when $k_{-1} > \bar{k}_{-1}$. $\bar{k}_{-1}$ is increasing in ω_0 and the technology growth rate G_1.*

Proof. *We find that there is a threshold value for the initial wealth distribution, ω_0^*:*

$$\omega_0^* = \frac{1}{\frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma}\frac{1}{(1-m)}\left[1 + \frac{1}{\beta}\left(m - \delta + \frac{\delta X_1}{\alpha}\right)\right]},$$

such that when $\omega_0 > \omega_0^*$, the threshold for binding collateral constraint, k_{-1}^{CC} derived in Proposition (4) is smaller than the expression of $\bar{k}_{-1}$ derived in Proposition (3) with natural borrowing limit. Since the collateral constraint is not binding when $k_{-1} > k_{-1}^{CC}$, the equilibrium outcomes of this economy are identical as those with natural borrowing limit, including the expression of the upper bound of capital, $\bar{k}_{-1}$. We have shown in Proposition (3) that in this region, $\bar{k}_{-1}$ increases with ω_0 and G.

When $\omega_0 < \omega_0^*$, k_{-1}^{CC} is larger than the expression of $\bar{k}_{-1}$ derived in Proposition (3). By Lemma 1, the ZLB is binding when k_{-1} is high enough. In this case, by setting $R_0 = 1$ and $X_0 \to \infty$ in (59) and (60), we have 2 unknowns: $\bar{k}_{-1}$ and R_1^K, which can be solved by the following two equations:

$$(1-\delta)\left(1 - \frac{\omega_0}{1+\gamma}\right)\bar{k}_{-1} = k_0 + \frac{1-\alpha}{\beta X_1} A_1^{\frac{1}{1-\alpha}} \left(\frac{X_1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \left(R_1^K - 1 + \delta\right)^{\frac{\alpha}{\alpha-1}}, \tag{61}$$

$$\left(1 - mR_1^K\right) k_0 = \frac{\gamma}{1+\gamma}(1-\delta)\,\omega_0 \bar{k}_{-1}. \tag{62}$$

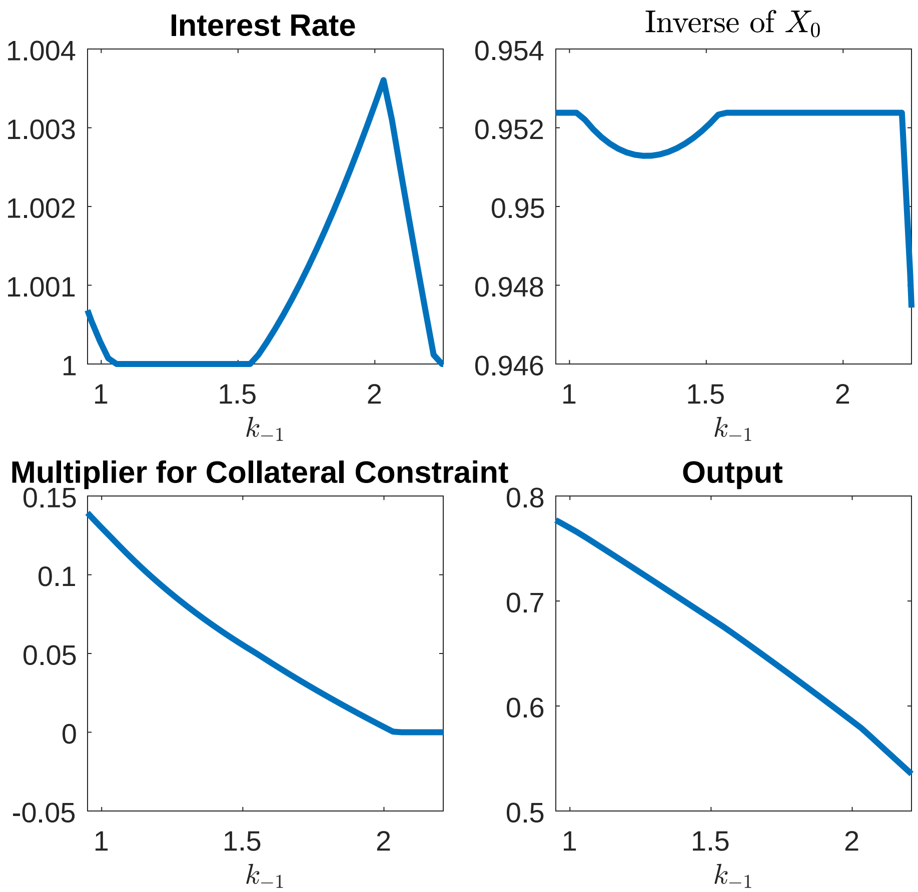


Figure 4: Policy Functions of k_{-1} with Increasing R_0

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.1$, $\delta = 0.025$, $g = 0$, $m = 0.6$ and $\epsilon = 21$.

in which k_0 is a function of R_1^K as in (53). In (62), the left-hand side is decreasing in R_1^K, and we can find a unique solution of R_1^K as a function of ω_0 and $\bar{k}_{-1}$. Insert the solved R_1^K into (61), we can pin down $\bar{k}_{-1}$. We also see that R_1^K is decreasing in ω_0 from (62). Using Implicit Function Theorem in (61), we can show that $\bar{k}_{-1}$ increases with ω_0. Using the Implicit Function Theorem in (61), we can also show that $\bar{k}_{-1}$ increases with A_1 (and thus G_1 if we fix A_0).

□

In Figure 5, we plot $\bar{k}_{-1}$ as a function of ω_0 under different m. The collateral constraint binds when ω_0 is small, and we see $\bar{k}_{-1}$ is smaller when m is smaller. The reason for the existence of $\bar{k}_{-1}$ is insufficient demand. When the collateral constraint binds with smaller m, the entrepreneurs' consumption and investment are further constrained. Thus from the feasibility condition (17), $\bar{k}_{-1}$ is smaller. For large ω_0, the collateral constraint is not binding, and $\bar{k}_{-1}$ is the same when the values of m are different.

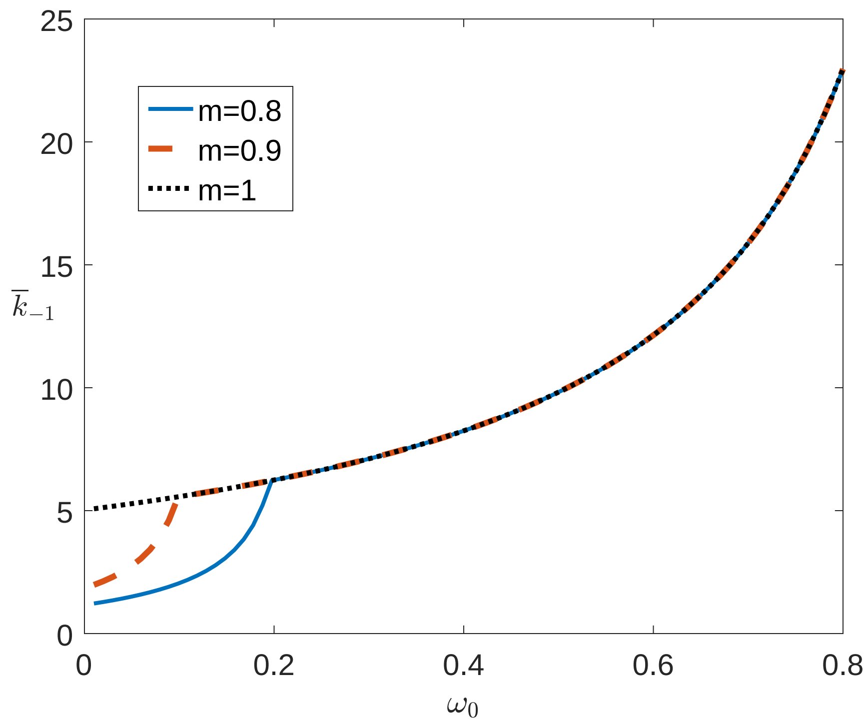


Figure 5: Upper Bound of Capital $\bar{k}_{-1}$

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $g = 0$ and $\epsilon = 21$.

The regions for the different cases are plotted in Figure 6 with $m = 0.9$. We see that ZLB binds when k_{-1} is high, and does not bind when k_{-1} is low. The Collateral constraint binds when the entrepreneurs' wealth share ω_0 is low, and when k_{-1} is low.

