Financial Frictions, Investment, and Tobin's q

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Abstract

We develop a model of investment with financial constraints and use it to investigate the relation between investment and Tobin's q. A firm is financed partly by insiders, who control its assets, and partly by outside investors. When insiders' wealth is scarce, they earn a rate of return higher than the market rate of return and thus the firm's value includes a quasi-rent on invested capital. This implies that two forces drive q: changes in the value of invested capital and changes in the value of the insiders' future rents per unit of capital. This weakens the correlation between q and investment, relative to the frictionless benchmark. We present a calibrated version of the model, which, due to this effect, can generate more realistic correlations between investment, q, and cash flow.

Keywords: Financial constraints, optimal financial contracts, investment, Tobin's q, limited enforcement.

JEL codes: E22, E30, E44, G30.

1 Introduction

Dynamic models of the firm imply that investment decisions and the value of the firm should both respond to expectations about future profitability of capital. In models with constant returns to scale and convex adjustment costs these relations are especially clean, as investment and the firm's value respond exactly in the same way to new information about future profitability. This is the main prediction of Tobin's q theory, which implies that current investment moves one-for-one with q, the ratio of the firm's financial market value to its capital stock. This prediction, however, is typically rejected in the data, where investment appears to correlate more strongly with current cash flow than with q.

In this paper, we investigate the relation between investment, q, and cash flow in a model with financial frictions. The presence of financial frictions introduces quasi-rents in the market valuation of the firm. These quasi-rents break the one-to-one link between investment and q. We study how the presence of these quasi-rents affects the statistical correlations between investment, q, and cash flow, and ask whether a model with financial frictions can match the correlations in the data.

Our main conclusion is that the presence of financial frictions can bring the model closer to the data, but that the model's implications depend crucially on the shock structure. The crucial observation is that in a model with financial frictions it is still true that investment and q respond to future profitability, but the two variables now respond differently to information at different horizons. Investment is particularly sensitive to current profitability, which determines current internal financing, and to near-term financial profitability, which determines collateral values. On the other hand, q is relatively more sensitive to profitability farther in the future, which will determine future growth and thus the size of future quasi-rents. Therefore, to break the link between investment and q, we need the presence of both short-lived shocks—which tend to move investment more and have relatively smaller effects on q—and long-lived shocks—which do the opposite.

To develop these points, we build a stochastic model of investment subject to limited enforcement, with fully state-contingent claims. We show that our limited enforcement constraint is equivalent to a state-contingent collateral constraint, so our model is essentially a stochastic version of Kiyotaki and Moore (1997) with adjustment costs and state-contingent claims.[1] We show that the model leads to a wedge between average q—which correspond to the q measured from financial market values—and marginal q—which captures the marginal incentive to invest and is related one-to-one to investment.[2] We then analyze two versions of the model and look at their implications for an investment regression in which the investment rate is regressed on average q and cash flow.

First, we focus on a version of the model with no adjustment costs, which, under some simplifying assumptions, can be linearized and studied analytically. We consider three different shock structures. In a case with a single persistent shock, the model has indeterminate predictions regarding investment regression coefficients. This simply follows because in this case q and cash flow are perfectly collinear. In a case with two shocks—a temporary shock and a persistent shock—the one-to-one relation between q and investment breaks down because investment is driven by productivity in periods t and $t+1$ while q responds to all future values of productivity. Finally, we consider a case with "news shocks", that is, we allow agents to observe J periods in advance the realization of productivity shocks. In this case, we show that increasing the length of the horizon J reduces the coefficient on q and increases the coefficient on cash flow in investment regressions. This is due again to the differential responses of investment and q to information on productivity at different horizons.

The model with no adjustment costs, while analytically tractable, is quantitatively unappealing, as it tends to produce too much short-run volatility and too little persistence in investment. Therefore, for a more quantitative evaluation of the model we introduce adjustment costs. We calibrate the model to data moments

[1]Related recent stochastic models that combine state-contingent claims with some form of collateral constraint include He and Krishnamurthy (2013), Rampini and Viswanathan (2013) and Di Tella (2016).

[2]The terminology goes back to Hayashi (1982), who shows that the two are equivalent in a canonical model with convex adjustment costs.

from Compustat and analyze its implications both in terms of impulse responses and in terms of investment regressions. Our baseline calibration is based on the two shocks structure, with temporary and persistent shocks. In this calibration we show that q responds relatively more strongly to the persistent shock while investment responds relatively more strongly to the transitory shock, in line with the intuition from the no-adjustment-cost case. This leads to investment regressions with a smaller coefficient on q and a larger coefficient on cash flow, relative to a model with no financial frictions, thus bringing us closer to empirical coefficients. However, the q coefficient is still larger than in the data and the cash flow coefficient is smaller than in the data. When adding the possibility of news shocks, the disconnect between q and investment increases, leading to further reductions in the q coefficient and increases in the cash flow coefficient.

Fazzari et al. (1988) started a large empirical literature that explores the relation between investment and q using firm-level data. The typical finding in this literature is a small coefficient on q and a positive and significant coefficient on cash flow.[3] Fazzari et al. (1988), Gilchrist and Himmelberg (1995) and most of the subsequent literature interpret these findings as a symptom of financial frictions at work. More recent work by Gomes (2001) and Cooper and Ejarque (2003) questions this interpretation. The approach taken in these two papers is to look at the statistical implications of simulated data generated by a model to understand the empirical correlations between investment, q and cash flow.[4] In their simulated economies with financial frictions q still explains most of the variability in investment, and cash flow does not provide additional explanatory power. In this paper, we take a similar approach but reach different conclusions. This is due to two main differences. First, Gomes (2001) and Cooper and Ejarque (2003) model financial frictions by introducing a transaction cost which is a function of the flow of outside finance issued each period, while we introduce a contractual imperfection that imposes an upper bound on the stock of outside liabilities as a fraction of total assets. Our approach adds a state variable to the problem, namely

[3] See Hubbard (1998) for a survey.

[4] An approach that goes back to Sargent (1980).

the stock of existing liabilities of the firm as a fraction of assets, thus generating slower dynamics in the gap between internal funds and the desired level of investment. Second, we explore a variety of shock structures, which, as we argue below, play an important role in our results.

A related strand of recent literature has focused on violations of q theory coming from decreasing returns or market power, leaving aside financial frictions.[5] We see our effort as complementary to this literature, since both financial frictions and decreasing returns determine the presence of future rents embedded in the value of the firm. Also in that literature the shock structure plays an important role in the results. For example, Eberly et al. (2008) show that it is easier to obtain realistic implications for investment regressions by assuming a Markov process in which the distribution from which persistent productivity shocks are drawn switches occasionally between two regimes. Abel and Eberly (2011) also show that in models with decreasing returns it is possible to obtain interesting dynamics in q with no adjustment costs, similarly to what we do in Section 3 in a model with constant returns to scale and financial constraints.

The simplest shock that breaks the link between q and investment in models with financial constraints is a purely temporary shock to cash flow, which does not affect capital's future productivity. Absent financial frictions this shock should have no effect on current investment. This idea is the basis of a strand of empirical literature that tests for financial constraints by identifying some source of purely temporary shocks to cash flow. This is the approach taken by Blanchard et al. (1994) and Rauh (2006), which provide reliable evidence of the presence of financial constraints. Our paper builds on a similar intuition, by showing that in general shocks affecting profitability at different horizons have differential effects on q and investment and asks whether, given a realistic mix of shocks, a model with financial frictions can produce the unconditional correlations observed in the data.

In this paper we use the simplest possible model with the features we need:

[5]See Schiantarelli and Georgoutsos (1990), Alti (2003), Moyen (2004), Eberly et al. (2008), Abel and Eberly (2011), Abel and Eberly (2012).

an occasionally binding financial constraint; a dynamic, stochastic structure; adjustment costs that can produce realistic investment dynamics. There is a growing literature that builds richer models that are geared more directly to estimation. In particular, Hennessy and Whited (2007) build a rich structural model of firms' investment with financial frictions, which is estimated by simulated method of moments. They find that the financial constraint plays an important role in explaining observed firms' behavior. In their model, due to the complexity of the estimation task, the financial friction is introduced in a reduced form manner, by assuming transaction costs associated to the issuance of new equity or debt, as in Gomes (2001) or Cooper and Ejarque (2003).[6] We see our effort as complementary, as we have a more stylized model, but with financial constraints coming from an explicitly modeled contractual imperfection. Lastly, relative to these papers, and the entire investment regression literature, we demonstrate the importance of flexible shock structures, beside financial frictions, in bringing the model implications closer to the data.

A growing number of papers uses recursive methods to characterize optimal dynamic financial contracts in environments with different forms of contractual frictions (Atkeson and Cole (2005), Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), DeMarzo et al. (2012)). The limited enforcement friction in this paper makes it closer to the models in Albuquerque and Hopenhayn (2004) and Cooley et al. (2004). Within this literature Biais et al. (2007) look more closely at the implications of the theory for asset pricing. In particular, they find a set of securities that implements the optimal contract and then study the stochastic behavior of the prices of these securities. Here, our objective is to examine the model's implication for q theory, therefore we simply focus on the total value of the firm, which includes the value of all the claims held by insiders and outsiders.[7]

[6] The difference in results, relative to these papers, appears due to the fact that Hennessy and Whited (2007) also match the behavior of a number of financial variables.

[7] Our formulation also allows for the exploration of flexible shock structures, which are absent in the aforementioned papers. For example, most continuous time models assume shocks under the form of Brownian motions, thus rule out interesting shock structures such as permanent versus temporary shocks and news shocks considered in our paper.

In Section 2 we present the model. In Section 3, we study the case of no adjustment costs, deriving analytical results. In Section 4, we study the model with adjustment costs, relying on numerical simulations.

2 The Model

Consider an infinite horizon economy, in discrete time, populated by a continuum of entrepreneurs who invest in physical capital and raise funds from risk neutral investors.

The entrepreneurs' technology is linear: K_{it} units of capital, installed at time $t-1$ by entrepreneur i, yield profits $A_{it}K_{it}$ at time t. We can think of the linear profit function $A_{it}K_{it}$ as coming from a constant returns to scale production function in capital and other variable inputs which can be costlessly adjusted. Therefore, changes in A_{it} capture both changes in technology and changes in input and output prices. For brevity, we just call A_{it} "productivity". Productivity is a function of the state s_{it}, $A_{it} = A(s_{it})$, where s_{it} is a Markov process with a finite state space $\mathbf{S}$ and transition probability $\pi(s_{it}|s_{it-1})$. There are no aggregate shocks, so the cross sectional distribution of s_{it} across entrepreneurs is constant.

Investment is subject to convex adjustment costs. The cost of changing the installed capital stock from K_{it} to K_{it+1} is $G(K_{it+1}, K_{it})$ units of consumption goods at date t. The function G includes both the cost of purchasing capital goods and the installation cost. We assume G is increasing and convex in its first argument, decreasing in the second argument, and displays constant returns to scale. For numerical results, we use the quadratic functional form

$$G(K_{it+1}, K_{it}) = K_{it+1} - (1-\delta)K_{it} + \frac{\xi}{2}\frac{(K_{it+1} - K_{it})^2}{K_{it}}. \tag{1}$$

All agents in the model are risk neutral. The entrepreneurs' discount factor is β and the investors' discount factor is $\hat{\beta}$, with $\hat{\beta} > \beta$. We assume investors have a large enough endowment of the consumption good each period so that the equilibrium interest rate is $1 + r = 1/\hat{\beta}$. Each period an entrepreneur retires with probability γ and is replaced by a new entrepreneur with an endowment of 1 unit of capital. When an entrepreneur retires, productivity A_{it} is zero from next period on. The retirement shock is embedded in the process s_{it} by assuming that there is an absorbing state s^r with $A(s^r) = 0$ and the probability of transitioning to s^r from any other state is γ.

Each period, entrepreneur i can issue one-period state contingent liabilities, subject to limited enforcement. The entrepreneur controls the firm's capital K_{it} and, at the beginning of each period, can default on his liabilities and divert a fraction $1 - \theta$ of the firm's capital. If he does so, he re-enters the financial market as a new entrepreneur, with capital $(1 - \theta) K_{it}$ and no liabilities. That is, the punishment for a defaulting entrepreneur is the loss of a fraction θ of the firm's assets.

2.1 Optimal investment

We formulate the optimization problem of the individual entrepreneur in recursive form, dropping the subscripts i and t. Let $V(K, B, s)$ be the expected utility of an entrepreneur in state s, who enters the period with capital stock K and current liabilities B. For now, we simply assume that the problem's parameters are such that the entrepreneur's optimization problem is well defined. In the following sections, we provide conditions that ensure that this is the case.[8] The function V satisfies the Bellman equation

$$V(K, B, s) = \max_{C \geq 0, K' \geq 0, \{B'(s')\}} C + \beta \mathbb{E}\left[V\left(K', B'\left(s'\right), s'\right) | s\right], \tag{2}$$

subject to

$$C + G\left(K', K\right) \leq A(s)K - B + \hat{\beta}\mathbb{E}\left[B'\left(s'\right) | s\right], \tag{3}$$

$$V\left(K', B'\left(s'\right), s'\right) \geq V\left((1-\theta) K', 0, s'\right), \forall s', \tag{4}$$

[8]In the Online Appendix we provide a general existence result.

where C is current consumption, K' is next period's capital stock , and $B'(s')$ are next period's liabilities contingent on s'. Constraint (3) is the budget constraint and $\hat{\beta}\mathbb{E}[B'(s')|s]$ are the funds raised by selling the state contingent claims $\{B'(s')\}$ to the investors. Constraint (4) is the enforcement constraint that requires the continuation value under repayment to be greater than or equal to the continuation value under default.