In Figure 7, we plot the several variables as functions of k_{-1} fixing ω_0. When $\omega_0 = 0.19$, the collateral constraint always binds, while when $\omega_0 = 0.38$, the collateral constraint never binds. The shapes of the policy functions look similar under these two values of ω_0. The interest rate R_0 decreases in k_{-1}, and when the ZLB binds, markup X_0 increases from its steady state value $\frac{\epsilon}{\epsilon-1}$. At their respective upper bound of capital, $\bar{k}_{-1}$, X_0 goes to infinity, and output drops to zero.[5]

In Figure 8, we plot the several variables as functions of ω_0 fixing k_{-1}. We see that the

[5]One observation in Figure 7 is that output is decreasing in k_{t-1} in some region even when the ZLB is not binding, which is counter-intuitive. Since $Y_0 = A_0 k_{-1}^{\alpha} L_0^{1-\alpha}$, as k_{-1} increases, L_0 must decrease faster to reduce Y_0. In our setup, since the Frisch elasticity of labor supply is 0, L_0 is determined by the wealth effect, and in this case, L_0 decreases in c_0'. When the households have lower wealth (i.e., smaller c_0'), they supply more labor to smooth consumption. As k_{-1} increases, interest rate R_0 drops, which increases c_0' and decreases L_0 by (32) and (33). If L_0 decreases very fast in k_{-1}, Y_0 may also decreases in k_{-1} as a result. Whether this result arises or not depending on the parameter choices, and is not robust. However, as ZLB binds, Y_0 drops in k_{-1} as proved in Proposition 2 is quite robust, and is consistent with the "paradox of toil" discussed in the literature.

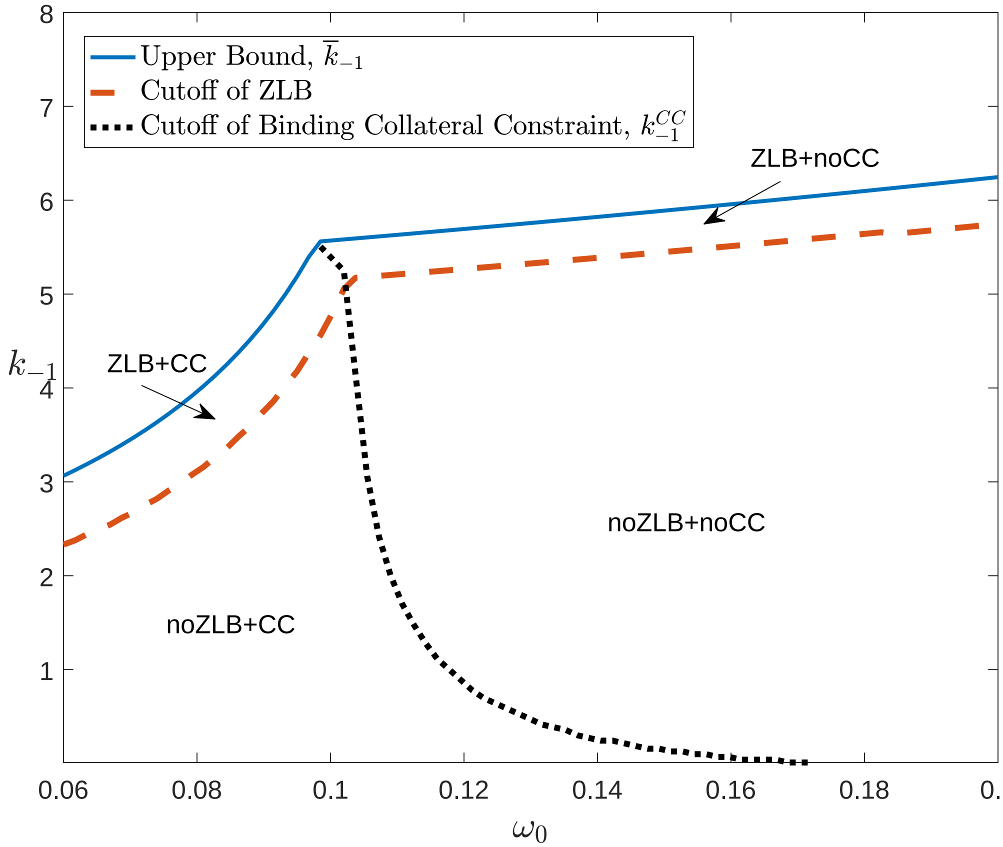


Figure 6: Regions for ZLB and Binding Collateral Constraint

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $g = 0$, $m = 0.9$ *and* $\epsilon = 21$.

interest rate and output are increasing in ω_0, and the multiplier of the collateral constraint is decreasing in ω_0.

3 Model with Irreversible Investment

3.1 Setup

Now we add one more component in the two period model that investment is irreversible. In addition, we assume that in each period, there is a market for old units of capital among the entrepreneurs. Now the entrepreneurs' budget constraint becomes

$$c_t + \frac{b_t}{R_t} + k_t - (1-\delta)\,\hat{k}_t + q_t\hat{k}_t \leq b_{t-1} + q_t k_{t-1} + \frac{1}{X_t}Y_t^e - w_t L_t,$$

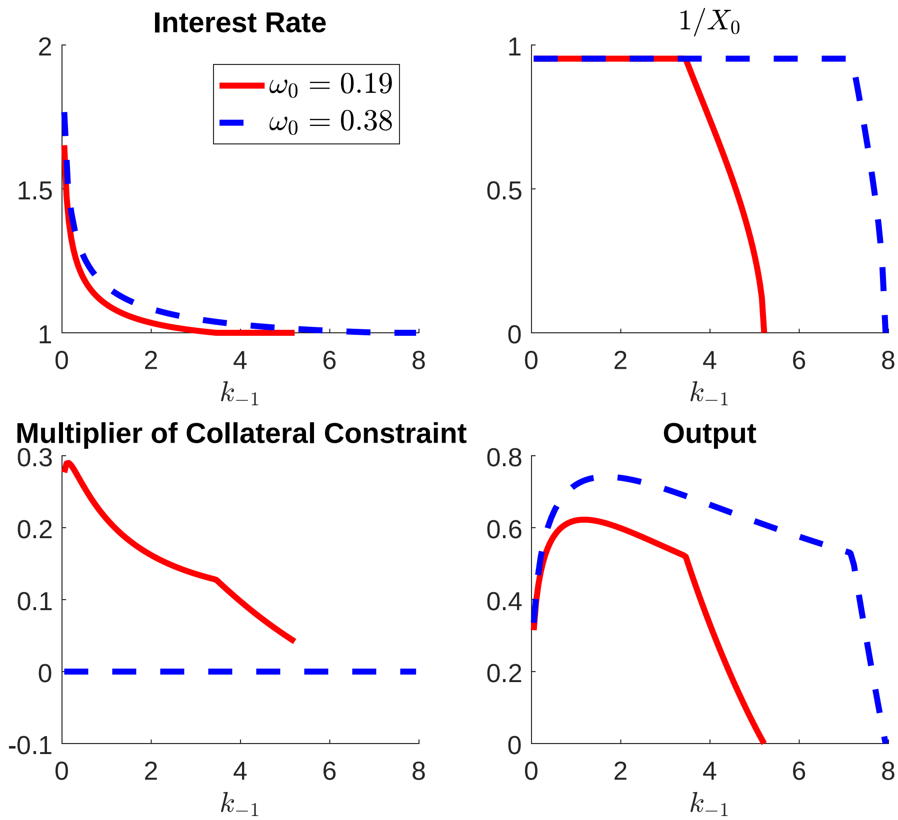


Figure 7: Policy Functions of k_{-1}

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $g = 0$, $m = 0.8$ and $\epsilon = 21$.

in which q_t is the price of used capital, and the investment is restricted to be nonnegative:

$$I_t = k_t - (1-\delta)\,\hat{k}_t \geq 0. \tag{63}$$

If we define the return to capital as

$$R_t^k = q_t + \frac{\alpha}{X_t} A_t \left(\frac{k_{t-1}}{L_t}\right)^{\alpha-1},$$

and use the definition of ω_t in (7), we can rewrite the entrepreneurs' budget as

$$c_t + \frac{b_t}{R_t} + k_t - (1-\delta)\,\hat{k}_t + q_t\hat{k}_t \leq \omega_t R_t^k k_{t-1}. \tag{64}$$

In equilibrium, we have one more market clearing condition for old units of capital:

$$\hat{k}_t = k_{t-1}.$$

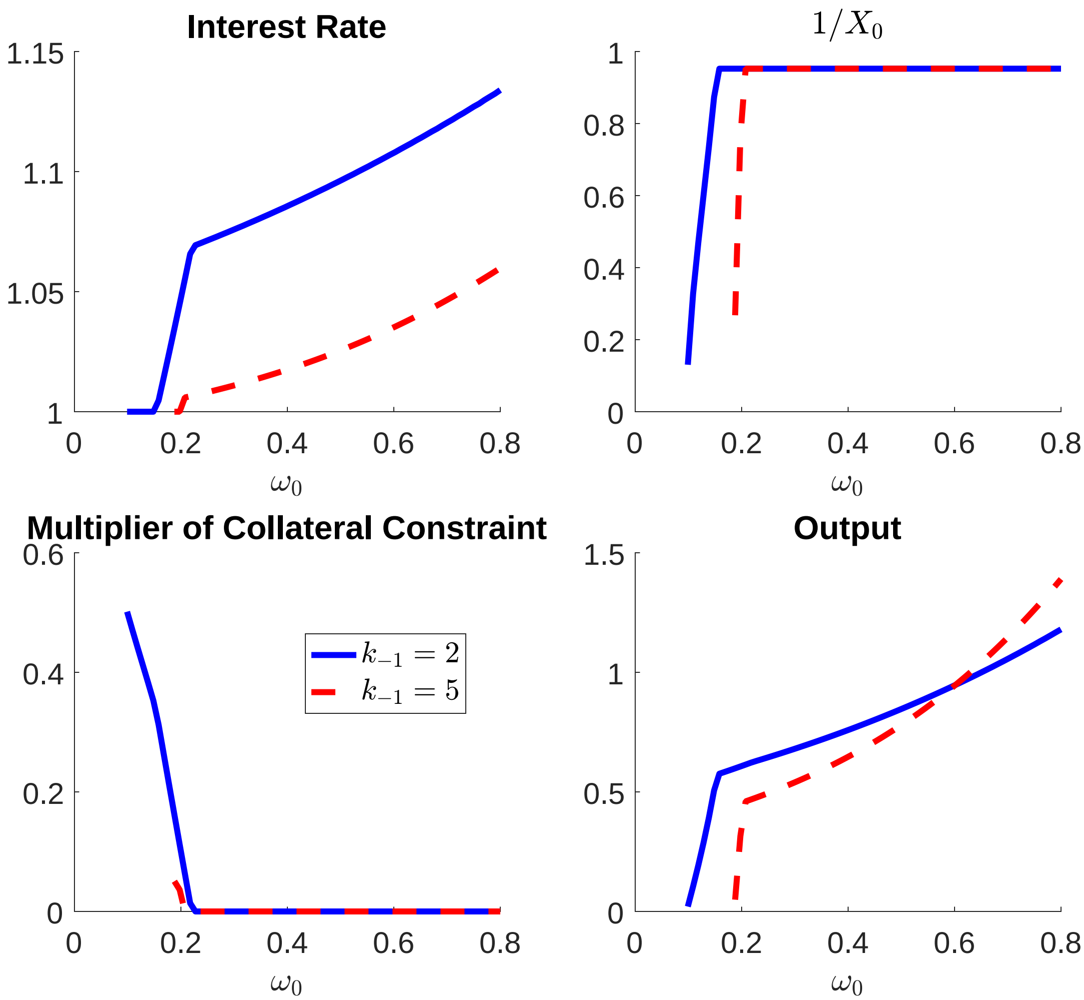


Figure 8: Policy Functions of k_{-1}

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $g = 0$, $m = 0.8$ *and* $\epsilon = 21$.

In addition, the entrepreneurs are still subject to the collateral constraint

$$b_t + mR^K_{t+1}k_t \geq 0. \tag{65}$$

Denote the Lagrangian multiplier for (63) as $\frac{\theta_t}{c_t}$, and the multiplier for (65) as $\frac{\mu_t}{c_t}$. In period 0, the FOC with respect to k_0 becomes

$$-\frac{1 - \theta_0 - mR^K_1\mu_0}{c_0} + \gamma\frac{R^k_1}{c_1} = 0, \tag{66}$$

$$\theta_0 \left[k_0 - (1 - \delta)\,\hat{k}_0\right] = 0. \tag{67}$$

The price of used capital, q_t, can be derived from the FOC with respect to $\hat{k}_t$:

$$q_t = (1-\theta_t)(1-\delta). \tag{68}$$

When the irreversibiliby constraint is slack, we have $q_t = 1-\delta$. Otherwise, q_t drops to clear the market.

3.2 Last Period

In the last period, there is no return of investment, and due to the irreversibility constraint, $k_1 = (1-\delta) k_0$, and the capital price drops to $q_1 = 0$. We still set the markup $X_1 = \frac{\epsilon}{\epsilon-1}$. By the market clearing condition (9), we have $c_1 + c_1' = Y_1$. And using (21), we have

$$R_1^k = \frac{\alpha}{X_1} A_1 \left(\frac{k_0}{L_1}\right)^{\alpha-1},$$

$$L_1 = \frac{\frac{1-\alpha}{X_1}}{1-\frac{\alpha}{X_1}\omega_1},$$

$$Y_1 = A_1 \left[\frac{\frac{1-\alpha}{X_1}}{1-\frac{\alpha}{X_1}\omega_1}\right]^{1-\alpha} k_0^\alpha,$$

and

$$c_1 = \omega_1 \frac{\alpha A_1}{X_1} \left[\frac{\frac{1-\alpha}{X_1}}{1-\frac{\alpha}{X_1}\omega_1}\right]^{1-\alpha} k_0^\alpha,$$

$$c_1' = A_1 \left(\frac{1-\alpha}{X_1}\right)^{1-\alpha} \left[\left(1-\omega_1\frac{\alpha}{X_1}\right) k_0\right]^\alpha.$$

3.3 Equilibrium with $m = 1$

Similar to Subsection 2.4, we focus on the case with natural borrowing limit first. The collateral constraint (65) does not bind in equilibrium, and $\mu_0 = 0$.

Denote $\lambda = \frac{1-\delta}{q_0}$. By (68), $\lambda \geq 1$. If the irreversibility condition (63) is not binding, $q_0 = 1-\delta$ from (68) and $\lambda = 1$. Otherwise, $\lambda > 1$. The gross return for holding capital k_0 is λR_1^k. By non-arbitrage condition, we have

$$R_0 = \lambda R_1^k.$$

Given k_0, R_0 and λ, in the last period, we have

$$\begin{aligned}
L_1 &= \left(\frac{X_1}{\alpha}\frac{R_0}{\lambda A_1}\right)^{\frac{1}{1-\alpha}} k_0, \\
Y_1 &= \frac{X_1}{\alpha}\frac{R_0}{\lambda}k_0, \\
c_1' &= \frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha}\frac{R_0}{\lambda A_1}\right)^{\frac{\alpha}{\alpha-1}}, \\
c_1 &= \frac{X_1}{\alpha}\frac{R_0}{\lambda}k_0 - \frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha}\frac{R_0}{\lambda A_1}\right)^{\frac{\alpha}{\alpha-1}}.
\end{aligned} \tag{69}$$

In the first period, we have

$$c_0 = \frac{1}{\gamma\lambda}\frac{X_1}{\alpha}k_0 - \frac{1}{\gamma R_0}\frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha}\frac{R_0}{\lambda A_1}\right)^{\frac{\alpha}{\alpha-1}}, \tag{70}$$

$$\begin{aligned}
c_0' &= \frac{1}{\beta R_0}\frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha}\frac{R_0}{\lambda A_1}\right)^{\frac{\alpha}{\alpha-1}}, \\
L_0 &= \left(\frac{G}{\beta R_0}\frac{X_0}{X_1}\right)^{-\frac{1}{\alpha}}\left(\frac{X_1}{\alpha}\frac{R_0}{\lambda A_1}\right)^{\frac{1}{1-\alpha}} k_{-1}, \\
R_0^k &= q_0 + (\beta R_0)^{\frac{1-\alpha}{\alpha}}\left(\frac{GX_0}{X_1}\right)^{-\frac{1}{\alpha}}\frac{R_0}{\lambda},
\end{aligned}$$

$$Y_0 = \frac{X_1}{\alpha G}\left(\frac{G}{\beta R_0}\frac{X_0}{X_1}\right)^{\frac{\alpha-1}{\alpha}}\frac{R_0}{\lambda}k_{-1}. \tag{71}$$

Lemma 2. *When $m = 1$, R_0 decreases in k_{-1} and increases in ω_0. The ratio $\frac{k_0}{k_{-1}}$ also decreases in k_{-1}.*

Proof. *See Appendix C.*

□

By Lemma 2, we see that given ω_0, both the ZLB and the irreversibility constraint tend to bind when k_{-1} is high. The question is, as k_{-1} increases, which constraint binds first. The following proposition summarizes the results in this subsection.