The assumption of constant returns to scale implies that the value function takes the form $V(K,B,s) = v(b,s)K$ for some function v, where $b = B/K$ is the ratio of current liabilities to the capital stock. We can then rewrite the Bellman equation as

$$v(b,s)K = \max_{\substack{C\geq 0, K'\geq 0\\ \{b'(s')\}}} C + \beta\mathbb{E}\left[v\left(b'\left(s'\right),s'\right)|s\right]K', \tag{5}$$

subject to

$$C + G\left(K',K\right) \leq A(s)K - bK + \hat{\beta}\mathbb{E}\left[b'\left(s'\right)|s\right]K', \tag{6}$$

$$v\left(b'\left(s'\right),s'\right) \geq (1-\theta)\,v\left(0,s'\right), \forall s'. \tag{7}$$

It is easy to show that v is strictly decreasing in b. We can then find state-contingent borrowing limits $\overline{b}(s')$ such that the enforcement constraint can be written as

$$b'\left(s'\right) \leq \overline{b}\left(s'\right), \forall s'. \tag{8}$$

So the enforcement constraint is equivalent to a state contingent upper bound on the ratio of the firm's liabilities to capital. Relative to existing models with collateral constraints, two distinguishing features of our model are that we allow for state-contingent claims and we derive the state-contingent bounds endogenously from limited enforcement.[9]

[9]Other recent models that allow for state-contingent claims include He and Krishnamurthy (2013) and Rampini and Viswanathan (2013). Cao (2013) develops a general model with an explicit stochastic structure that studies collateral constraints with non-state-contingent debt.

2.2 Average and Marginal q

To characterize the solution to the entrepreneur's problem let us start from the first order condition for K':

$$\lambda G_1\left(K', K\right) = \lambda\hat{\beta}\mathbb{E}\left[b'|s\right] + \beta\mathbb{E}\left[v'|s\right], \tag{9}$$

where λ is the Lagrange multiplier on the budget constraint (6), or the marginal value of wealth for the entrepreneur. The expressions $\mathbb{E}\left[b'|s\right]$ and $\mathbb{E}\left[v'|s\right]$ are shorthand for $\mathbb{E}\left[b'\left(s'\right)|s\right]$ and $\mathbb{E}\left[v\left(b'\left(s'\right), s'\right)|s\right]$. Optimality for consumption implies that $\lambda \geq 1$ and the non-negativity constraint on consumption is binding if $\lambda > 1$.

To interpret condition (9) rewrite it as:

$$\lambda = \frac{\beta\mathbb{E}\left[v'|s\right]}{G_1\left(K', K\right) - \hat{\beta}\mathbb{E}\left[b'|s\right]} \geq 1. \tag{10}$$

When the inequality is strict the entrepreneur strictly prefers reducing current consumption to invest in new units of capital. If C was positive the entrepreneur could reduce it and use the additional funds to increase the capital stock. The marginal cost of an extra unit of capital is $G_1(K', K)$ but the extra unit of capital increases collateral and allows the entrepreneur to borrow $\hat{\beta}\mathbb{E}\left[b'|s\right]$ more from the consumers. So a unit reduction in consumption leads to a levered increase in capital invested of $1/(G_1 - \hat{\beta}\mathbb{E}\left[b'|s\right])$. Since capital tomorrow increases future utility by $\beta E\left[v'|s\right]$, we obtain (10).

Condition (9) can be used to derive our main result on average and marginal q. The value of all the claims on the firm's future earnings, held by investors and by the entrepreneur at the end of the period, is

$$\hat{\beta}\mathbb{E}\left[B'\left(s'\right)|s\right] + \beta\mathbb{E}\left[V\left(K', B'\left(s'\right), s'\right)|s\right].$$

Dividing by total capital invested gives us average q:

$$q^a \equiv \hat{\beta}\mathbb{E}\left[b'|s\right] + \beta\mathbb{E}\left[v'|s\right].$$

Marginal q, on the other hand, is just the marginal cost of one unit of new capital, $q^m \equiv G_1\left(K', K\right)$. We can then rearrange equation (9) and express it in terms of q^a and q^m as:

$$q^a = q^m + \frac{\lambda - 1}{\lambda}\beta\mathbb{E}\left[v'|s\right]. \tag{11}$$

Since $\lambda > 1$ if only if the non-negativity constraint on consumption is binding, we have proved the following result.

Proposition 1. *Average q is greater than or equal to marginal q, with strict equality if and only if the non-negativity constraint on consumption is binding.*

The difference between average and marginal q is larger if either the Lagrange multiplier λ is larger or the future value of entrepreneurial equity $\mathbb{E}\left[v'|s\right]$ is larger, as we can see from equation (11). As we shall see in the numerical part of the paper, an increase in indebtedness b increases λ but reduces the future value of entrepreneurial equity, so in general the relation between b and $q^a - q^m$ can be non-monotone. There is a cutoff for b such that $\lambda = 1$ below the cutoff and $\lambda > 1$ above the cutoff, so we know the relation is increasing in some region.

The fact that the only Lagrange multiplier appearing in (11) is λ, does not mean that the collateral constraint is not relevant in determining the gap between average and marginal q. Consider the first order condition for b'

$$\hat{\beta}\lambda + \beta v_b\left(b'\left(s'\right), s'\right) = \mu(s'),$$

where $\mu(s')$ is the Lagrange multiplier on the enforcement constraint (8) (expressed as a ratio of $\pi(s'|s)K'$ for convenience). Using the envelope condition for b to substitute for v_b and using time subscripts we can then write

$$\lambda_t = \frac{\beta}{\hat{\beta}}\lambda_{t+1} + \frac{1}{\hat{\beta}}\mu_{t+1}. \tag{12}$$

This condition shows that λ_t is a forward looking variable determined by current and future values of μ_{t+1}. Positive values of this Lagrange multiplier in the future induce the entrepreneur to reduce consumption today to increase internal funds available. The forward looking nature of λ_t will be useful to interpret some of our numerical results about news shocks.

If $\beta = \hat{\beta}$, condition (12) implies that if, at some date t, the entrepreneur's consumption is positive and $\lambda_t = 1$, then the non-negativity constraint and the collateral constraint can not be binding at any future date. In other words, once the entrepreneur is unconstrained he can never go back to being constrained. This is due to the assumption of complete state contingent markets. Assuming $\beta < \hat{\beta}$ ensures that entrepreneurs can alternate between positive and zero consumption.

We conclude this section by introducing some asset pricing relations that will be used to characterize the equilibrium. We use the notation $G_{1,t}$ and $G_{2,t}$ as shorthand for $G_1(K_{t+1}, K_t)$ and $G_2(K_{t+1}, K_t)$.

Proposition 2. *The following conditions hold in equilibrium*

$$\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} \frac{A_{t+1} - G_{2,t+1} - b_{t+1}}{G_{1,t} - \hat{\beta}\mathbb{E}_t b_{t+1}} \right], \tag{13}$$

and

$$\hat{\beta}\mathbb{E}_t \left[\frac{A_{t+1} - G_{2,t+1}}{G_{1,t}} \right] \geq 1 \geq \mathbb{E}_t \left[\frac{\beta\lambda_{t+1}}{\lambda_t} \frac{A_{t+1} - G_{2,t+1}}{G_{1,t}} \right]. \tag{14}$$

The last two conditions hold with strict inequality if the collateral constraint is binding with positive probability.

Notice that

$$\frac{A_{t+1} - G_{2,t+1} - b_{t+1}}{G_{1,t} - \hat{\beta}\mathbb{E}_t b_{t+1}}$$

represents the levered rate of return on capital. Condition (13) further illustrates the forward-looking nature of λ_t. In particular, it shows that λ_t is a geometric cumulate of all future levered returns on capital. Condition (13) can also be interpreted as a standard asset pricing condition, dividing both sides by λ_t and observing that $\beta\lambda_{t+1}/\lambda_t$ is the stochastic discount factor of the entrepreneur.

The expression

$$\frac{A_{t+1} - G_{2,t+1}}{G_{1,t}}$$

is the unlevered return on capital. When the collateral constraint is binding the first inequality in (14) is strict and this implies that the expected rate of return on capital is higher than the interest rate $1 + r$. This implies that the levered return on capital is higher than the unlevered return. The entrepreneurs will borrow up to the point at which the discounted levered rate of return is 1, by condition (13). At that point the discounted unlevered return will be smaller than 1, by the second inequality in (14). This second inequality can also be interpreted as capturing the fact that investing in physical capital has the additional benefit of relaxing the collateral constraint.

Define the finance premium as the difference between the expected return on entrepreneurial capital and the interest rate (which is equal to $1/\hat{\beta}$):

$$fp_t \equiv \mathbb{E}_t \left[\frac{A_{t+1} - G_{2,t+1}}{G_{1,t}} \right] - (1 + r) \,. \tag{15}$$

The first inequality in (14) shows that the finance premium is positive whenever the collateral constraint is binding. We will use this definition of the finance premium in Section 4.5.

3 Model with No Adjustment Costs: Analytical Results

We now consider the case of no adjustment costs, which arises when

$$G(K_{t+1}, K_t) = K_{t+1} - (1 - \delta) K_t.$$

In this case, we can derive some analytical results that help build the intuition for the numerical results in the following sections. For this section we assume a strict inequality between the discount factors of entrepreneurs and investors, $\beta < \hat{\beta}$, so that we can focus on cases in which the collateral constraint is always binding.

Absent adjustment costs, the value function takes the linear form

$$V\left(K, B, s\right) = \Lambda\left(s\right)\left[R\left(s\right)K - B\right], \tag{16}$$

where R is the gross return on capital defined by

$$R\left(s\right) \equiv A\left(s\right) + 1 - \delta.$$

Notice that $R\left(s\right)K - B$ is the total net worth of the entrepreneur at the beginning of the period, the total value of the capital stock minus the entrepreneur's liabilities. With a linear value function the borrowing limits are

$$\overline{b}(s) = \theta R\left(s\right), \tag{17}$$

and they have a natural interpretation: the entrepreneur can pledge a fraction θ of the firm's gross returns.

We now make assumptions that ensure that the problem is well defined and that the collateral constraint is always binding in equilibrium. Assume the following three inequalities hold for all s:

$$\beta\mathbb{E}\left[R\left(s'\right)|s\right] > 1, \tag{18}$$

$$\theta\hat{\beta}\mathbb{E}\left[R\left(s'\right)|s\right] < 1, \tag{19}$$

$$\frac{\left(1-\gamma\right)\left(1-\theta\right)\beta\mathbb{E}\left[R\left(s'\right)|s, s' \neq s^{r}\right]}{1-\theta\hat{\beta}\mathbb{E}\left[R\left(s'\right)|s\right]} < \zeta, \tag{20}$$

for some $\zeta < 1$. Condition (18) implies that the expected rate of return on capital is greater than the inverse discount factor of the entrepreneur, so the entrepreneur prefers investment to consumption. Condition (19) implies that pledgeable returns are insufficient to finance the purchase of one unit of capital, i.e., investment cannot be fully financed with outside funds. This condition ensures that investment is finite. Finally, condition (20) ensures that the entrepreneur's utility is bounded. The last condition allows us to use the contraction mapping theorem to fully characterize the equilibrium marginal value of wealth $\Lambda(s)$ in the following proposition. The proof of this lemma and of the following results in this section are in the appendix.

Lemma 1. *If conditions (18)-(20) hold there is a unique function $\Lambda : \mathbf{S} \to [1, \infty)$ that satisfies the recursion*

$$\Lambda(s) = \frac{\beta(1-\theta)\mathbb{E}[\Lambda(s')R(s')|s]}{1-\theta\hat{\beta}\mathbb{E}[R(s')|s]}, \text{ for all } s \neq s^r, \tag{21}$$

and $\Lambda(s) = 1$ for $s = s^r$.

Notice that (21) is a special case of condition (13), in which the constraint is always binding. The following proposition characterizes an equilibrium.

Proposition 3. *If conditions (18)-(20) hold and $\Lambda(s)$ satisfies*

$$\Lambda(s) > \frac{\beta}{\hat{\beta}}\Lambda(s'), \tag{22}$$

for all $s, s' \in \mathcal{S}$, then the collateral constraint is binding in all states, consumption is zero until the retirement shock, investment in all periods before retirement is given by

$$\frac{K' - (1-\delta)K}{K} = \frac{(1-\theta)R(s)}{1-\theta\hat{\beta}\mathbb{E}[R(s')|s]} - (1-\delta), \tag{23}$$

and average q is

$$q^a = \mathbb{E}\left[\left((1-\theta)\beta\Lambda(s') + \theta\hat{\beta}\right)R(s')|s\right]. \tag{24}$$

Condition (22) ensures that entrepreneurs never delay investment. Namely, it implies that they always prefer to invest in physical capital today rather than buying a state-contingent security that pays in some future state.

The entrepreneur's problem can be analyzed under weaker versions of (18)-(22), but then the constraint will be non-binding in some states. It is useful to remark that we could embed our model in a general equilibrium environment with a constant returns to scale production function in capital and labor and a fixed supply of labor. In this general equilibrium model $A(s)$ is replaced by the endogenous value of the marginal product of capital. It is then possible to derive conditions (18)-(22) endogenously if shocks are small and the non-stochastic steady state features a binding collateral constraint.

We now assume conditions (18)-(22) hold and analyze the model assuming that there are small shocks to A around the level $\bar{A}$ and linearizing the equilibrium conditions (23)-(24) around the non-stochastic steady state. The investment rate is defined as investment over assets and is denoted by

$$IK_t \equiv \frac{K_{t+1} - (1-\delta) K_t}{K_t}.$$

We will use a bar to denote steady state values and a tilde to denote deviations from the steady state.

In steady state equation (21) yields

$$\bar{\Lambda} = \frac{\beta (1-\theta) \gamma \bar{R}}{1 - \left(\theta\hat{\beta} + (1-\theta)(1-\gamma)\beta\right) \bar{R}}.$$

and the investment rate is

$$\bar{IK} = \frac{(1-\theta) \bar{R}}{1 - \theta\hat{\beta}\bar{R}} - (1-\delta).$$

The following proposition charaterizes the dynamics of investment and Tobin's Q around the steady state.

Proposition 4. *If the economy satisfies (18)-(22) a linear approximation gives the follow-*

ing expressions for investment and average q:

$$\tilde{IK}_t = \frac{1-\theta}{1-\theta\hat{\beta}\bar{R}}\left[\tilde{A}_t + \frac{\theta\hat{\beta}\bar{R}}{1-\theta\hat{\beta}\bar{R}}\mathbb{E}_t\left[\tilde{A}_{t+1}\right]\right], \tag{25}$$

$$\begin{aligned}\tilde{q}^a_t = &\left[\beta\left(1-\theta\right)\left(\gamma+\left(1-\gamma\right)\bar{\Lambda}\right)+\theta\hat{\beta}\right]\mathbb{E}_t\left[\tilde{A}_{t+1}\right]+\\ &+\beta\left(1-\theta\right)\left(1-\gamma\right)\bar{R}\mathbb{E}_t\left[\tilde{\Lambda}_{t+1}\right],\end{aligned} \tag{26}$$

where

$$\tilde{\Lambda}_t = \frac{\bar{\Lambda}/\bar{R}}{1-\theta\hat{\beta}R}\sum_{j=0}^{\infty}\left(\frac{\left(1-\gamma\right)\bar{\Lambda}}{\gamma+\left(1-\gamma\right)\bar{\Lambda}}\right)^j\mathbb{E}_t\left[\tilde{A}_{t+j}\right], \tag{27}$$

conditional on $s_t \neq s^r$.

Equations (25)-(26) express investment and average q in terms of current and future expected values of productivity. Since A_t is equal to profits over capital, we match it to cash flow over assets in the empirical literature. Given assumptions about the process for A_t, equations (25) and (26) give us all the information about the variance-covariance matrix of $\left(\tilde{IK}_t, \tilde{q}^a_t, \tilde{A}_t\right)$ and thus about investment regression coefficients.

The crucial observation is that average q is affected by the marginal value of entrepreneurial net worth, which is a forward looking variable that reflects expectations about all future excess returns on entrepreneurial capital.[10] Through this channel, average q responds to information about future values of A_t at all horizons. At the same time, investment is only driven by the current and next period value of A_t. The current value determines internal funds, the next period value determines collateral values. Putting these facts together implies that shocks that affect profitability differentially at different horizons will break the link between average q and investment.

We now turn to a few examples that show how different shock structures lead to different implications for the variance-covariance matrix of investment, average q and cash flow and thus for investment regressions.

[10] See the discussion following Proposition 2.

Example 1. *Productivity $\tilde{A}_t$ follows the AR(1) process:*

$$\tilde{A}_t = \rho \tilde{A}_{t-1} + \varepsilon_t,$$

where ε_t is an i.i.d. shock.

In this example, we have $\mathbb{E}_t \left[\tilde{A}_{t+j}\right] = \rho^j \tilde{A}_t$ so all future expected values of $\tilde{A}_t$ are proportional to the current value. Substituting in (25)-(26), it is easy to show that both $\tilde{q}_t^a$ and $\tilde{IK}_t$ are linear functions of $\tilde{A}_t$. Therefore, in this case cash flow and average q are both, separately, sufficient statistics for investment. This is true even though there is a financial constraint always binding, simply due to the fact that a single shock is driving both variables.

In this example, the coefficients of a regression of investment on average q and cash flow are indeterminate due to perfect collinearity, but adding cash flow to a univariate regression of investment on average q alone does not increase the regression's explanatory power.

Example 2. *Productivity $\tilde{A}_t$ has a persistent component x_t and a temporary component η_t:*

$$\tilde{A}_t = x_t + \eta_t$$

with

$$x_t = \rho x_{t-1} + \varepsilon_t.$$

In this example, we have $\mathbb{E}_t \left[\tilde{A}_{t+j}\right] = \rho^j x_t$, and substituting in (25)-(26), we arrive at:

$$\tilde{IK}_t = \frac{(1-\theta)\left(1-(1-\rho)\bar{R}\theta\hat{\beta}\right)}{\left(1-\theta\hat{\beta}\bar{R}\right)^2} x_t + \frac{1-\theta}{1-\theta\hat{\beta}\bar{R}}\eta_t,$$

$$\tilde{q}_t^a = \left[\left(\beta\left(1-\theta\right)\left(\gamma+\left(1-\gamma\right)\bar{\Lambda}\right)+\theta\hat{\beta}\right)\rho + \frac{\beta\left(1-\theta\right)\left(1-\gamma\right)\left(\gamma+\left(1-\gamma\right)\bar{\Lambda}\right)}{\left(1-\theta\hat{\beta}\bar{R}\right)\left(\gamma+\left(1-\gamma\right)\left(1-\rho\right)\bar{\Lambda}\right)}\bar{\Lambda}\rho\right] x_t.$$

If we now run a regression of investment on average q and cash flow, cash flow is the only variable that can capture variations in η_t, so the coefficient on cash flow will be positive and equal to

$$\frac{1-\theta}{1-\theta\hat{\beta}\bar{R}},$$

and cash flow improves the explanatory power of the investment regression. The coefficient on cash flow here is bigger than 1, but that's clearly due to the absence of adjustment costs. In the next section we will build on the logic of this example, to analyze quantatively the effect of financial constraints on investment regressions.

Notice that in this example, investment, q and cash flow are fully determined by the two random variables x_t and η_t and the coefficients are independent of the variance parameters. This implies that, given all the other parameters, the coefficients of the investment regression are independent of the values of the variances σ_ε^2 and σ_η^2, as long as both are positive. As we shall see, this result does not extend to the general model with adjustment costs.

As an aside, notice that in this example, the coefficient on cash flow is higher for firms with larger values of θ, i.e., for firms that can finance a larger fraction of investment with external funds. These firms respond more because they can lever more any temporary increase in internal funds. This is reminiscent of the observation in Kaplan and Zingales (1997) that the coefficient on cash flow in an investment regression should not be used as measure of the tightness of the financial constraint.

We now turn to our last example, in which we introduce news shocks.

Example 3. *The productivity process is as in Example 2 but the value of the permanent component x_t is known J periods in advance, with $J \geq 1$.*

In the appendix, we show that in this example investment and q dynamics are given by

$$\tilde{q}_t^a = \left\{ \begin{array}{c} \beta\left(1-\theta\right)\left(\gamma+\left(1-\gamma\right)\bar{\Lambda}\right)+\theta\hat{\beta} \\ +\frac{\beta(1-\theta)(1-\gamma)\bar{\Lambda}}{\left(1-\theta\hat{\beta}R\right)\left(1-\frac{(1-\gamma)\bar{\Lambda}\rho}{\gamma+(1-\gamma)\bar{\Lambda}}\right)} \end{array} \right\} x_{t+1}+\widetilde{\varepsilon}_t \tag{28}$$

where[11]

$$\widetilde{\varepsilon}_t = \sum_{j=1}^{J-1} \frac{\beta\left(1-\theta\right)\left(1-\gamma\right)\bar{\Lambda}}{\left(1-\theta\hat{\beta}R\right)\left(1-\frac{\left(1-\gamma\right)\bar{\Lambda}\rho}{\gamma+\left(1-\gamma\right)\bar{\Lambda}}\right)} \left(\frac{\left(1-\gamma\right)\bar{\Lambda}}{\gamma+\left(1-\gamma\right)\bar{\Lambda}}\right)^{j} \varepsilon_{t+1+j},$$

and

$$\tilde{IK}_t = \frac{1-\theta}{1-\theta\hat{\beta}R}\left(x_t+\eta_t\right) + \frac{\left(1-\theta\right)R\theta\hat{\beta}}{\left(1-\theta\hat{\beta}R\right)^2} x_{t+1}.$$

We can then show that increasing J affects the coefficients and the R^2 of the investment regression as follows.

Proposition 5. *In the economy of Example 3, all else equal, increasing the horizon J at which shocks are anticipated decreases the coefficient on average q, increases the coefficient on cash flow, and reduces the R^2 of the investment regression.*

The proof of this result is in the appendix. Investment, as in the previous example, is just a linear function of productivity at times t and $t+1$, which fully determine current cash flow and collateral values. On the other hand, q is a function of all future values of A_t and, given the presence of news, these values are driven by anticipated future shocks which have no effect on investment. This weakens the relation between q and investment. Moreover, since q is the only source of information about x_{t+1}, and, with news shocks, it becomes a noisier source of information, this also reduces the joint explanatory power of q and cash flow.

Notice that news shocks here are acting very much like measurement error in q, by adding a shock to it that is unrelated to the shocks driving investment. However, financial frictions are essential in introducing this source of error. Absent financial frictions future values of productivity should not affect q, and it is only because q includes future quasi-rents that the relation arises.

In the next section, we will see that the forces identified in these three examples carry over to a more general model with adjustment costs.

[11] When $J=1$, $\widetilde{\varepsilon}_t = 0$.

4 Model with Adjustment Costs: Quantitative Analysis

We now turn to the full model with adjustment costs and analyze its implications using numerical simulations. While the no adjustment cost model analyzed above is useful to build intuition, it has a number of unrealistic implications in particular for the inertial behavior of investment. The full model with adjustment costs, on the other hand, can be calibrated to match some moments of the observed processes for profits and investment, so that we can look at its quantitative implications.

We start by describing our choice of parameters and characterize the equilibrium in terms of policy functions and impulse responses. We then run investment regressions on the simulated output and explore the model's ability to replicate empirical investment regressions.

4.1 Calibration

The time period in the model is one year. The baseline parameter values are summarized in Table 1. The first three parameters are pre-set, the remaining parameters are calibrated on Compustat data. We now describe their choice in detail.

The investors' discount factor $\hat{\beta}$ is chosen so that the implied interest rate is 8.7%. As argued by Abel and Eberly (2011) the interest rate used in this type of exercise should correspond to a risk-adjusted expected return. The number we choose is in the range of rates of return used in the literature.[12] The entrepreneurs' discount factor β has effects similar to the parameter γ which governs their exit rate. In particular, both affect the incentives of entrepreneurs to accumulate wealth and become financially unconstrained and both affect the forward looking component of q. Therefore, we fix β at a level lower than $\hat{\beta}$ and

[12] Abel and Eberly (2011) and DeMarzo et al. (2012) choose numbers near 10%, while Moyen (2004) and Gomes (2001) use $r = 6.5\%$.

Table 1: Parameters

Preset	β	$\hat{\beta}$	θ	
	0.90	0.92	0.3	
Calibrated to cash flow moments	μ_a	ρ_x	σ_ε	σ_η
	0.246	0.743	0.0713	0.0375
Calibrated to investment and q moments	δ	ξ	γ	
	0.0250	1.75	0.095	

calibrate γ.[13] Regarding the fraction of non-divertible assets θ, there is only indirect empirical evidence, and existing simulations in the literature have used a wide range of values. Here we choose $\theta = 0.3$ in line with evidence in Fazzari et al. (1988) and Nezafat and Slavik (2013). In particular, Fazzari et al. (1988) report that 30% of manufacturing investment is financed externally. Nezafat and Slavik (2013) use US Flow of Funds data for non-financial firms to estimate the ratio of funds raised in the market to fixed investment, and find a mean value of 0.284.

The parameters in the second line of Table 1 are calibrated to match moments of the firm-level cash flow time series in Compustat. We assume that profits per unit of capital A_t are the sum of a persistent and a temporary component. Namely,

$$\begin{aligned} A_{it} &= x_{it} + \eta_{it} \\ x_{it} &= (1-\rho_x)\mu_a + \rho_x x_{it-1} + \varepsilon_{it} \end{aligned}$$

where η_{it} and ε_{it} are i.i.d. Gaussian shocks with variances σ_η^2 and σ_ε^2. We identify profits per unit of capital in the model, A_{it}, with cash flow per unit of capital in the data, denoted by CFK_{it}.[14] The parameter μ_a is set equal to average cash flow per unit of capital in the data. The values of ρ_x, σ_ε and σ_η are chosen to match the first and second order autocorrelation and the standard deviation of cash flow

[13] Changing the chosen value of β in a reasonable range does not affect the results significantly.
[14] Cash flow is equal to net income before extraordinary items plus depreciation.

Table 2: Target moments and model values

Moment	$\rho_1(CFK)$	$\rho_2(CFK)$	$\sigma(CFK)$	$\mu(IK)$	$\sigma(IK)$	$\mu(q^a)$
Target value	0.60	0.41	0.113	0.17	0.111	2.5
Model value	0.60	0.41	0.113	0.23	0.098	2.5

per unit of capital in the data, denoted, respectively, by $\rho_1(CFK)$, $\rho_2(CFK)$ and $\sigma(CFK)$. These moments are estimated using the approach of Arellano and Bond (1991) and Arellano and Bover (1995) and are reported in Table 1.[15] Notice that simply computing raw autocorrelations in the data—as sometimes done in the literature—would lead to biased estimates, given the short sample length.[16] In terms of sample, we use the same sub-sample of Compustat used in Gilchrist and Himmelberg (1995) so that we can compare our simulated regressions to their results.[17]

The next three parameters in Table 1, δ, ξ, and γ, are chosen to match three moments from the Compustat sample: the mean and standard deviation of the investment rate, $\mu(IK)$ and $\sigma(IK)$, and the mean of average q, $\mu(q^a)$. The reason why δ and ξ help determine the level and volatility of the investment rate is intuitive, as these two parameters determine the depreciation rate and the slope of the adjustment cost function. The parameter γ controls the speed at which entrepreneurs exit, so it affects the discounted present value of the quasi-rents they expect to

[15]We estimate the firm-specific variation in cash-flow by first taking out the aggregate mean for each year and then applying the function xtabond2 in STATA. This implements the GMM approach of Arellano and Bover (1995). This approach avoids estimating individual fixed effects affecting both the dependent variable (cash flow) and one of the independent variables (lagged cash flow), by first-differencing the law of motion for cash flow, and then using both lagged differences and lagged levels as instruments. We use the first three available (non-autocorrelated) lags in differences as instruments, with lags chosen separately for the 1st and 2nd order autocorrelation estimation. One lagged level is also used as an instrument.

[16]This type of bias was first documented in Nickell (1981). The bias is non-negligible in our sample. For the first-order autocorrelation, the Arellano and Bond (1991) approach gives $\rho_1(CFK) = 0.60$, while the raw autocorrelation in the data is 0.42.

[17]In particular, we restrict attention to the sample period 1978-1989 and use the same 428 listed firms used in their paper.

receive in the future and thus average q. However, the three parameters interact, so we choose them jointly—by a grid search—in order to minimizes the average squared percentage deviation between the three model-generated moments and their targets. The target moments from the data and the model generated moments are reported in Table 2.[18]

Notice that there is a tension between hitting the targets for $\mu(IK)$ and $\sigma(IK)$. Increasing any of the parameters, δ, ξ, γ reduces $\mu(IK)$, bringing it closer to its target value, but also decreases $\sigma(IK)$, bringing it farther from its target. Notice also that it is important for our purposes that the model generates a realistic level of volatility in the investment rate, given that IK is the dependent variable in the regressions we will present in Section 4.3 below.