Proposition 6. *With the the investment irreversibility constraint and $m = 1$, an equilibrium* *always exists. Denote ω_0^* as*

$$\omega_0^* = \frac{(1+\gamma)\frac{X_1}{\alpha}\left[1-(1-\delta)\left(\frac{G}{\beta}\right)^{\frac{1}{\alpha}}\right]}{\left[1+\beta(1-\delta)\left(\frac{G}{\beta}\right)^{\frac{1}{\alpha}}\right]\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)}. \tag{72}$$

Given ω_0:

(1) if $\omega_0 \geq \omega_0^*$, the ZLB never binds in equilibrium. We give a threshold value of initial capital, $k_{-1}^*(\omega_0)$ in equation (86) in Appendix D. When $k_{-1} < k_{-1}^*$, the irreversibility constraint does not bind and $q_0 = 1-\delta$. When $k_{-1} > k_{-1}^*$, capital price q_0 decreases in k_{-1}, but R_0 is fixed.

(2) if $\omega_0 < \omega_0^*$, given ω_0, we identify a threshold value for binding ZLB, $\hat{k}_{-1}(\omega_0)$ in (87), and a threshold value for binding irreversibility $k_{-1}^{**}(\omega_0)$ in (88) in Appendix D. When $k_{-1} < \hat{k}_{-1}$, neither of the ZLB and irreversibility constraint binds. When $k_{-1} \in \left[\hat{k}_{-1}, k_{-1}^{**}\right]$, the ZLB binds, and X_0 increases with k_{-1}. The capital price $q_0 = 1-\delta$ in this region. When $k_{-1} > k_{-1}^{**}$, both the ZLB and the irreversibility constraint bind. X_0 remains constant with k_{-1}, and q_0 decreases in k_{-1}.

Proof. See the proof in Appendix D.

□

It is a bit surprising to see that R_0 remains constant when the irreversibility constraint binds. The intuition can be seen by comparing output Y_0 and Y_1 in equations (71) and (69). As k_{-1} increases, k_0 increases proportionally, and the ratio of outputs in both periods stay constant. Since R_0 measures the relative scarcity of resources in both periods, it remains constant.

See an example of the policy functions in Figure 9. The ZLB starts to bind at $k_{-1} = 0.08$, and the irreversibility constraint starts to bind at $k_{-1} = 0.103$.

3.4 Equilibrium with $m < 1$

Proposition 7. *When $m < 1$, with the the investment irreversibility constraint, an equilibrium always exists. In particular, given ω_0, as k_{-1} increases, the ratio $\frac{k_0}{k_{-1}}$ decreases. When the irreversibility constraint is binding and the collateral constraint is not binding, the equilibrium properties of the economy is the same as the economy with natural borrowing limit as in Proposition 6; when both the collateral constraint and irreversibility constraint bind, as k_{-1} increases, q_0 decreases, but μ_0, R_0 and X_0 remain constant.*

Proof. See the proof in Appendix E.

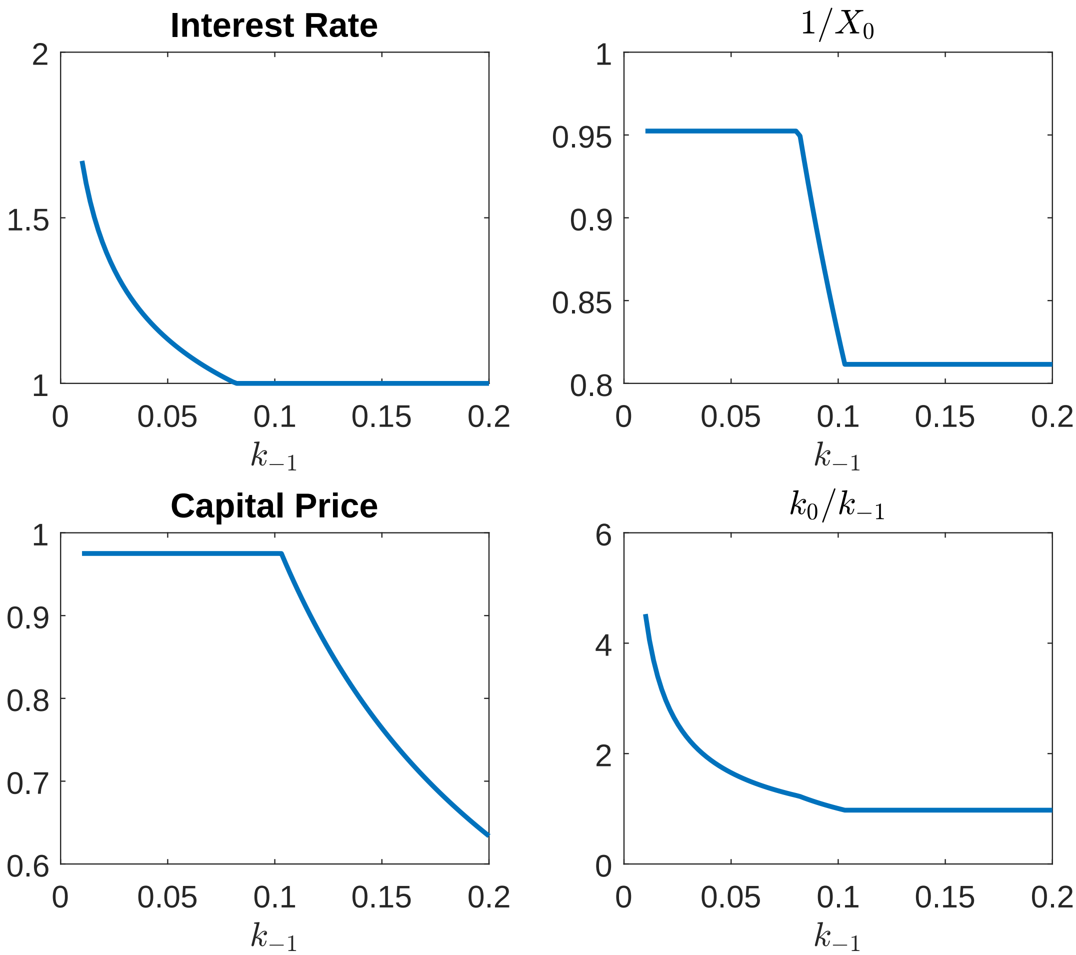


Figure 9: Policy Functions of k_{-1} with $m = 1$

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $G = 0.9$, $m = 1$ and $\epsilon = 21$. ω_0 is set to 0.11.

$\square$

See an example of the policy functions in Figure 10, in which the ZLB starts to bind at $k_{-1} = 0.105$, and the irreversibility constraint starts to bind at $k_{-1} = 0.107$.