Our calibration also determines the average size of the wedge between average and marginal q. In particular, $\mu(q^a) = 2.5$ is the mean value of average q while ξ and $\mu(IK)$ determine the mean value of marginal q, which is $1 + \xi(\mu(IK) - \delta) = 1.25$. Therefore, the average wedge between average and marginal q is 1.25. Since the presence of the wedge is what breaks the sufficient statistic property of q it is useful that our calibration imposes some discipline on the wedge's size.

All the simulations assume that entrepreneurs enter the economy with a unit endowment of capital and zero financial wealth (i.e., zero current profits and zero debt). Since the entrepreneurs' problem is invariant to the capital stock and all our empirical targets are normalized by total assets, the choice of the initial capital endowment is just a normalization. We have experimented with different initial conditions for financial wealth, but they have small effects on our results given that—with our parameters—the state variable b converges quickly to its stationary distribution.

It is useful to compare our results to those of a benchmark model with no financial frictions. To make the parametrization of the two models comparable, we re-calibrate the parameters δ, ξ and γ for the frictionless case. The moments and associated parameters are reported in Table 3. Notice that the frictionless model generates a low value of $\mu(q^a)$. For given IK, increasing ξ would increase

[18] The target standard deviation $\sigma(IK)$ is a pooled estimate.

Table 3: Calibration of frictionless model

Parameter	δ	ξ	γ
	0.05	1.50	0.125
Moment	$\mu(IK)$	$\mu(q^a)$	$\sigma\,(IK)$
Target value	0.17	2.5	0.111
Model value	0.18	1.2	0.116

marginal and average q (which are the same in the frictionless case), but it would reduce the volatility of investment.

In Section 4.5 we consider an alternative calibration approach, that targets the average finance premium, as defined in equation (15).

4.2 Model dynamics

We now characterize the optimal solution to the entrepreneurs' problem, first describing optimal choices and values as function of the state variables and next showing what this behavior implies for the responses of endogenous variables to different shocks.

4.2.1 Characterization

To illustrate the model behavior, it helps intuition to use as state variables A and n, where n is defined as

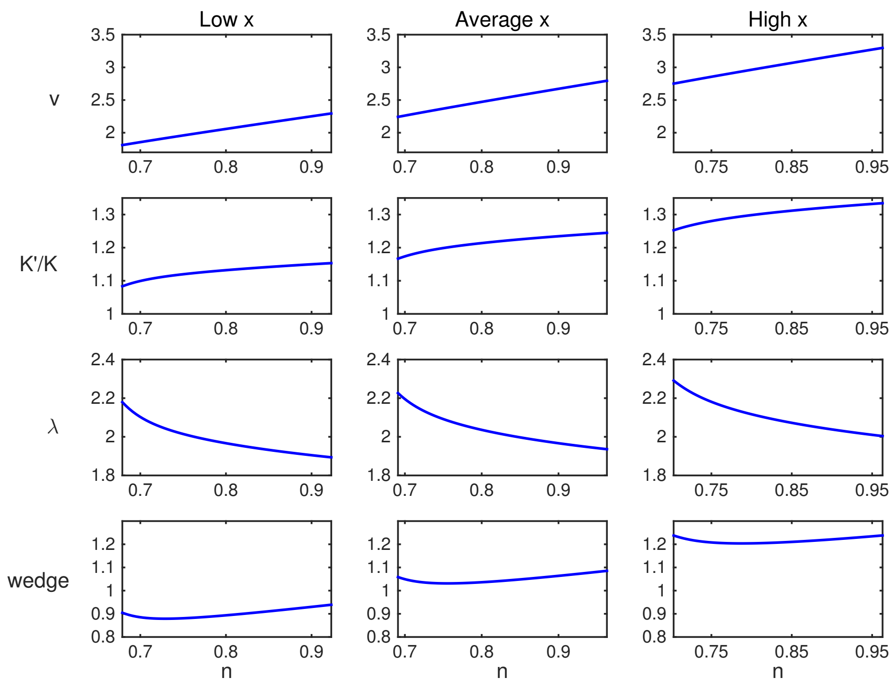
$$n \equiv A + 1 - \delta - b, \tag{29}$$

rather than using A and b. The variable n is a measure of net worth over assets. Net worth excluding adjustment costs is $AK + (1-\delta)K - B$. Dividing by K leads to (29).[19]

On each row of Figure 1 we plot, respectively, the value function (per unit of

[19] An alternative is to evaluate installed capital at its shadow value, thus getting net worth equal to $AK - G_2(K', K)K - B$. The figures are similar.

Figure 1: Characterization of equilibrium



Note: The three columns correspond to the 20th, 50th, and 80th percentile of the persistent component of productivity x. The range for the net worth variable n is between the 10th and 90th percentiles of the distribution of n conditional on x.

capital) v, the optimal investment ratio K'/K, the Lagrange multiplier λ on the entrepreneur's budget constraint, and the wedge between average q and marginal q. Each column corresponds to different values of persistent component of productivity x. In particular, we report three values corresponding to the the 20th, 50th and 80th percentile of the unconditional distribution of x. On the horizontal axis we have n, but the domain differs between columns as we plot values between the 10th to 90th percentile of the conditional distribution of n, conditional on the reported value of x.[20]

[20]The joint distribution of (n, x) is computed numerically as the invariant joint distribution gen-

A higher level of n leads to a higher value v and a higher level of investment K'/K. Moreover, the value function is concave in n. The Lagrange multiplier λ is equal to the derivative of the value function and therefore is decreasing in n. The fact that λ is decreasing in n reflects the fact that a higher ratio of net worth to capital allows firms to invest more, leading to a higher shadow cost of capital G_1 and thus to a lower expected returns on investment. Eventually, for very high values of n we reach $\lambda = 1$. However, as the figures show this does not happen for the range of n values more frequently visited in equilibrium.

The bottom row documents how the wedge varies with the level of net worth n and with the persistent component of productivity x. Let us first look at the effect of n. Even though λ is decreasing in n, the wedge, $q^a - q^m$, does not vary much with n for a given value of x. Our analytical derivations in Section 2 help explain this outcome. Recall from equation (11) that the wedge is equal to

$$\frac{\lambda - 1}{\lambda} \beta \mathbb{E}\left[v'|s\right].$$

When we reach the unconstrained solution and $\lambda = 1$ the wedge disappears. However, for lower levels of n, for which the constraint is binding, the relation is in general non-monotone. An increase in n reduces the marginal gain from an extra unit of net worth. However, at the same time it increases the future growth rate of firm's capital stock and so it increases the base to which this marginal quasi-rent is applied. This second effect is captured by the expression $\mathbb{E}[v'|s]$, because the value per unit of capital v' embeds the future growth of the firm and is increasing in n. The plots in the bottom row of Figure 1 show that in the relevant range of n these two effects roughly cancel.

On the other hand, comparing the values of the wedge across columns, shows that persistent component of productivity x has large effects on the wedge and that the wedge is increasing in x. The reason is that higher values of x lead both to higher values of λ, as the marginal benefits of extra internal funds increase with productivity, and to higher values of K'/K and v, because higher productiv-

erated by the optimal policies.

ity allows the firm to raise more external funds and grow faster. Therefore both elements of the wedge increase with higher values of x.

4.2.2 Impulse response functions

We now present impulse response functions that illustrate the model dynamics following the two shocks. To construct these impulse response functions, we take a firm starting at the median values of the state variables n and x. We then subject the firm to a shock at time t, simulate 10^6 paths following the shock, and report the difference between the average simulated paths, with and without the initial shock. Given the non-linearity of the model, the initial conditions for n and x in general affect the responses. However, in our simulations these non-linear effects are relatively small, so the plots below are representative.

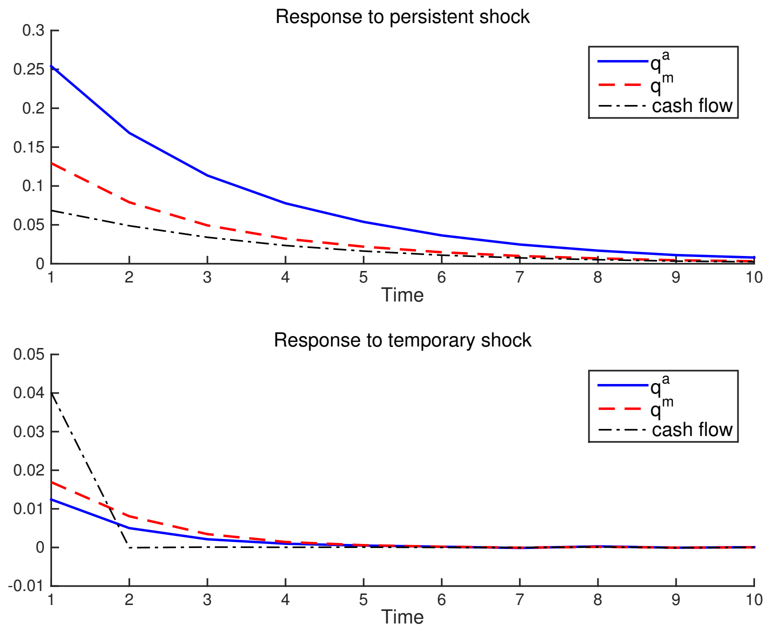
In the top panel of Figure 2 we plot the responses of marginal and average q, and cash flow per unit of capital to a 1-standard-deviation persistent shock ε.[21] Following a persistent shock all variables increase and return gradually to trend. The response of average q is larger than that of marginal q, thus producing an increase in the wedge.

In the bottom panel of Figure 2 we plot the responses of the same variables to a 1-standard-deviation temporary shock η. Also in this case all three variables respond positively, but the response is more short-lived. Moreover, now the response of average q is slightly smaller than the response of marginal q, so the wedge shows a small decrease after the shock.

Notice that average q is a forward-looking variable that incorporates the quasi-rents that the entrepreneur is expected to receive in the future. It is not surprising that these quasi-rents are only marginally affected by a temporary shock. In the model with no adjustment costs, the effect is zero, as shown in Section 3 above. Here, because of adjustment costs, there is a slight positive effect, due to the fact that the investment response displays a small but positive degree of persistence

[21] The response of investment K'/K is always proportional to the response of marginal q and is thus omitted.

Figure 2: Impulse response functions



Note: Average paths following a shock at time 1, in (level) deviations from average paths following no shock. Cash flow is cash flow per unit of capital.

and high investment in the future increases the future value of installed capital. But this effect is small. In the case of a persistent shock, instead, future quasi-rents are directly affected by higher future productivity, which is going to lead to faster growth (as shown in Figure 1), thus explaining the large increase in q^a in the top panel of Figure 2.

The discussion following Figure 1, helps to explain the response of the wedge $q^a - q^m$. A temporary shock, by increasing A temporarily, leads to a pure increase in net worth per unit of capital, as $n = A - b$. As we argued when presenting Figure 1, the effect of such an increase on the wedge is in general ambiguous and, with our parameter choices, close to zero. In the case of a persistent shock, instead, the effect is unambiguously to increase the wedge, as the increase in x leads to a higher λ and to a higher $\mathbb{E}[v'|s]$.

The relative responses of cash flow and marginal q are also different across the two shocks. In particular, we have a larger response of marginal q relative to the cash flow response in the case of a persistent shock. The reason is that in the case of a persistent shock the collateral value of capital increases, thus amplifying the effect on investment.

4.3 Investment regressions

We now turn to investment regressions, and ask whether the model can replicate the coefficients on q and cash flow observed in the data. In particular, we ask to what extent does the presence of a financial friction help in obtaining a smaller coefficient on q and a positive and large coefficient on cash flow. To answer this question, we generate simulated data from our model and run investment regressions on it. In line with the empirical literature, we generate a balanced panel of 500 firms for 20 periods, and run the following investment regression:[22]

$$IK_{it} = a_{i0} + a_1 q^a_{it} + a_2 CFK_{it} + e_{it}, \tag{30}$$

where we allow for firm-level fixed effects. All reported results are the mean values for 50 simulated panels.

The regression coefficients for the baseline model are presented in the first row of Table 4. As reference points, in the second row we report the coefficients that arise in the model without financial frictions and in the last row the empirical estimates in Gilchrist and Himmelberg (1995), which are representative of the orders of magnitude obtained in empirical studies.[23] We also report coefficients of univariate regressions of investment on average q and cash-flow separately.

The results for the frictionless benchmark are reported in the second line of Ta-

[22] The model features random exit, so to generate a balanced panel we only keep firms for which exit does not occur for 20 periods.

[23] We do not report standard errors, but they are small (less than 0.04) for both coefficients in our simulated data. They are also small in the empirical estimates of Gilchrist and Himmelberg (1995).

Table 4: Investment regressions

	a_1	a_2	R^2	Univariate q^a coefficient	R^2	Univariate CFK coefficient	R^2
Baseline model	0.22	0.15	0.98	0.27	0.98	0.81	0.89
Frictionless model	0.67	0.00	1.00	0.67	1.00	0.95	0.86
GH (1995)	0.033	0.24		0.05			

ble 4. In this case, average q is a sufficient statistic for investment, the coefficient on cash flow is zero and the coefficient on q is equal to the inverse of the adjustment cost coefficient ξ, which is calibrated to 1.5. This line shows the standard empirical failure of the benchmark adjustment cost model.

Adding financial frictions helps to get a smaller coefficient on q and a positive coefficient on cash flow. The effect is sizable, although the coefficient on q is still large compared to the very small numbers found in empirical regressions. Notice also that the R^2 of the regression is very close to 1. This is not surprising given that we have a simple two-shock structure and two explanatory variables.[24] Given that the model is non-linear, the R^2 can in general be smaller than 1. However, by experimenting with impulse responses for different initial values of the state variables we have confirmed that, given our parameter values, the model is close to linear in its responses to the two shocks, which helps to explain the high R^2 in Table 4.