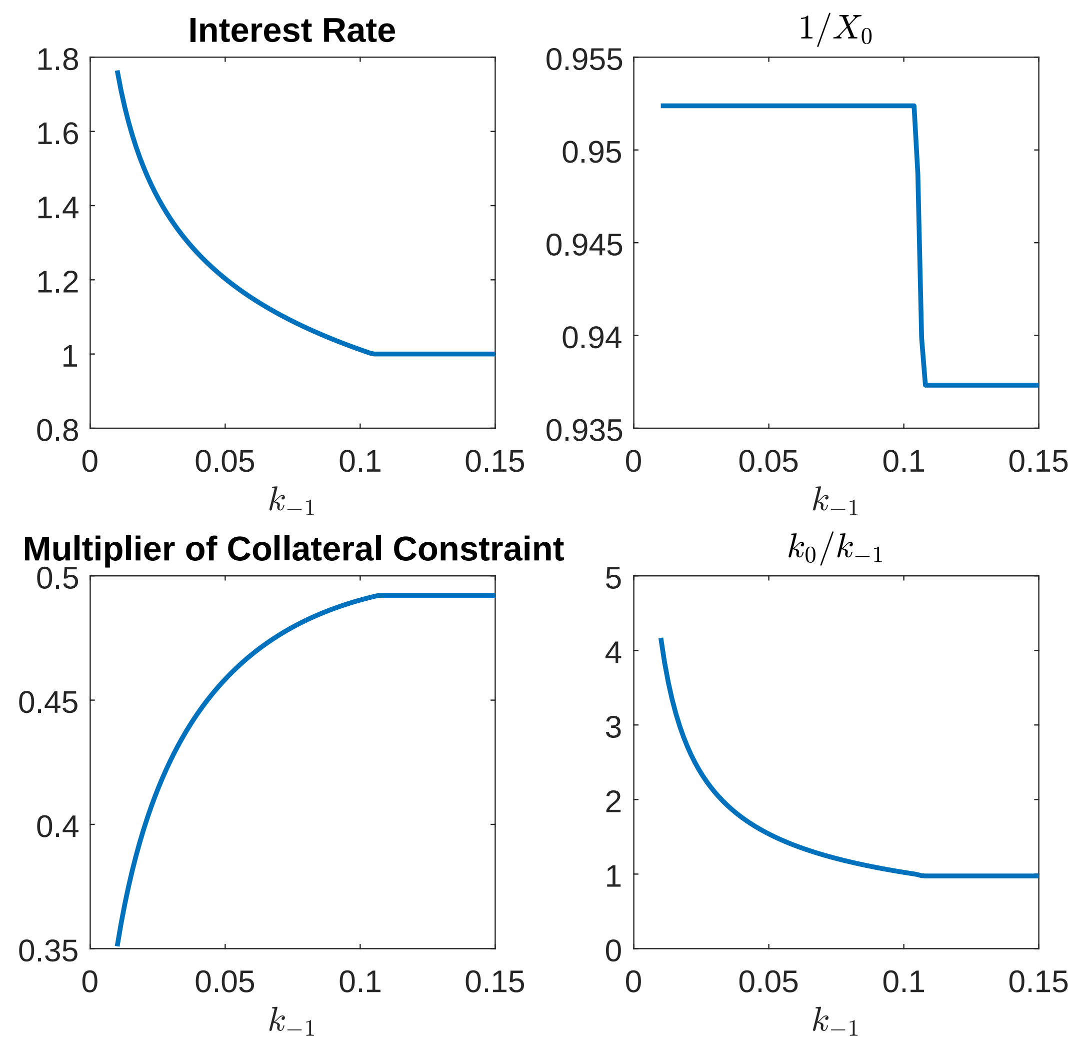


Figure 10: Policy Functions of k_{-1} with Collateral Constraint

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $g = 0$, $m = 0.8$ and $\epsilon = 21$. ω_0 is set to 0.11.

4 Infinite-Horizon Model

In this section, we extend the two period model with investment irreversibility in Section 3 into an infinite-horizon model with Markov shocks to the growth rate of the TFP.

The households maximize a lifetime utility function given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln c_t' - \frac{1}{\eta} (L_t')^\eta \right\}, \tag{73}$$

The budget constraint of the households is

$$c_t' + \frac{b_t'}{R_t} \leq b_{t-1}' + w_t L_t' + \int_0^1 \Xi_t(z)dz. \tag{74}$$

The entrepreneurs use a constant-returns-to-scale technology that uses capital and labor as inputs. They produce consumption good Y_t according to

$$Y_t = K_{t-1}^{\alpha} \left(A_t L_t\right)^{1-\alpha}, \tag{75}$$

where A_t is the aggregate productivity which depends on the aggregate state s_t. We assume that there are two sources of uncertainty in the labor productivity A_t, which follows a stochastic process with both level shocks, a_t, and growth shocks G_t

$$A\left(s^t\right) = a\left(s_t\right) G\left(s^t\right) \tag{76}$$

and $G\left(s^t\right)$ evolves according to the process

$$\frac{G\left(s^{t+1}\right)}{G\left(s^t\right)} = 1 + g\left(s_{t+1}\right), \tag{77}$$

where $a\left(s_t\right)$ are level shocks and $g\left(s_{t+1}\right)$ are growth shocks.

The entrepreneurs maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \frac{\left(c_t\right)^{1-\sigma_1} - 1}{1-\sigma_1} \tag{78}$$

subject to the budget constraint

$$c_t + \frac{b_t}{R_t} + k_t - (1-\delta)\hat{k}_t + q_t \hat{k}_t \leq b_{t-1} + q_t k_{t-1} + \frac{1}{X_t} Y_t^e - w_t L_t. \tag{79}$$

The remaining part of the infinite-horizon model is the same as the two-period model in Section 3. In order to solve a stationary system of the model, we normalize the variables by the growth shock G_t.

Table 1: Baseline Parameter Values

Parameters	Value	
β	0.99	discount factor of household
γ	0.98	discount factor of entrepreneur
α	0.35	land share in production
η	1	labor supply elasticity
δ	0.025	depreciation rate
ϵ	21	steady state markup is 5%
σ_1	1	CRRA parameter of entrepreneur

5 Calibration and Numerical Results

5.1 Parameters

Parameter values are given in Table 1. Most of the variables are from Cao and Nie (2017).

At present, we assume there is no shock to a_t, and set $a_t \equiv \bar{a} = 1$. The growth process g_t is from Elenev et al. (2016). Their annual growth process is

$$g_t = (1-\rho_g)\bar{g} + \rho_g g_{t-1} + \epsilon_{g,t},\ \epsilon_{g,t} \sim iid\mathcal{N}(0, \sigma_g^2)$$

with $\bar{g}^a = 0.02$, $\sigma_g^a = 2.85\%$ and $\rho_g^a = 0.22$. Converting to quarterly frequency, we have $\bar{g} = 0.005$, $\rho_g = 0.6849$ and $\sigma_g = 2.13\%$.[6] We use Tauchen's method to discretize the growth process into a 5-state Markov process.

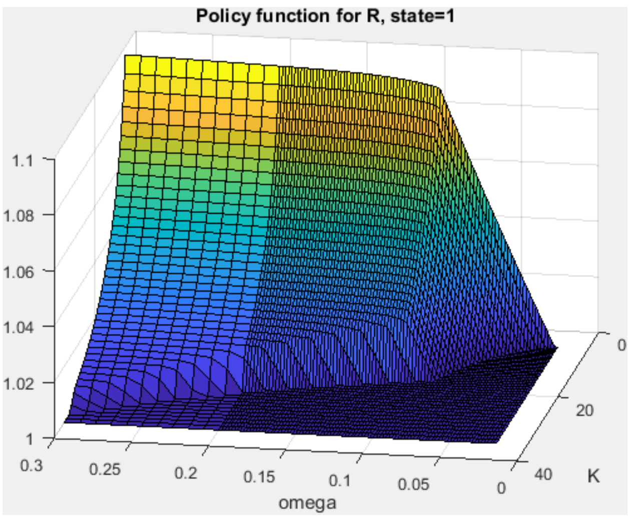
5.2 Numerical Results

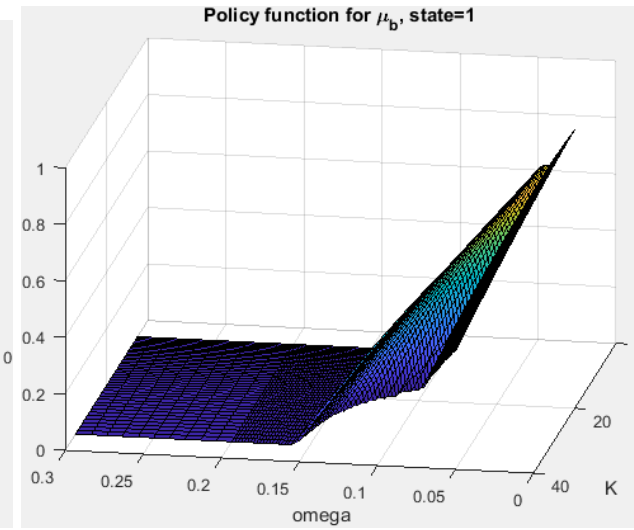
In Table 2, show the average duration of ZLB, as well as the probabilities of binding ZLB and binding collateral constraint by state under different values of m. A higher m means the entrepreneurs' collateral constraint is more slack, and thus they have access to more credit in general. The results are generated by simulations based on 24 samples and 10000 periods for each sample, with the the first 5000 periods dropped. Several observations from the table: 1. In the stationary distribution, the ZLB is mostly binding when the growth rate is low. 2. With higher m, the probability of binding collateral constraint is is lower, but the probability of binding ZLB is higher. 3. The mean duration of the ZLB is decreasing in m. The intuition is, with better access to the credit market, the entrepreneurs tend to use higher leverage, and thus the ZLB period triggered by the adverse shock and the deleveraging process tends to be longer. This is similar to the phenomenon of

[6] $\bar{g} = \bar{g}^a/4$, $\rho_g = (\rho_g^a)^{\frac{1}{4}}$,and $\sigma_g^2 = (\sigma_g^a)^2/(1+\rho_g^2+\rho_g^4+\rho_g^6)$.

volatility paradox as suggested in Brunnermeier and Sannikov (2014).

We also plot interest rate and the multiplier of the collateral constraint as functions of capital k_{t-1} and wealth distribution ω_t. We plot the policy function in state 1 where $g_t = g_{LL}$. We see that the ZLB is binding when k_{t-1} is high, or when ω_t is low. On the other hand, the collateral constraint tends to bind when ω_t is low.