The presence of the wedge breaks the one-to-one relation between q and investment and allows for cash flow to have explanatory power in the the investment regression. In particular, as we saw in Figure 2 the wedge responds in opposite directions to the two shocks, while q^m respond positively to both. So the wedge plays a role somewhat similar to measurement error in dampening the regression coefficient. Notice however that the model still features a strong positive relation between q^a and investment, as documented by the fifth and sixth columns

[24]For the same reason, in the linear model of Example 2, Section 3, the R^2 is 1.

Table 5: Investment regressions: changing shock variances

σ_ε	σ_η	σ_η^2/σ_A^2	a_1	a_2	R^2	Univariate q^a coefficient	R^2	Univariate CFK coefficient	R^2
0.113	0.000	0.00	0.38	-0.48	0.99	0.25	0.99	0.90	0.98
0.071	0.037	0.11	0.22	0.15	0.98	0.27	0.98	0.81	0.89
0.033	0.080	0.50	0.28	0.11	0.96	0.32	0.95	0.48	0.56
0.006	0.107	0.90	0.34	0.10	0.84	0.38	0.75	0.18	0.32
0.000	0.113	1.00	2.47	0.01	0.92	2.53	0.92	0.11	0.37

of Table 4, which show that a univariate regression between investment and average q produces a large coefficient and a large R^2 in simulated data (unlike in actual data). In the rest of the paper we investigate shock structures that can potentially weaken this relation.

4.3.1 The role of the shock structure

It is useful to look at how the shock structure affects investment regressions. In Table 5 we report regression coefficients and R^2 for different combinations of σ_ε and σ_η, keeping constant the total volatility of A_t. The second row corresponds to the baseline case of Table 4. In the third column, we report the fraction of variance due to the temporary shock. Here we keep all remaining parameters at their baseline level, since we want to focus on how variance parameters affect our result.

The first row of Table 5 shows an extreme case with no temporary shocks. In this case, the coefficient on q is larger than in our baseline and the coefficient on cash flow is actually negative. The last row of the table shows the opposite extreme, with only temporary shocks. Interestingly, also this row displays a larger coefficient on q. The coefficient on cash flow in this case is close to zero. So going to a one-shock model, worsens the model performance in terms of replicating investment regressions. In this case q and investment tend to comove simply because they are driven by the same shock. In these cases, we get close to the sufficient statistic result obtained in the one-shock linear model of Example 1. Example 1 has indeterminate implications for the coefficients, due to the perfect collinearity of q and cash flow. Here, the perfect collinearity result does not hold for two reasons: first, the model displays inertia so past values of x_t determine investment and q, which complicates the correlation structure of investment, q and cash flow; second, the model is non-linear. For these reasons, the bivariate coefficients are determinate even with a single shock, and, in particular, the model prefers to assign a large coefficient on q.[25]

The remaining rows of Table 5 illustrate intermediate cases in which both shocks are present. As argued above, both shocks increase investment but they have opposite effects on the wedge and that is what reduces the predictive power of q. So there is some intermediate mix of shocks that adds maximum noise to the information contained in average q and reduces the overall explanatory power of the investment regression. In the table this is visible in the non-monotone relation between the ratio σ_η^2/σ_A^2 and the R^2 of the regression.

While it is intuitive that mixing the two shocks affects the total explanatory power of investment regressions and reduces R^2, the quantitative effects on the two coefficients a_1 and a_2 are more complex to interpret, as they also depend on the magnitudes of the responses of investment, cash flow, and q to the underlying shocks. In particular, persistent shocks tend to affect more, in relative terms, q than investment, due to the forward looking nature of q and the presence of the financial constraint which dampens the response of investment (see Figure 2). [26] Persistent shocks lead to a smaller response of investment for a given response of q, when compared to temporary shocks. This is immediately visible in the mono-

[25]The results in this table may help reconcile our results with the results of Gomes (2001). In particular, although Gomes (2001) uses a different model of financial frictions, it is possible that his result—that q is almost a sufficient statistic for investment—could be driven by his one-shock structure.

[26]The same two reasons identified above (inertia and non-linearity) for one-shock models, explain why in the two-shock model the relative size of the two variances matter for the regression coefficients, unlike in the simple linearized model with no adjustment costs of Section 3, Example 2.

tone increase in the univariate coefficient with σ_η^2/σ_A^2. The effect on the bivariate coefficient a_1 is more complex as, at the same time, the presence of temporary shocks increases the coefficient on cash flow. Therefore, the relation between each of the coefficients a_1 and a_2 and the variance ratio σ_η^2/σ_A^2 is non-monotone.

The overall take out from Table 5 is that, given all other model parameters, the relative variance of temporary and persistent shocks matter for both the explanatory power and for the individual coefficients in investment regressions.

4.3.2 The role of parameters

To illustrate how the results depend on the model's parameters, we now experiment with different parameter configurations. In the exercises below we keep all other parameters fixed, i.e., we do not recalibrate the model. Alternative calibrations are discussed in Section 4.5. Table 6 documents the investment regression results for these alternative specifications.[27]

The first observation is that our main result is robust to a range of parameter values: financial frictions reduce the coefficient on average q, a_1, (which is equal to $1/\xi$ in the frictionless case) and produce a positive and sizeable value for the coefficient on cash flow, a_2. Notice also that for all parameter values explored in this table R^2 remains very high for both the multivariate regression and the univariate regression with average q.

Quantitatively, there are some interesting details. Two parameterizations stand out: higher values for $\hat{\beta}$ or low values for γ both yield a lower a_1 and a higher a_2, bringing the model implied regression coefficients closer to their empirical counterparts. The reason for these effects is that they magnify the forward-looking component of q, thus further breaking the link with current investment. However, notice that these values also produce a counterfactually high levels of q on average.[28] Furthermore, low values of θ or ρ_x and high values of σ_A, δ or ξ yield

[27]When we experiment with different values of $\hat{\beta}$ we vary β at the same time, keeping the difference between constant at $\hat{\beta} - \beta = 0.02$, as in the baseline.

[28]When we re-calibrate our model starting from $\hat{\beta} = 0.93$, the calibration compensates with a higher value of γ, to hit the average level of q and thus produces coefficients $a_1 = 0.20$ and

Table 6: Investment regressions: changing parameters

	a_1	a_2	R^2	Univariate q^a coefficient	R^2	Univariate CFK coefficient	R^2
Baseline	0.22	0.15	0.98	0.27	0.98	0.81	0.89
$\hat{\beta} = 0.910$	0.35	0.21	0.98	0.46	0.97	0.80	0.90
$\hat{\beta} = 0.930$	0.06	0.16	0.99	0.07	0.98	0.81	0.90
$\theta = 0.200$	0.24	0.21	0.98	0.32	0.97	0.79	0.91
$\theta = 0.400$	0.16	0.10	0.99	0.18	0.99	0.84	0.87
$\rho_x = 0.700$	0.24	0.20	0.98	0.32	0.98	0.74	0.91
$\sigma_A = 0.090$	0.24	0.18	0.97	0.30	0.96	0.76	0.84
$\sigma_A = 0.130$	0.20	0.13	0.99	0.23	0.99	0.84	0.91
$\delta = 0.015$	0.12	0.14	0.99	0.14	0.98	0.81	0.89
$\delta = 0.035$	0.31	0.17	0.98	0.39	0.97	0.82	0.89
$\xi = 1.500$	0.15	0.11	0.99	0.17	0.99	0.90	0.88
$\xi = 2.000$	0.27	0.18	0.98	0.34	0.97	0.75	0.90
$\gamma = 0.085$	0.08	0.16	0.99	0.10	0.98	0.81	0.90
$\gamma = 0.105$	0.33	0.18	0.98	0.41	0.97	0.81	0.89

higher values for both a_1 and a_2. Finally, it is interesting to note that our model implies that a_1 is increasing in ξ, which is the opposite of what happens with no financial frictions.

4.4 News shocks

We now turn to news shocks. Example 3 in Section 3 shows that in the case of no adjustment costs news shocks introduce additional noise in average q, thus reducing its predictive power. Here we want to investigate whether the same forces are at work in our full model with adjustment costs and see their quantitative implications.

Introducing news shocks increases the number of state variables, since we need to keep track of anticipated values of x_t. Therefore, to simplify computations,

$a_2 = 0.15$, which are very close to our baseline results.

Table 7: News shocks: calibration

	Parameters			Moments			
	δ	ξ	γ	$\mu(IK)$	$\frac{\sigma(IK)}{\sigma(CFK)}$	$\mu(q^a)$	$\sigma(q^a)$
Targets				0.17	0.98	2.5	0.97
No news (7 states)	0.0250	2.00	0.09	0.22	0.79	2.49	0.27
No news (2 states)	0.0200	2.00	0.10	0.22	0.94	2.24	0.33
$J = 1$	0.0275	3.00	0.08	0.21	0.86	2.39	0.42
$J = 2$	0.0225	3.50	0.08	0.20	0.85	2.24	0.45
$J = 3$	0.0225	3.50	0.08	0.19	0.91	2.67	0.59
$J = 4$	0.0275	3.50	0.08	0.19	0.95	2.48	0.59
$J = 5$	0.0300	3.50	0.08	0.19	0.97	2.50	0.63

we employ a coarser description of the permanent component of the productivity process, using a two-state Markov process for x_t. The stochastic process for A_t is specified and calibrated as in our baseline but we assume agents observe x_t J periods in advance as in Example 3 in Section 3. We experiment with $J = 1, 2, ..., 5$, re-calibrating the parameters δ, ξ and γ for each value of J. In Table 7 we report the calibrated parameters for each value of J. In the Table we also report our baseline calibration (no news, 7 states) and a calibration with no news and a 2 states Markov chain, which help to evaluate the effect of news on our results.

Table 7 shows that introducing news shocks improves the model's ability to match the empirical level of the investment rate, reducing the value of $\mu(IK)$, while producing similar values for $\sigma(IK)$ and $\mu(q^a)$. In the table, we also report the volatility of q^a (which is not used as a target for our calibration), and the table shows that introducing news improves the model's realism in this dimension. The analytical derivations in Section 3 (Example 3) suggest a reason for this: anticipated shocks seem to introduce an additional source of volatility in q^a.

Turning to investment regressions, Table 8 shows regression coefficients and R^2 for different values of J. The coefficient on q^a and the R^2 behave in a similar way as suggested by Example 3: increasing the horizon adds noise in q^a thus reducing the coefficient and the overall R^2. The cash flow coefficient goes down

Table 8: News shocks: investment regressions

	a_1	a_2	R^2
GH (1995)	0.033	0.24	
No news (7 states)	0.2047	0.1530	0.984
No news (2 states)	0.2434	0.0829	0.985
$J = 1$	0.1920	−0.0121	0.982
$J = 2$	0.1774	0.0161	0.974
$J = 3$	0.1417	0.0502	0.978
$J = 4$	0.1467	0.0628	0.976
$J = 5$	0.1394	0.0824	0.971

when going from no news to 1 period anticipation, and then increases monotonically in J.

Comparing the cases of no news and the case $J = 5$, the overall take away from Tables 7 and 8 is that news shocks improve the model's ability to match the observed behavior of investment, q and cash flow, both in terms of levels and volatility and in terms of the cross-correlations captured by investment regressions. The central intuition is that news shocks introduce a source of variation in q due to anticipated future shocks, which have little bearing on the contemporaneous movements in investment.

Due to the use of a 2 state Markov chain, the model with news does worse than the baseline in terms of the cash flow coefficient, so it is an open quesiton for future work whether increasing the state space and possibly using alternative models of anticipated news that economize on state variables can further improve the model's empirical performance.[29]

4.5 Targeting the mean finance premium

In this section we consider an alternative calibration in which we add to our target moments the mean finance premium, $\mu(fp)$. Following Bernanke et al. (1999) we

[29] See for example the information structure in Blanchard et al. (2013).

Table 9: Targeting the finance premium

Parameters:				
	δ	ξ	γ	
	0.1300	2.00	0.005	
Moments:				
	$\mu(IK)$	$\mu(q^a)$	$\sigma(IK)$	$\mu(fp)$
Target	0.17	2.5	0.111	0.020
Model	0.24	2.2	0.096	0.024
Investment regression:				
	a_1	a_2	R^2	
	0.19	0.22	0.99	

choose a target for μ_{fp} of 2%. In particular, we now choose the parameters δ, ξ, and γ to minimize the average squared percentage deviation of the four moments targeted. The main reason for this robustness check is to ensure that our results do not rely on an implausibly high value of the external finance premium.

Parameter values, model moments and regression results for this calibration are reported in Table 9. Overall, the results are similar to the baseline, except this calibration delivers a larger coefficient on a_2. In particular, a useful observation is that the model does not need to rely on a high external finance premium to produce a large wedge between average and marginal q.

5 Conclusions

The paper shows that financial frictions can help dynamic investment models move closer to the correlations observed in the data. The model in this paper is stylized, but we think our main conclusions on the role of different shocks will extend to more complex models. In particular, we think it is a promising avenue to build models where a substantial fraction of the volatility in q is associated to news about profitability relatively far in the future and where these news have relatively small effects on current investment decisions. By assuming risk neutrality, we have omitted an important source of volatility in asset prices, namely volatility in discount factors and risk premia. It is an important open question how these additional sources of volatility affect the correlations investigated here, especially because these factors are likely to correlate with the stringency of financial constraints for individual firms.

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A Appendix

A.1 Proofs for Section 2

Proof of Proposition 2. The envelope condition for K is

$$v\left(b,s\right)=\lambda\left(A\left(s\right)-G_{2}\left(K',K\right)-b\right).$$

Substituting in (9) and using time subscripts, we get

$$\lambda_{t}G_{1,t}=\hat{\beta}\lambda_{t}\mathbb{E}_{t}\left[b_{t+1}\right]+\beta\mathbb{E}_{t}\left[\lambda_{t+1}\left(A_{t+1}-G_{2,t+1}-b_{t+1}\right)\right], \tag{31}$$

which, rearranged, gives (13). Notice that (12) and $\mu_t \geq 0$ imply

$$\mathbb{E}_{t}\left[(\hat{\beta}\lambda_{t}-\beta\lambda_{t+1})b_{t+1}\right]\geq 0.$$

So (31) implies

$$G_{1,t}\lambda_{t}\geq\beta\mathbb{E}_{t}\left[\lambda_{t+1}\left(A_{t+1}-G_{2,t+1}\right)\right],$$

which yields the first inequality in (14). Moreover, (12) also implies

$$\mathbb{E}_{t}\left[\hat{\beta}\lambda_{t}\left(A_{t+1}-G_{2,t+1}-b_{t+1}\right)\right]\leq\mathbb{E}_{t}\left[\beta\lambda_{t+1}\left(A_{t+1}-G_{2,t+1}-b_{t+1}\right)\right],$$

which, together with (31), gives the second inequality in (14). $\square$

A.2 Proofs for Section 3

Proof of Lemma 1. Let $\tilde{B}$ be the space of bounded functions $f:\mathbf{S}/s^{r}\rightarrow[1,\infty)$. Define the map $T:\tilde{B}\rightarrow\tilde{B}$ as follows

$$Tf\left(s\right)=\left(1-\theta\right)\beta\frac{(1-\gamma)\mathbb{E}\left[f\left(s'\right)R\left(s'\right)|s,s'\neq s^{r}\right]+\gamma R\left(s^{r}\right)}{1-\theta\hat{\beta}\mathbb{E}\left[R(s')|s\right]}.$$

Let us first check that $Tf \in \tilde{B}$ if $f \in \tilde{B}$, so the map is well defined. Notice that conditions (18)-(19) and $\beta < \hat{\beta}$ imply that

$$\frac{(1-\theta)\,\beta\mathbb{E}\left[R\left(s'\right)|s\right]}{1-\theta\hat{\beta}\mathbb{E}\left[R\left(s'\right)|s\right]} > 1.$$

Then for any $f \in \tilde{B}$ we have

$$Tf(s) \geq \frac{(1-\theta)\,\beta\mathbb{E}\left[R\left(s'\right)|s\right]}{1-\theta\hat{\beta}\mathbb{E}\left[R\left(s'\right)|s\right]} > 1, \tag{32}$$

showing that $Tf\,(s) \geq 1$.

Next, we show that T satisfies Blackwell's sufficient conditions for a contraction. The monotonicity of T is easily established. To check that it satisfies the discounting property notice that if $f' = f + a$, then

$$Tf'\,(s) - Tf\,(s) = \frac{(1-\gamma)\,(1-\theta)\,\beta\mathbb{E}\left[R\left(s'\right)|s, s \neq s^r\right]}{1-\theta\hat{\beta}\mathbb{E}\left[R\left(s'\right)|s\right]}a < \zeta a,$$

where the inequality follows from assumption (20). Since T is a contraction a unique fixed point f exists. Set $\Lambda(s) = f(s)$ for all $s \neq s^r$. Inequality (32) shows that $\Lambda\,(s) > 1$ for all $s \neq s^r$, completing the proof.

□

Proof of Proposition 3. Let Λ be defined as in Lemma 1. We proceed by guessing and verifying that the value function has the form (16). Under this conjecture, the no-default condition (4) can be rewritten in the form

$$B'(s') \leq \theta R(s')K'.$$

Therefore, we can rewrite problem (2) as

$$\max_{C,K',B'(.)} C + \beta \sum_{s'} \pi\left(s'|s\right)\left[\Lambda\left(s'\right)\left(R\left(s'\right)K' - B'\left(s'\right)\right)\right]$$

$$
\begin{aligned}
\text{s.t.} \quad & C + K' \leq R(s)\,K - B + \hat{\beta}\sum_{s'} \pi(s'|s)B'(s'), \quad (\lambda) \\
& B'\left(s'\right) \leq \theta R\left(s'\right) K' \text{ for all } s', \quad \left(\mu\left(s'\right)\pi\left(s'|s\right)\right) \\
& C \geq 0, \quad (\tau_c) \\
& K \geq 0, \quad (\tau_k)
\end{aligned}
$$

where, in parenthesis, we report the Lagrange multiplier associated to each constraint. The multipliers of the no-default constraints are normalized by the probabilities $\pi\left(s'|s\right)$. The first-order conditions for this problem are

$$
\begin{aligned}
& 1 - \lambda + \tau_c = 0, \\
& \beta \mathbb{E}\left[\Lambda(s')R(s')|s\right] - \lambda + \theta \mathbb{E}\left[\mu(s')R(s')|s\right] + \tau_k = 0, \\
& -\beta\Lambda(s')\pi\left(s'|s\right) + \lambda\hat{\beta}\pi\left(s'|s\right) - \mu\left(s'\right)\pi\left(s'|s\right) = 0.
\end{aligned}
$$

We want to show that the values for C, K', B' in the statement of the proposition are optimal. It is immediate to check that they satisfy the problem's constraints. To show that they are optimal we need to show that $\tau_c = \lambda - 1 > 0$, $\tau_k = 0$, and $\mu\left(s'\right) > 0$ for all s'. Setting $\tau_k = 0$ the second and third first-order conditions give us

$$
\lambda = \frac{(1-\theta)\,\beta\mathbb{E}\left[\Lambda(s')R(s')\right]}{1 - \theta\hat{\beta}\mathbb{E}\left[R'(s')\right]}
$$

which, by construction, is equal to $\Lambda(s)$. Then we have

$$
\tau_c = \Lambda(s) - 1 > 0,
$$

which follows from Lemma 1,

$$
\mu\left(s'\right) = \hat{\beta}\Lambda(s) - \beta\Lambda\left(s'\right) > 0,
$$

which follows from condition (22). Substituting the optimal values in the objective function we obtain $\Lambda(s)\left(R(s)\,K - B\right)$ confirming our initial guess.

□

Proof of Proposition 5. First we derive the formula (28) for average q. From formula (26) we have

$$\begin{aligned}
\tilde{q}_t^a &= \left[\beta\left(1-\theta\right)\left(\gamma+\left(1-\gamma\right)\bar{\Lambda}\right)+\theta\hat{\beta}\right]\mathbb{E}_t\left[\tilde{A}_{t+1}\right] \\
&\quad +\beta\left(1-\theta\right)\left(1-\gamma\right)\bar{R}\mathbb{E}_t\left[\tilde{\Lambda}_{t+1}\right] \\
&= \left[\beta\left(1-\theta\right)\left(\gamma+\left(1-\gamma\right)\bar{\Lambda}\right)+\theta\hat{\beta}\right]x_{t+1} \\
&\quad +\beta\left(1-\theta\right)\left(1-\gamma\right)\bar{R}\bar{\Lambda}\mathbb{E}_t\left[\frac{1}{1-\theta\hat{\beta}\bar{R}}\sum_{j=0}^{\infty}\left(\frac{\left(1-\gamma\right)\bar{\Lambda}}{\gamma+\left(1-\gamma\right)\bar{\Lambda}}\right)^j\mathbb{E}_{t+1}\left[\frac{x_{t+1+j}}{\bar{R}}\right]\right],
\end{aligned}$$

where the last equality comes from formula (27) and the fact that x_{t+1} is known at time t and η_{t+1+j} is not known at t for all $j \geq 0$. Now simplify the second term using the dynamic equation for x_{t+1+j}:

$$x_{t+1+j} = \rho^j x_{t+1} + \sum_{j'=1}^{j}\rho^{j-j'}\varepsilon_{t+1+j'}.$$

By using this expression and simplify the algebra, we obtain:

$$\begin{aligned}
&\mathbb{E}_t\left[\frac{1}{1-\theta\hat{\beta}\bar{R}}\sum_{j=0}^{\infty}\left(\frac{\left(1-\gamma\right)\bar{\Lambda}}{\gamma+\left(1-\gamma\right)\bar{\Lambda}}\right)^j\mathbb{E}_{t+1}\left[x_{t+1+j}\right]\right] \\
&\qquad = \frac{1}{1-\theta\hat{\beta}\bar{R}}\frac{1}{1-\frac{(1-\gamma)\bar{\Lambda}\rho}{\gamma+(1-\gamma)\bar{\Lambda}}}x_{t+1} \\
&+\mathbb{E}_t\left[\frac{1}{1-\theta\hat{\beta}\bar{R}}\sum_{j=0}^{\infty}\left(\frac{\left(1-\gamma\right)\bar{\Lambda}}{\gamma+\left(1-\gamma\right)\bar{\Lambda}}\right)^j\sum_{j'=1}^{j}\rho^{j-j'}\varepsilon_{t+1+j'}\right].
\end{aligned}$$

Using the fact that $\mathbb{E}_t\left[\varepsilon_{t+1+j'}\right] = 0$ for all $j' > J-1$ and $\mathbb{E}_t\left[\varepsilon_{t+1+j'}\right] = \varepsilon_{t+1+j'}$ for

$j' \leq J-1$, the second term becomes

$$\mathbb{E}_t\left[\frac{1}{1-\theta\hat{\beta}\bar{R}}\sum_{j=0}^{\infty}\left(\frac{(1-\gamma)\,\bar{\Lambda}}{\gamma+(1-\gamma)\,\bar{\Lambda}}\right)^j\sum_{j'=1}^{j}\rho^{j-j'}\varepsilon_{t+1+j'}\right]$$

$$=\mathbb{E}_t\left[\frac{1}{1-\theta\hat{\beta}\bar{R}}\sum_{j'=1}^{\infty}\sum_{j=j'}^{\infty}\left(\frac{(1-\gamma)\,\bar{\Lambda}}{\gamma+(1-\gamma)\,\bar{\Lambda}}\right)^j\rho^{j-j'}\varepsilon_{t+1+j'}\right]$$

$$=\frac{1}{1-\theta\hat{\beta}\bar{R}}\sum_{j'=1}^{J-1}\left(\frac{(1-\gamma)\,\bar{\Lambda}}{\gamma+(1-\gamma)\,\bar{\Lambda}}\right)^{j'}\varepsilon_{t+1+j'}\frac{1}{1-\frac{(1-\gamma)\bar{\Lambda}\rho}{\gamma+(1-\gamma)\bar{\Lambda}}}.$$

This equality combined with the derivation for $\tilde{q}_t^a$ above implies (28).

Now we compute regression coefficients and R^2 for the regression of $I\tilde{K}_t$ on $\tilde{A}_t$ and $\tilde{q}_t^a$. Let $y_t = I\tilde{K}_t$ and $\mathbf{X}_t = \begin{bmatrix}\tilde{A}_t & \tilde{q}_t^a\end{bmatrix}$. To simplify the algebra write the equations for $I\tilde{K}_t$ and $\tilde{q}_t$ as follows:

$$I\tilde{K}_t = \alpha_{i1}\tilde{A}_t + \alpha_{i2}x_{t+1}$$
$$\tilde{q}_t^a = \alpha_q x_{t+1} + \tilde{\epsilon}_t$$

We can then compute

$$E[y_t\mathbf{X}_t] = \left[(\alpha_{i1}+\alpha_{i2}\rho)\,\sigma^2 + \alpha_1\sigma_\eta^2 \quad (\alpha_{i1}+\alpha_{i2}\rho)\,\alpha_q\rho\sigma^2 + \alpha_{i2}\alpha_q\tilde{\sigma}_\epsilon^2\right],$$

$$E[\mathbf{X}_t\mathbf{X}_t'] = \begin{bmatrix}\sigma^2+\sigma_\eta^2 & \alpha_q\rho\sigma^2\\ \alpha_q\rho\sigma^2 & \alpha_q^2\sigma^2+\tilde{\sigma}_\epsilon^2\end{bmatrix},$$

and

$$E[y_t^2] = (\alpha_{i1}+\alpha_{i2}\rho)^2\,\sigma^2 + \alpha_{i2}^2\tilde{\sigma}_\epsilon^2 + \alpha_{i1}\sigma_\eta^2,$$

where $\sigma^2 = var(x_t) = (1-\rho^2)\sigma_\epsilon^2$ and $\tilde{\sigma}_\epsilon^2 = var(\tilde{\epsilon}_t)$.

The coefficients on cash flow and $\tilde{q}_t^a$ are given by:

$$\frac{1}{\det(E[\mathbf{X}_t\mathbf{X}_t'])}\begin{bmatrix}\alpha_q^2\sigma^2+\tilde{\sigma}_\epsilon^2 & -\alpha_q\rho\sigma^2\\ -\alpha_q\rho\sigma^2 & \sigma^2+\sigma_\eta^2\end{bmatrix}E[y_t\mathbf{X}_t],$$

and, after some algebra, we get the coefficient on q^a, which is

$$\frac{\alpha_{i2}\alpha_q\left(\sigma_\eta^2\rho^2\sigma^2+\left(\sigma^2+\sigma_\eta^2\right)\sigma_\epsilon^2\right)}{\alpha_q^2\sigma^4(1-\rho^2)+\sigma_\eta^2(\alpha_q^2\sigma^2+\tilde{\sigma}_\epsilon^2)+\sigma^2\tilde{\sigma}_\epsilon^2},$$

and is immediately decreasing in $\tilde{\sigma}_\epsilon^2$. Similarly, we can derive the coefficient on $\tilde{A}_t$, which is

$$\frac{\tilde{\sigma}_\epsilon^2\left(\left(\alpha_{i1}+\alpha_{i2}\rho\right)\sigma^2+\alpha_{i1}\sigma_\eta^2\right)+\alpha_{i1}\alpha_q^2\sigma^2\left(\tilde{\sigma}_\epsilon^2+\sigma_\eta^2\right)}{\alpha_q^2\sigma^4(1-\rho^2)+\sigma_\eta^2(\alpha_q^2\sigma^2+\tilde{\sigma}_\epsilon^2)+\sigma^2\tilde{\sigma}_\epsilon^2}.$$

Rewrite this expression as

$$\frac{A_1+A_2\tilde{\sigma}_\epsilon^2}{A_3+A_4\tilde{\sigma}_\epsilon^2}=\frac{A_2}{A_4}+\frac{A_1A_4-A_2A_3}{A_4\left(A_3+A_4\tilde{\sigma}_\epsilon^2\right)},$$

where $A_1, A_2, A_3, A_4 > 0$. Direct algebra yields

$$A_1A_4-A_2A_3=-\alpha_q^2\sigma^2\rho\alpha_{i2}\sigma_\epsilon^2\left(\sigma^2+\sigma_\eta^2\right)-\alpha_q^2\sigma^2\alpha_{i2}\rho\sigma^2\rho^2\sigma_\eta^2<0.$$

Therefore the coefficient on cash flow is strictly increasing in $\tilde{\sigma}_\epsilon^2$.