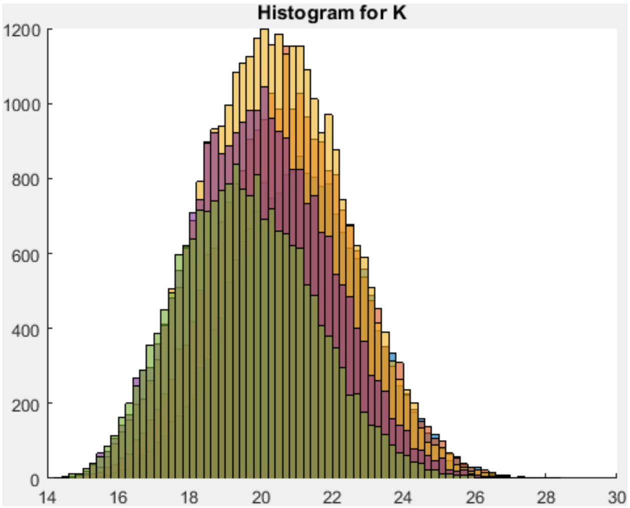


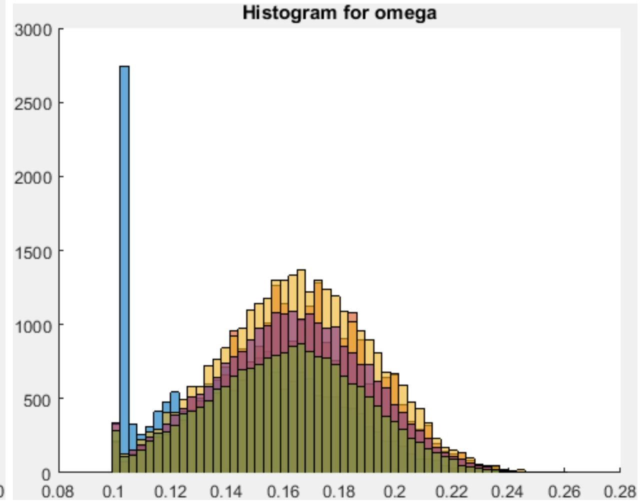


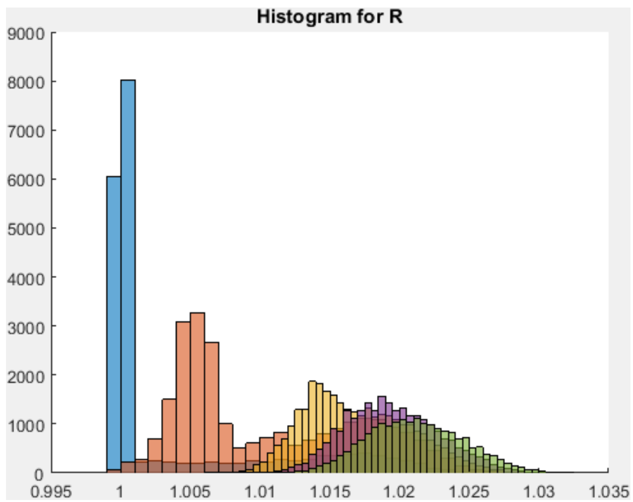
In the following graphs, we plot the ergodic distributions of capital, wealth share as well as the interest rate.

Table 2: ZLB Duration and Probability of Binding Constraints

m=0.90					
mean(ZLB duration)	1.9094				
State	$g = g_{LL}$	$g = g_L$	$g = \bar{g}$	$g = g_H$	$g = g_{HH}$
prob ZLB by state	0.69793	0.013694	0	0	0
prob cc by state	0.20874	0.020459	0.011387	0.00691	0.00776
prob ZLB & cc	0.20226	0	0	0	0
m=0.80					
mean(ZLB duration)	1.8603				
State	$g = g_{LL}$	$g = g_L$	$g = \bar{g}$	$g = g_H$	$g = g_{HH}$
prob ZLB by state	0.64511	0.003234	0	0	0
prob cc by state	0.65698	0.40617	0.27844	0.2404	0.25551
prob ZLB & cc	0.46345	0	0	0	0
m=0.70					
mean(ZLB duration)	1.8574				
State	$g = g_{LL}$	$g = g_L$	$g = \bar{g}$	$g = g_H$	$g = g_{HH}$
prob ZLB by state	0.34906	0.000105	0	0	0
prob cc by state	0.91707	0.81368	0.69628	0.68584	0.70155
prob ZLB & cc	0.31573	0	0	0	0
m=0.60					
mean(ZLB duration)	1.8732				
State	$g = g_{LL}$	$g = g_L$	$g = \bar{g}$	$g = g_H$	$g = g_{HH}$
prob ZLB by state	0.093646	1.91E-05	0	0	0
prob cc by state	0.98594	0.96441	0.92685	0.9288	0.92436
prob ZLB & cc	0.089977	0	0	0	0
m=0.50					
mean(ZLB duration)	1.6842				
State	$g = g_{LL}$	$g = g_L$	$g = \bar{g}$	$g = g_H$	$g = g_{HH}$
prob ZLB by state	0.006475	0	0	0	0
prob cc by state	0.99934	0.99629	0.9916	0.97411	0.90676
prob ZLB & cc	0.006272	0	0	0	0







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Appendix

A Proofs for Subsection 2.4

Proof for Proposition 1. The expression for $\hat{k}_{-1}$ is given by

$$\hat{k}_{-1} = \frac{\frac{1+\beta}{\beta}\frac{1-\alpha}{X_1}A_1\left(\frac{\delta X_1}{A_1\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \left(1-\frac{\alpha}{X_1}\right)\frac{\delta X_1}{\alpha}\Gamma_0}{\Gamma_1\left(1-\frac{\alpha}{X_1}\right)X_1^{\frac{\alpha-1}{\alpha}} + \left(1-\frac{\alpha}{X_1}\right)\frac{\delta X_1}{\alpha}\frac{\left(1-\delta+\Gamma_1 X_1^{\frac{\alpha-1}{\alpha}}\right)}{\frac{1-\delta+\frac{\delta X_1}{\alpha}}{\gamma}+1} + \left[1-\delta+\delta\beta^{\frac{1-\alpha}{\alpha}}G^{-\frac{1}{\alpha}}\right](1-\omega_0)}, \tag{80}$$

in which $\Gamma_0 = \frac{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{1-\alpha}{X_1}A_1\left(\frac{\delta X_1}{A_1\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{\frac{1-\delta+\frac{\delta X_1}{\alpha}}{\gamma}+1}$, and $\Gamma_1 = \left(\frac{G_1}{\beta X_1}\right)^{\frac{\alpha-1}{\alpha}}\frac{\delta X_1}{G_1\alpha}$.

If the ZLB is not binding, i.e., $R_0 > 1$, by the monetary policy rule (24), $X_0 = \frac{\epsilon}{\epsilon-1}$. A careful examination of equation (39) shows that R_0 is decreasing in k_{-1}. To see how R_0 responds to ω_0, we can rewrite (39) as a function in the form of $F(R_0, \omega_0) = 0$. It is easy to check that F is increasing in R_0 and decreasing in ω_0. Using the implicit function theorem, we have $\partial R_0/\partial\omega_0 > 0$. To derive the expression of $\hat{k}_{-1}$, we insert $R_0 = 1$ and $X_0 = X_1$ into (39), and we can get (80).

□

B Proof of Proposition 4

Since k_{-1}^{CC} is the threshold for the collateral constraint to be binding, at $k_{-1} = k_{-1}^{CC}$, we should have $R_0 = R_1^k$, $\mu_0 = 0$ and $\omega_0 = 1 - m$. Insert these expressions into equations (59) and (60), we have the following two equations:

$$k_0 = \frac{\gamma}{1+\gamma}\frac{R_0^k\omega_0}{1-m}k_{-1}^{CC}, \tag{81}$$

$$k_0 = \left[(1-\delta)\left(1-\frac{\omega_0}{1+\gamma}\right) + \left(\frac{X_0}{\alpha}-\frac{\omega_0}{1+\gamma}\right)\left(\frac{X_1}{GX_0}\right)^{\frac{1}{\alpha}}(\beta R_0)^{\frac{1-\alpha}{\alpha}}(R_0-1+\delta)\right]k_{-1}^{CC} \tag{82}$$
$$- \frac{1-\alpha}{X_1}A_1^{\frac{1}{1-\alpha}}\left(\frac{X_1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\left[\frac{1}{\beta R_0}(R_0-1+\delta)^{\frac{\alpha}{\alpha-1}}\right].$$

in which k_0 and R_0^k are functions of R_0 given in (53) and (56).