Finally, the R^2 is

$$R^2=\frac{E[y_t\mathbf{X}_t]E[\mathbf{X}_t\mathbf{X}_t']^{-1}E[\mathbf{X}_t'y_t]}{E[y_t^2]},$$

which can be written as

$$R^2=\frac{B_1+B_2\tilde{\sigma}_\epsilon^2}{B_3+B_4\tilde{\sigma}_\epsilon^2}=\frac{B_2}{B_4}+\frac{B_1B_4-B_2B_3}{B_4\left(B_3+B_4\tilde{\sigma}_\epsilon^2\right)},$$

where $B_1, B_2, B_3, B_4 > 0$. In order to show that R^2 is decreasing in $\tilde{\sigma}_\epsilon^2$, we only need to show that $B_1B_4 - B_2B_3 > 0$. After some algebra we obtain

$$B_1B_4 - B_2B_3 =$$
$$\sigma_\eta^2\alpha_{i2}^2\rho^2\sigma^2\alpha_q^2\sigma_\eta^2\rho^2\sigma^2 + \left(\sigma^2 + \sigma_\eta^2\right)\alpha_{i2}^2\sigma_\epsilon^2\alpha_q^2\left(\sigma_\eta^2\rho^2\sigma^2 + \left(\sigma^2(1-\rho^2) + \sigma_\eta^2\right)\sigma_\epsilon^2 + \rho^2\sigma^2\left(\sigma^2 + \sigma_\eta^2\right)\right) > 0.$$

□

A.3 Numerical Algorithm

With slight abuse of notation, let $v(n, s)$ denote the value function as a function of net worth net of adjustment costs $n = A + 1 - \delta - b$ instead of as a function of b.

The main complication in the computation is that at each iteration we have to solve for an optimal state-contingent portfolio choice because the entrepreneur can choose $b'(s')$ for each s'. Let λ denote the multiplier on the budget constraint (6). The envelope condition implies $v_n(s, n) = 1 + \lambda$. Moreover

$$\begin{aligned} \hat{\beta}(1+\lambda) &\geq \beta v_n\left(A\left(s'\right) + 1 - \delta - b'\left(s'\right), s'\right) \\ v\left(A\left(s'\right) + 1 - \delta - b'\left(s'\right), s'\right) &\geq (1-\theta)\, v\left(A\left(s'\right) + 1 - \delta, s'\right). \end{aligned} \tag{33}$$

with at least one of the two inequalities holds with equality. These two equations determine $b'(s')$ as a function of λ. To determine k' we have:

$$G_1\left(k', 1\right)(1+\lambda) = \hat{\beta}\mathbb{E}\left[b'\left(s'\right)|s\right](1+\lambda) + \beta\mathbb{E}\left[v\left(A\left(s'\right) + 1 - \delta - b'\left(s'\right), s'\right)|s\right] \tag{34}$$

so k' is a function of λ. Notice that in order to solve for the optimal decisions $b(s')$ and k' we need to invert the first derivative: v_n. Numerical derivative is computationally time consuming and imprecise. Thus in each iteration of the algorithm, we solve for both v and v_n.

Lastly, to compute λ, we use

$$A(s) + 1 - \delta - b + \hat{\beta}\mathbb{E}\left[b'\left(s'\right)|s\right]k' - G\left(k', 1\right)\begin{cases} = 0 & \text{if } \lambda > 0 \\ > 0 & \text{if } \lambda = 0 \end{cases},$$

where $b'(s')$ an k' are determined from (33) and (34) given λ.

Financial Frictions, Investment and Tobin's Q: Online Appendix

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Abstract

In this online appendix, we show the existence of the value function under limited commitment defined in the main paper. First, we show that given a set of state-contingent borrowing constraints, the value function of the entrepreneurs is the unique solution to a Bellman equation and is continuous. Then, we show that the exist state-contingent borrowing limits such that, at the limits the entrepreneurs are indifferent between defaulting and not defaulting. In order to derive these results, we require the entrepreneurs to have a strictly lower discount rate than the lenders do, and every period the entrepreneurs are hit by I.I.D exit shocks.

1 Equilibrium Existence

In this appendix, we prove the existence of the value function with endogenous outside option. The proof proceeds as follow.

Step 1: Given exogenous state dependent borrowing limits $\{\bar{b}(s)\}_{s\in\mathcal{S}}$, $(B'(s') \leq \bar{b}(s')K'(s))$ we show that the value function $V(K,B,s;\bar{b})$ exists and solves the Bellman equation (2) in the main paper. Moreover, the value function is continuous in K, B, and $\bar{b}$.

Step 2: Then we use the Schauder's Fixed Point Theorem to show that there exist borrowing limits $\bar{b}$ such that

$$V(K,\bar{b}(s)K,s;\bar{b}) = V((1-\theta)K,0,s;\bar{b})$$

for all $K > 0$ and for all $s \in \mathcal{S}$.

Step 1 is shown in Subsection 1.1. Step 2 is shown in Subsection 1.2. Subsection 1.3 provides supporting results and detailed proofs.

1.1 Step 1: Existence and Properties of the Value Function

It is not straightforward to show the existence and properties of the value function since the Bellman operator in (2) is not a contraction mapping and the objective function is unbounded. Therefore, we rely on the homogeneity of the problem, as in Alvarez and Stokey (1998), and work directly with the sequence problem.

Towards establishing the existence and important properties of the value function, we need to make the following assumption on discount factors and exit probability.

Assumption 1. *$\beta < \hat{\beta}$ and $\gamma > 0$.*

Moreover, the adjustment cost function satisfies the following assumption.

Assumption 2. *$G(K',K)$ is linearly homogenous, weakly convex, strictly increasing in K' and strictly decreasing in K. Moreover, G satisfies:*

A1) For each $K > 0$, there exist $K' > 0$ such that $G(K',K) = 0$.

A2) G satisfies the Inada condition: for each $K > 0$,

$$\lim_{K'\to 0} G_1(K',K) = 0, \;\; \textit{and} \;\; \lim_{K'\to\infty} G_1(K',K) = \infty.$$

We partition the state space $\mathcal{S}$ into two subsets. Let $\iota(s)$ denote the indicator whether the entrepreneur does not retire at s, i.e. $\iota(s) = 0$ if the entrepreneur retires at s. Let $\mathcal{S}^* = \{s \in \mathcal{S} : \iota(s) = 1\}$.

For each $s \in \mathcal{S}^*$, π^* is the transition probabilities conditional on not exiting next period:

$$\pi^*\left(s'|s\right) = \begin{cases} \frac{\pi(s'|s)}{1-\gamma} & \text{if } s' \in \mathcal{S}^* \\ \frac{\pi(s'|s)}{\gamma} & \text{if } s' \in \mathcal{S}\backslash\mathcal{S}^* \end{cases}.$$

For an history s^t such that $s^{t-1} \in \mathcal{S}^{*,t-1}$:

$$\pi^*(s^t) = \frac{\pi^*(s^{t-1})}{(1-\gamma)^{t-1}}\pi^*(s_t|s_{t-1}).$$

To be consistent with the notations for the states at which the entrepreneurs does not retire ($\mathcal{S}^*$), we assume that for $s \in \mathcal{S}\backslash\mathcal{S}^*$, the present discounted value of cash flow per unit of capital to the entrepreneur is $A(s) > 0$. This assumption is more general than the one assumed in the main paper, in which at s^r, $A(s^r) = 0$, but the total value include the scrapping value of capital $G(0,1) > 0$.

It is easier to work with bond holdings of the entrepreneurs instead of debt holdings as in the main paper (bond holding is minus debt holding). Given the debt limits, $\left\{\bar{b}(s)\right\}_{s\in\mathcal{S}}$, with $\bar{b}(s) < A(s)$, for $s \in \mathcal{S}\backslash\mathcal{S}^*$, an entrepreneur starts with initial capital K_0 and initial debt B_0 and make state-contingent decision on consumption, capital, and bond holdings, $\left\{C_t\left(s^t\right), K_{t+1}\left(s^t\right), B_{t+1}\left(s^{t+1}\right)\right\}_{t,s^t}$ to solve

$$V(K_0, B_0, s_0; \bar{b}) = \sup_{\left\{C_t(s^t), K_{t+1}(s^t), B_{t+1}\left(s^{t+1}\right)\right\}_{t,s^t}} u\left(\left\{C_t\left(s^t\right), K_{t+1}\left(s^t\right), B_{t+1}\left(s^{t+1}\right)\right\}_{t,s^t}\right) \tag{1}$$

$$u = \sum_{t=0}^{\infty} \sum_{s^t:s^{t-1}\in\mathcal{S}^{*,t-1}} \beta^t(1-\gamma)^{t-1}\pi^*\left(s^t\right)\left(\begin{array}{c}(1-\gamma)\mathbf{1}_{\{\iota(s_t)=1\}}C_t\left(s^t\right)+ \\ \gamma\mathbf{1}_{\{\iota(s_t)=0\}}V_0\left(K_t(s^{t-1}), B_t\left(s^t\right), s_t\right)\end{array}\right)$$

$s.t.$

$$C_t\left(s^t\right) + G\left(K_{t+1}\left(s^t\right), K_t\left(s^{t-1}\right)\right) \leq A\left(s_t\right)K_t\left(s^{t-1}\right) + B_t\left(s^t\right) - \hat{\beta}\sum_{s^{t+1}|s^t}\pi\left(s^{t+1}|s^t\right)B_{t+1}\left(s^{t+1}\right) \tag{2}$$

$$C_t(s^t) \geq 0, K_{t+1}\left(s^t\right) > 0, \tag{3}$$

$$B_{t+1}(s^{t+1}) \geq -\bar{b}\left(s_{t+1}\right)K_{t+1}\left(s^t\right), \tag{4}$$

where for each $s \in \mathcal{S}\backslash\mathcal{S}^*$:

$$V_0\left(K, B, s\right) = A\left(s\right)K + B.$$

We would like to show that V is continuous in both K_0, B_0 and $\bar{b}$. In addition, the value $V(K_0, B_0, s_0; \bar{b})$ is also finite, and can be achieved by some allocation, i.e. the sup operator in (1) can be replaced by the max operator.

The Bellman equation associated to the optimization problem of the entrepreneurs is, for $s \in S^*$:

$$V(K,B,s) = \sup_{C,K',B'(s')} (1-\gamma)\,C + \beta(1-\gamma)\sum_{s'\in S^*}\pi^*(s'|s)V(K',B',s') + \beta\gamma\sum_{s'\in\mathcal{S}\setminus\mathcal{S}^*}\pi^*(s'|s)V_0(K',B'(s'),s'). \tag{5}$$

subject to

$$C \geq 0, K' > 0, \tag{6}$$

and

$$B'(s') \geq -\bar{b}(s')K' \tag{7}$$

and

$$C + G(K',K) \leq A(s)K + B - \hat{\beta}\sum_{s'\in S^*}\pi(s'|s)B'(s'). \tag{8}$$

This Bellman equation does not give rise to a contraction mapping. Moreover, the objective function is not bounded. Therefore, we cannot directly apply the Contraction Mapping Theorem, e.g. as in Lucas and Stokey (1989). Instead, we directly work with the value function arises from the sequence problem (1) in order to show the existence of a solution to this Bellman equation and to derive the properties of the solution.[1]

First, to show that the value function $V(K,B,s;\bar{b})$ is bounded, we need to define the first best problem as following:

$$V^{FB}(K_0,B_0,s_0) = \max_{\left\{C_t(s^t),K_{t+1}(s^t),B_{t+1}\left(s^{t+1}\right)\right\}_{t,s^t}} \tag{9}$$

$$\sum_{t=0}^{\infty}\sum_{s^t:s^{t-1}\in\mathcal{S}^{*,t-1}}\hat{\beta}^t(1-\gamma)^{t-1}\pi^*\left(s^t\right)\begin{pmatrix}(1-\gamma)\mathbf{1}_{\{\iota(s_t)=1\}}C_t\left(s^t\right)+\\ \gamma\mathbf{1}_{\{\iota(s_t)=0\}}V_0^{FB}\left(K_t(s^{t-1}),s_t\right)\end{pmatrix}$$

$s.t.$

$$C_t\left(s^t\right)+G\left(K_{t+1}\left(s^t\right),K_t\left(s^{t-1}\right)\right)\leq A\left(s_t\right)K_t\left(s^{t-1}\right),\ \text{for all } t>0.$$

$$K_{t+1}\left(s^t\right)>0,$$

[1]Different directions to tackle Bellman equations with unbounded returns and value functions can be found in Alvarez and Stokey (1998),Van and Morhaim (2002), and Rincon-Zapatero and Rodriguez-Palmero (2003)

and

$$C_0(s_0) + G(K_1, K_0) \leq A(s_0)K_0 + B_0.$$

Lastly, for $s \in \mathcal{S} \backslash \mathcal{S}^*$

$$V_0^{FB}(K, s) = A(s)K.$$

Notice that, relative to the second best problem, $C_t(s^t)$ can be negative. This first best problem corresponds to the classical problem of investment under adjustment cost in Hayashi (1982), except for the exit shock that forces the entrepreneurs to liquidate.

We assume that the first best problem yields finite objective.

Assumption 3. *The value of the first best problem is finite.*

It is easy to show that the value function of the first best problem is linear in K, B in particular $V^{FB}(K, B, s) = v^{FB}(s)K + B$. Moreover $v^{FB}(s)$ is determined by the following Bellman equation

For $s \in \mathcal{S}^*$

$$\begin{aligned} v^{FB}(s) = &\max_{k' > 0} A(s) - G(k', 1) + \hat{\beta}(1 - \gamma) \sum_{s' \in S^*} \pi^*(s'|s) v^{FB}(s') k' \\ &+ \hat{\beta}\gamma \sum_{s' \in \mathcal{S} \backslash \mathcal{S}^*} \pi^*(s'|s) A(s') k'. \end{aligned} \tag{10}$$

The lemma below show that when the first best problem yields finite value, the value $V(K_0, B_0, s_0; \bar{b})$ is also finite.