We find the ZLB binds if and only if $\omega_0 \leq \omega_0^*$:

$$\omega_0^* = \frac{\left[\frac{X_1}{\alpha} - \frac{\left(\frac{X_1}{\alpha}-1\right)(1-\delta)}{\left(1-\delta+\delta G^{-\frac{1}{\alpha}}\beta^{\frac{1-\alpha}{\alpha}}\right)}\right](1-m)\,k_0}{\frac{(1-m)k_0}{1+\gamma} + \frac{\gamma}{1+\gamma}\left(c_0' + k_0\right)}.$$

When $\omega_0 \leq \omega_0^*$, both c_0' and k_0 in (81) and (82) are constant, and we can solve for k_{-1}^{CC} as a function of ω_0 by the following equation:

$$\frac{X_1}{G}\left[\frac{\frac{(1-m)\Upsilon_1}{\frac{\gamma}{1+\gamma}\omega_0 k_{-1}^{CC}} - (1-\delta)}{\delta\beta^{\frac{1-\alpha}{\alpha}}}\right]^{-\alpha} = \frac{\alpha\left[\Upsilon_0 + \Upsilon_1 - (1-\delta)\left(1-\frac{\omega_0}{1+\gamma}\right)k_{-1}^{CC}\right]}{\frac{(1-m)\Upsilon_1}{\frac{\gamma}{1+\gamma}\omega_0} - (1-\delta)\,k_{-1}^{CC}} + \frac{\alpha\omega_0}{1+\gamma}, \tag{83}$$

in which $\Upsilon_0 = \frac{1}{\beta}\frac{1-\alpha}{X_1}A_1\left(\frac{\delta X_1}{\alpha A_1}\right)^{\frac{\alpha}{\alpha-1}}$, and $\Upsilon_1 = \frac{\frac{1-\alpha}{X_1}\left(\frac{\delta X_1}{\alpha A_1}\right)^{\frac{1}{\alpha-1}}}{\frac{m\alpha(1-\delta)}{\delta X_1}+1-\frac{\alpha}{X_1}(1-m)}$.

When $\omega_0 > \omega_0^*$, the ZLB is not binding, and we can solve for the two unknowns, k_{-1}^{CC} and R_0 by (81) and (82) with $X_0 = \frac{\epsilon}{\epsilon-1}$.

C Proof of Lemma 2

Since $c_0 = \frac{1}{1+\gamma}\omega_0 R_0^k k_{-1}$, by (70) we have

$$\frac{1}{1+\gamma}\omega_0 R_0^k k_{-1} = \frac{1}{\gamma R_0}\left[\frac{X_1}{\alpha}\frac{R_0}{\lambda}k_0 - \frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha}\frac{R_0}{\lambda A_1}\right)^{\frac{\alpha}{\alpha-1}}\right]. \tag{84}$$

By the feasibility condition, we have

$$\begin{aligned}\frac{1}{\gamma\lambda}\frac{X_1}{\alpha}k_0 - \left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{1}{R_0}\frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha}\frac{R_0}{\lambda A_1}\right)^{\frac{\alpha}{\alpha-1}} + k_0 \\ = (1-\delta)\,k_{-1} + \frac{X_1}{\alpha G}\left(\frac{G}{\beta R_0}\frac{X_0}{X_1}\right)^{\frac{\alpha-1}{\alpha}}\frac{R_0}{\lambda}k_{-1}.\end{aligned} \tag{85}$$

Assuming that both the ZLB and the irreversibility condition are not binding, we get the following equation to pin down R_0:

$$\left[\frac{1}{\gamma}-\Pi_0\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\right]\frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha A_1}\right)^{\frac{\alpha}{\alpha-1}}R_0^{\frac{1}{\alpha-1}}$$
$$=\left[(1-\delta)\left(\Pi_0-\frac{1}{1+\gamma}\omega_0\right)+\left(\Pi_0\frac{X_0}{\alpha}-\frac{1}{1+\gamma}\omega_0\right)\beta^{\frac{1-\alpha}{\alpha}}\left(\frac{GX_0}{X_1}\right)^{-\frac{1}{\alpha}}R_0^{\frac{1}{\alpha}}\right]k_{-1}.$$

in which $\Pi_0=\frac{\frac{1}{\gamma}\frac{X_1}{\alpha}}{1+\frac{1}{\gamma}\frac{X_1}{\alpha}}$.

It is easy to see that R_0 decreases in k_{-1} and increases in ω_0.

$$k_0=\frac{\left[(1-\delta)+\frac{X_0}{\alpha}\beta^{\frac{1-\alpha}{\alpha}}\left(\frac{GX_0}{X_1}\right)^{-\frac{1}{\alpha}}R_0^{\frac{1}{\alpha}}\right]k_{-1}+\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha A_1}\right)^{\frac{\alpha}{\alpha-1}}R_0^{\frac{1}{\alpha-1}}}{1+\frac{1}{\gamma}\frac{X_1}{\alpha}}.$$

We can show that the ratio $\frac{k_0}{k_{-1}}$ decreases in k_{-1}.

D Proof of Proposition 6

In a model without the ZLB, given ω_0, we need to solve for the threshold value of initial capital where the irreversibility condition starts to bind. Define the threshold value as $k^*_{-1}(\omega_0)$. To solve $k^*_{-1}(\omega_0)$, insert $k_0=(1-\delta)k^*_{-1}$ and $\lambda=1$ into (84) and (85), we get

$$k^*_{-1}(\omega_0)=\frac{\frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha A_1}\right)^{\frac{\alpha}{\alpha-1}}\left(\frac{G}{\beta}\right)^{\frac{1}{\alpha-1}}\left[\frac{(1-\delta)\left[\frac{X_1}{\alpha}+\beta\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\omega_0}{1+\gamma}\right]}{\frac{X_0}{\alpha}-\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\omega_0}{1+\gamma}}\right]^{\frac{\alpha}{\alpha-1}}}{\frac{X_1}{\alpha}(1-\delta)-\frac{1}{1+\gamma}\left[(1-\delta)+\frac{1}{\beta}\frac{(1-\delta)\left[\frac{X_1}{\alpha}+\beta\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\omega_0}{1+\gamma}\right]}{\frac{X_0}{\alpha}-\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\omega_0}{1+\gamma}}\right]\omega_0}, \tag{86}$$

and the interest rate at $k^*_{-1}(\omega_0)$ is

$$R_0^*(\omega_0)=\frac{G}{\beta}\left[\frac{(1-\delta)\left[\frac{X_1}{\alpha}+\beta\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\omega_0}{1+\gamma}\right]}{\frac{X_0}{\alpha}-\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\omega_0}{1+\gamma}}\right]^{\alpha},$$

which is increasing in ω_0.

Now in the model with ZLB, the question is whether $R_0^*(\omega_0)$ is larger or smaller than one. By setting $R_0^*(\omega_0) = 1$, we get the corresponding threshold value of ω_0 as

$$\omega_0^* = \frac{(1+\gamma)\frac{X_1}{\alpha}\left[1-(1-\delta)\left(\frac{G}{\beta}\right)^{\frac{1}{\alpha}}\right]}{\left[1+\beta(1-\delta)\left(\frac{G}{\beta}\right)^{\frac{1}{\alpha}}\right]\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)}.$$

If $\omega_0 \geq \omega_0^*$, given ω_0, when $k_{-1} > k_{-1}^*(\omega_0)$, the irreversibility constraint binds. We can insert $k_0 = (1-\delta)k_{-1}$ into equations (84) and (85) to pin down the two unknowns: R_0 and λ. Surprisingly, the interest rate remains unchanged in this region, i.e.,

$$R_0 = R_0^*(\omega_0),$$

and

$$\lambda = \left[\frac{\frac{1}{\gamma}(1-\delta) - \frac{1}{G}\left(\frac{G}{\beta}\right)^{\frac{\alpha-1}{\alpha}} R_0^{\frac{1}{\alpha}}}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{1-\alpha}{X_1}}\right]^{1-\alpha} \frac{X_1 R_0}{\alpha A_1} k_{-1}^{1-\alpha},$$

which is increasing in k_{-1}, and $q_0 = \frac{1-\delta}{\lambda}$ is decreasing in k_{-1}. The ZLB never binds in this case.

If $\omega_0 < \omega_0^*$, given ω_0 and increase k_{-1}, ZLB will bind first. Insert $R_0 = 1$, $X_0 = X_1$ and $\lambda = 1$ into (84) and (85), we get the cutoff of initial capital for a binding ZLB as

$$\hat{k}_{-1}(\omega_0) = \frac{\left(\frac{\alpha}{X_1}+\frac{1}{\beta}\right)\frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha A_1}\right)^{\frac{\alpha}{\alpha-1}}}{1-\delta+\frac{X_1}{\alpha}\beta^{\frac{1-\alpha}{\alpha}}G^{-\frac{1}{\alpha}} - \left(\frac{\alpha}{X_1}+\frac{1}{\gamma}\right)\frac{\gamma}{1+\gamma}\omega_0\left(1-\delta+\beta^{\frac{1-\alpha}{\alpha}}G^{-\frac{1}{\alpha}}\right)}, \tag{87}$$

When $k_{-1} \geq \hat{k}_{-1}$, X_0 starts to adjust. Inserting $\lambda = 1$ and $R_0 = 1$ into (84) and (85), we can solve the two unknowns: X_0 and k_0. Using Implicit Function Theorem, we can show that when ZLB binds and as k_{-1} increases, X_0 increases, and $\frac{k_0}{k_{-1}}$ decreases. In this case, denote the threshold of k_{-1} when the irreversibility condition starts to bind as $k_{-1}^{**}(\omega_0)$. We get

$$k_{-1}^{**}(\omega_0) = \frac{\frac{1-\alpha}{X_1}A_1\left(\frac{X_1}{\alpha A_1}\right)^{\frac{\alpha}{\alpha-1}}}{\frac{X_1}{\alpha}(1-\delta) - \frac{\gamma}{1+\gamma}\omega_0\left((1-\delta)+\beta^{\frac{1-\alpha}{\alpha}}\left(\frac{G X_0^{**}(\omega_0)}{X_1}\right)^{-\frac{1}{\alpha}}\right)}, \tag{88}$$

in which $X_0^{**}(\omega_0)$ can be solved by the following equation:

$$(1-\delta)\frac{X_1}{\alpha\beta} = \frac{1}{\alpha}\beta^{\frac{1-\alpha}{\alpha}}\left(\frac{G}{X_1}\right)^{-\frac{1}{\alpha}}\left(X_0^{**}\right)^{\frac{\alpha-1}{\alpha}} - \left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\gamma}{1+\gamma}\omega_0\left((1-\delta)+\beta^{\frac{1-\alpha}{\alpha}}\left(\frac{G}{X_1}\right)^{-\frac{1}{\alpha}}\left(X_0^{**}\right)^{-\frac{1}{\alpha}}\right). \tag{89}$$

If we impose $X_0^{**} = X_1$ in (89), the RHS is larger than the LHS. Otherwise, setting $X_0^{**} = \infty$, the RHS is smaller than the LHS. This suggests that a finite solution of X_0^{**} always exists.

Given ω_0, when $k_{-1} > k_{-1}^{**}(\omega_0)$, we find that X_0 becomes an constant, i.e.,

$$X_0 = X_0^{**}(\omega_0),$$

and

$$\lambda = \left[\frac{\frac{X_1}{\alpha}(1-\delta) - \frac{\gamma}{1+\gamma}\omega_0 \left[(1-\delta) + \beta^{\frac{1-\alpha}{\alpha}} \left(\frac{GX_0^{**}}{X_1} \right)^{-\frac{1}{\alpha}} \right]}{\frac{1-\alpha}{X_1} A_1 \left(\frac{X_1}{\alpha A_1} \right)^{\frac{\alpha}{\alpha-1}}} \right]^{1-\alpha} k_{-1}^{1-\alpha},$$

suggesting that q_0 is decreasing in k_{-1}.

E Proof of Proposition 7

We prove this proposition by solving the equilibrium explicitly. Given the state variables $\{k_{-1}, \omega_0\}$, we solve the equilibrium by dropping the collateral constraint (65) first following the equations in Subsection 3.3. Then we go back to (65) to check whether the collateral constraint is satisfied.

If the collateral constraint is violated, then with $\{k_{-1}, \omega_0\}$, we solve the equilibrium assuming that the collateral constraint is binding. In the last period, we have

$$b_0 = -mR_1^K k_0,$$

and the wealth share is

$$\omega_1 = 1 - m.$$

The other variables can be expressed as functions of k_0 as follows:

$$c_1 = (1-m)\frac{\alpha A_1}{X_1}\left[\frac{\frac{1-\alpha}{X_1}}{1 - \frac{\alpha}{X_1}(1-m)}\right]^{1-\alpha} k_0^{\alpha},$$

$$c_1' = A_1 \left(\frac{X_1}{1-\alpha}\right)^{\alpha-1} \left[\left(1 - \frac{\alpha}{X_1}(1-m)\right) k_0\right]^{\alpha}.$$

$$
\begin{aligned}
R_1^k &= \frac{\alpha}{X_1} A_1 \left[\frac{\frac{1-\alpha}{X_1}}{1 - \frac{\alpha}{X_1}(1-m)} \right]^{1-\alpha} k_0^{\alpha-1}, \\
L_1 &= \frac{\frac{1-\alpha}{X_1}}{1 - \frac{\alpha}{X_1}(1-m)}, \\
Y_1 &= A_1 \left[\frac{\frac{1-\alpha}{X_1}}{1 - \frac{\alpha}{X_1}(1-m)} \right]^{1-\alpha} k_0^{\alpha}.
\end{aligned} \tag{90}
$$

In the first period, given R_0, X_0 and λ, we have

$$
\begin{aligned}
c_0' &= \frac{1}{\beta R_0} A_1 \left(\frac{X_1}{1-\alpha} \right)^{\alpha-1} \left[\left(1 - \frac{\alpha}{X_1}(1-m) \right) k_0 \right]^{\alpha}, \\
L_0 &= \left(\beta R_0 \frac{X_1}{G X_0} \right)^{\frac{1}{\alpha}} \frac{\frac{1-\alpha}{X_1} k_{-1}}{\left(1 - \frac{\alpha}{X_1}(1-m) \right) k_0},
\end{aligned}
$$

$$
\begin{aligned}
R_0^k &= \frac{1-\delta}{\lambda} + \frac{\alpha}{X_0} A_0 \left(\frac{1}{\beta R_0} \frac{G X_0}{X_1} \right)^{\frac{\alpha-1}{\alpha}} \left[\frac{X_1}{1-\alpha} \left(1 - \frac{\alpha}{X_1}(1-m) \right) k_0 \right]^{\alpha-1}, \\
Y_0 &= A_0 \left(\beta R_0 \frac{X_1}{G X_0} \right)^{\frac{1-\alpha}{\alpha}} \left[\frac{\frac{1-\alpha}{X_1}}{\left(1 - \frac{\alpha}{X_1}(1-m) \right) k_0} \right]^{1-\alpha} k_{-1}, \\
c_0 &= \frac{1}{1+\gamma} \omega_0 R_0^k k_{-1}.
\end{aligned} \tag{91}
$$

We use the following two equations to solve the equilibrium. The first one is derived by the feasibility condition:

$$
\begin{aligned}
&\frac{1}{1+\gamma} \omega_0 R_0^k k_{-1} + \frac{1}{\beta R_0} A_1 \left(\frac{X_1}{1-\alpha} \right)^{\alpha-1} \left[\left(1 - \frac{\alpha}{X_1}(1-m) \right) k_0 \right]^{\alpha} + k_0 \\
&= (1-\delta) k_{-1} + A_0 \left(\beta R_0 \frac{X_1}{G X_0} \right)^{\frac{1-\alpha}{\alpha}} \left[\frac{\frac{1-\alpha}{X_1}}{\left(1 - \frac{\alpha}{X_1}(1-m) \right) k_0} \right]^{1-\alpha} k_{-1},
\end{aligned} \tag{92}
$$

and the second one is derive by the FOC of the entrepreneurs, (20) and (66):

$$
\left(\frac{1}{\lambda} - \frac{m \frac{\alpha}{X_1} A_1 \left[\frac{\frac{1-\alpha}{X_1}}{1 - \frac{\alpha}{X_1}(1-m)} \right]^{1-\alpha} k_0^{\alpha-1}}{R_0} \right) k_0 = \frac{\gamma}{1+\gamma} \omega_0 R_0^k k_{-1}, \tag{93}
$$

in which R_0^k is given by (91).

When both the ZLB and the irreversibility investment constraint bind, we have the following equation to pin down X_0:

$$\frac{\frac{1}{1+\gamma}\omega_0}{1-\frac{\gamma}{1+\gamma}\omega_0}\left[\frac{\gamma}{1+\gamma}\omega_0\frac{\alpha}{X_0}\left(\frac{GX_0}{\beta X_1}\right)^{\frac{\alpha-1}{\alpha}}L_1\left(1-\delta\right)^{\alpha-1}+m\frac{\alpha}{X_1}GL_1\left(1-\delta\right)^{\alpha}\right]$$
$$=\left(1-\frac{1}{1+\gamma}\omega_0\left(1-\delta\right)^{\alpha-1}\frac{\alpha}{X_0}\right)\left(\frac{GX_0}{\beta X_1}\right)^{\frac{\alpha-1}{\alpha}}L_1-\frac{G}{\beta}\frac{(1-\alpha)}{X_1}\left(1-\delta\right)^{\alpha}. \tag{94}$$

in which L_1 is constant as in (90). Notice that the value of X_0 is independent of k_1, suggesting that when both ZLB and the irreversibility constraint bind, only the capital price q_0 adjust but not the markup X_0. An equilibrium always exists in this case.