Lemma 1. *The value of the first best problem always exceed (or equal) the value of the entrepreneurs' optimization problem* (1). *In particular,*

$$V(K, B, s; \bar{b}) \leq V^{FB}(K, B, s)$$

for all K, B.

Proof. See Subsection 1.3. □

This lemma together with Assumption 3 implies that V is finite. We now establish additional properties of V.

We notice that the support of V has a particular form. We define the support of V, $X_{\bar{b}}$ as following.

Definition 1. $X_{\bar{b}}$ denote the set of (s_0, K_0, B_0) with $K_0 > 0$ and such that there exists a sequence

$$\left\{ C_t(s^t), K_{t+1}(s^t), B_{t+1}(s^{t+1}) \right\}_{t=0}^{\infty}$$

with (2), (3), and (4) are satisfied for all t, s^t.

It is easy to verify that $X_{\bar{b}}$ has the following properties.

Lemma 2. *$X_{\bar{b}}$ is a cone, i.e. for each $x = (s, K, B) \in X_{\bar{b}}$ and $\lambda > 0$*

$$(s, \lambda K, \lambda B) \in X_{\bar{b}}.$$

In addition

1. $x = (s, K, B) \in X_{\bar{b}}$ if and only if there exist C and K' and $(B'(s'))_{s' \in \mathcal{S}}$ such that $(s', K', B'(s')) \in X_{\bar{b}}$ and (6),(7), and (8) are satisfied.

2. $x = (s, K, B) \in X_{\bar{b}}$ if and only if $B(s) \geq \underline{b}(s)K(s)$ for some $\underline{b}(s)$.

3. When $B = \underline{b}(s)K$, $V(K, B, s) \geq \underline{v}(s)K$ for some $\underline{v}(s) > 0$.

Proof. See Subsection 1.3. □

Because $X_{\bar{b}}$ is a cone,

$$\left\{K, B : (s, K, B) \in X_{\bar{b}}\right\} = \left\{\lambda\,(1, b) : \lambda > 0 \text{ and } (s, 1, b) \in X_{\bar{b}}\right\}.$$

For convenience, we also use $X^0_{\bar{b}}$ as:

$$X^0_{\bar{b}} = \left\{(s, b) : (s, 1, b) \in X_{\bar{b}}\right\}.$$

Now we return analyzing the value function of the entrepreneurs. One difficulty in working with the value function V directly is that a sequence

$$\left\{C_t(s^t), K_{t+1}(s^t), B_{t+1}(s^{t+1})\right\}_{t=0}^{\infty}$$

that satisfy (2), (3), and (4) can potentially have $\frac{K_{t+1}}{K_t} \to 0$ and $\frac{B_{t+1}}{K_{t+1}} \to \infty$ (which makes it difficult to show that V is continuous, and the supremum can be attained). Therefore, we work with an alternative optimization problem in which

$$\frac{B_{t+1}(s^{t+1})}{K_{t+1}(s^t)} \leq M, \tag{11}$$

and

$$\overline{M} \geq \frac{K_{t+1}(s^t)}{K_t(s^t)} \geq \underline{M}, \tag{12}$$

where $M > 0$. Later we will show that these bounds do not bind as $M, \overline{M} \to \infty$, and $\underline{M} \to$

0. It is easy to see that $X_{\bar{b}}$ and $X_{\bar{b}}^0$ do not change with these two additional constraints if $M, \overline{M}$ are sufficiently large and $\underline{M}$ sufficiently small.

Let $V^M(s_0, K_0, B_0; \bar{b}, M, \underline{M}, \overline{M})$ denote the value function of the sequence problem (1) with the additional constraints (11) and (12).

Lemma 3. *V^M has the following properties:*

1.V^M is upper semi-continuous in (K, B).

2. V^M is linearly homogenous, strictly increasing and weakly concave in (K, B).

Proof. See Subsection 1.3. □

Due to homogeneity, $V^M(K, B, s) = Kv^M(\frac{B}{K}, s)$ (to simplify the notations we dropped the arguments $M, \overline{M}, \underline{M}$ in V^M). The following lemma shows that v^M must satisfy a Bellman equation.

Lemma 4. *A upper semi-continuous function $v : X_{\bar{b}}^0 \to \mathbb{R}^+$, such that $Kv(\frac{B}{K}, s)$ is the value function V^M (of the sequence problem (1) with the additional constraints (11) and (12)) if and only if v satisfies the Bellman equation:*

$$v(b,s) = \sup_{c,k',b'(s')} c + \\ + \beta k' \left(\gamma \sum_{s' \in \mathcal{S} \setminus S^*} \pi^*(s'|s) \left(A(s') - \bar{b}(s') \right) + (1-\gamma) \sum_{s' \in S^*} \pi^*(s'|s) v(b'(s'), s') \right) \tag{13}$$

subject to

$$c \geq 0,$$

and

$$k' \in \left[\underline{M}, \overline{M} \right] \tag{14}$$

and, for $s' \in \mathcal{S}^*$

$$b'(s') \in \left[\max\left\{ -\bar{b}(s'), \underline{b}(s') \right\}, M \right] \tag{15}$$

and

$$c + G(k', 1) + \hat{\beta}(1-\gamma)k' \sum_{s' \in S^*} \pi^*(s'|s) b'(s') - \hat{\beta}\gamma k' \sum_{s' \in \mathcal{S} \setminus S^*} \pi^*(s'|s) \bar{b}(s') \\ \leq A(s) + b. \tag{16}$$

Moreover,

1. *The* $\sup$ *operator in* (13) *can be attained, i.e. it can be replaced by* $\max$.
2. V^M *can be achieved by some consumption, investment, and borrowing plan* $\{C_t, K_{t+1}, B_{t+1}\}$.

Proof. See Subsection 1.3. □

The proof of this lemma shows that V^M can be achieved.

In the following two lemmas we show that if M is sufficiently high, the upper bound on $b'(s')$ in the constraint (11) and the bounds on k' in the constraint (14) do not bind.

Lemma 5. *Given* $\left(c, k', (b'(s'))_{s' \in \mathcal{S}}\right)$ *that solves of the optimization problem* (13)*. Fix* M*, when* $\underline{M}$ *and* $\overline{M}$ *are sufficiently small and sufficiently large respectively these bound do not bind, i.e.*

$$\underline{M} < k' < \overline{M}.$$

Proof. See Subsection 1.3. □

Lemma 6. *Given* $\left(c, k', (b'(s'))_{s' \in \mathcal{S}}\right)$ *that solves of the optimization problem* (13)*. When* M *is sufficiently large, and* $\underline{M}$ *and* $\overline{M}$ *are chosen as in Lemma 5 (as functions of* M*) so that the constraints* (14) *on* k' *do not bind, the upper bound on* $b'(s')$*, equation* (15)*, does not bind.*

Proof. See Subsection 1.3. □

Theorem 1. *Let* $M, \underline{M}, \overline{M}$ *denote the bound chosen as in Lemmas 5 and 6 such that these bounds do not bind in the Bellman equation* (13)*, then the function* $Kv^M(\frac{B}{K}, s)$ *is the value function of the initial optimization problem of the entrepreneurs* (1)*. Consequently, the* $\sup$ *operator in* (1) *can be attained.*

Proof. Because the bounds on k and b' do not bind, by Lemma 4, $Kv^M(\frac{B}{K}, s)$ can be achieved by a sequence $\left\{C_t^*, K_{t+1}^*, B_{t+1}^*\right\}$ such that for all t, s^t

$$\frac{B_{t+1}^*(s^{t+1})}{K_{t+1}^*(s^t)} < M$$

and

$$\overline{M} > \frac{K_{t+1}^*(s^t)}{K_t^*(s^t)} > \underline{M}.$$

Therefore, the sequence is a local maximum to the problem of the entrepreneurs (1). Since this is a concave maximization problem, it is also a global maximum. □

Lemma 7. *The value function* $V(K, B, s; \bar{b})$ *is continuous in* (K, B)*.*

Proof. We just have to show that the solution $v(b, s; \bar{b})$ to Bellman equation (13) without the exogenous bounds on k' and b' is continuous in b. Indeed, for $\Delta > 0$ and sufficiently small. Let $(k', b'(s'))$ denote the maximizer at (b, s). By reducing k', we can show that $v(b - \Delta, s) \geq v(b, s) - const * \Delta$. Moreover, $v(b - \Delta, s) \leq v(b, s) - \Delta$. Therefore $\lim_{\tilde{b} \uparrow b} v(\tilde{b}, s) = v(b, s)$. Similarly, we can show that $\lim_{\tilde{b} \downarrow b} v(\tilde{b}, s) = v(b, s)$. □

Lemma 8. *The value function $V(K, B, s; \bar{b})$ is continuous in $\bar{b}$.*

Proof. Similarly to the proof of Lemma 3, we can show that $V^M(K, B, s; \bar{b})$ is continuous in $\bar{b}$. We can choose $M, \overline{M}, \underline{M}$ as in Lemmas 5 and 6 locally independent of $\bar{b}$. So $V^M(K, B, s; \bar{b}) = V(K, B, s; \bar{b})$ is continuous in $\bar{b}$. □

1.2 Step 2: Existence of Endogenous Borrowing Limits

Given the existence and properties of the value function $V(K, B, s; \bar{b})$, we show that there exists $\bar{b}$ such that it is consistent with the value function, i.e. at $b = \bar{b}$, the entrepreneurs are indifferent between defaulting and not defaulting. In order to show the existence of such $\bar{b}$, we use the Schauder Fixed Point Theorem.[2]

Let $\mathcal{B} = \left\{ \bar{b}(s)_{s \in \mathcal{S}^*} : \bar{b}(s) \in \left[0, v^{FB}(s)\right] \right\}$.

Definition 2. Define the function $T : \mathcal{B} \to \mathbb{R}^{S^*}$ as following. For each $\bar{b}$ defined over $\mathcal{S}^*$, we extend $\bar{b}$ to $\mathcal{S}$ by defining $\bar{b}(s) = \theta A(s)$ for $s \in \mathcal{S} \backslash \mathcal{S}^*$, and

$$T(\bar{b})(s) = \sup \left\{ b : (1, -b, s) \in X_{\bar{b}} \text{ and } v(-b, s; \bar{b}) \geq (1 - \theta) v(0, s; \bar{b}) \right\}.$$

Lemma 9. *The function T satisfies $T(\mathcal{B}) \subset \mathcal{B}$ and is continuous.*

Proof. If $\left(1, -v^{FB}(s), s\right) \in X_{\bar{b}}$, $v(-v^{FB}(s), s; \bar{b}) \leq v^{FB}(s) - v^{FB}(s) = 0 < (1 - \theta) v(0, s; \bar{b})$. Therefore, $T(\bar{b})(s) \leq v^{FB}(s)$, i.e. $T(\mathcal{B}) \subset \mathcal{B}$. The continuity of T comes from the continuity of $v(b, s; \bar{b})$ in $\bar{b}$. □

Theorem 2. *There exists $b^* \in \mathcal{B}$ such that, for all $K > 0$*

$$V\left(K, -b^* K, s; b^*\right) \geq V\left(\left(1 - \theta\right) K, 0, s; b^*\right), \tag{17}$$

with strict equality if there exists $(K, B, s) \in X_{b^*}$ such that

$$V(K, B, s; b^*) \leq V\left(\left(1 - \theta\right) K, 0, s; b^*\right).$$

[2]Similar proof technique can be found in Krueger and Uhlig (2006)

Proof. Applying the Schauder's Fixed Point Theorem for T, there exists b^*, such that $T(b^*) = b^*$. We show that b^* has the desired property. Indeed, (17) follows immediately from the definition of T and from the fact that V is continuous in B. Moreover, if there exists $(K, B, s) \in X_{b^*}$ such that

$$V(K, B, s) \leq V((1-\theta)\,K, 0, s; b^*).$$

Given that $V(K, 0, s; b^*) > V((1-\theta)K, 0, s; b^*)$, by Intermediate Value Theorem and continuity of V, there exists $\tilde{B} \leq 0$ such that

$$V(K, \tilde{B}, s; b^*) = V((1-\theta)\,K, 0, s; b^*).$$

Because V is increasing in B, $\tilde{B} = -b^* K$. $\square$

1.3 Supporting Results

Proof of Lemma 1. Consider the sequence of consumption $\{\hat{C}_t(s^t)\}$ generated by a consumption, investment, and borrowing plan of the entrepreneur $\{C_t, K_{t+1}, B_{t+1}\}$: and for each $s^t \in \mathcal{S}^{*t}$

$$\hat{C}_t(s^t) = C_t(s^t) - B_t(s^t) + \hat{\beta} \sum_{s_{t+1} \in \mathcal{S}} \pi(s^{t+1}|s^t) B_t(s^{t+1}).$$

Then, from the entrepreneur's budget constraint:

$$\hat{C}_t + G(K_{t+1}(s^t), K_t(s^{t-1})) \leq A(s_t) K_t(s^{t-1}).$$

Moreover, because $C_t \geq 0$,

$$\begin{aligned}
&\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^{*,t}} \hat{\beta}^t (1-\gamma)^{t-1} \pi^*\left(s^t\right) (1-\gamma) \hat{C}_t\left(s^t\right) \\
&= \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^{*,t}} \hat{\beta}^t (1-\gamma)^{t-1} \pi^*\left(s^t\right) (1-\gamma) C_t\left(s^t\right) \\
&- B_0 + \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^{*,t-1} \times (\mathcal{S} \setminus \mathcal{S}^*)} \hat{\beta}^t (1-\gamma)^{t-1} \pi^*\left(s^t\right) \gamma B_t\left(s^t\right) \\
&\geq \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^{*,t}} \beta^t (1-\gamma)^{t-1} \pi^*\left(s^t\right) (1-\gamma) C_t\left(s^t\right), \\
&- B_0 + \sum_{t=1}^{\infty} \sum_{s^t \in \mathcal{S}^{*,t-1} \times (\mathcal{S} \setminus \mathcal{S}^*)} \hat{\beta}^t (1-\gamma)^{t-1} \pi^*\left(s^t\right) \gamma B_t\left(s^t\right)
\end{aligned}$$

because $C_t \geq 0$. Therefore

$$\sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^{*,t}} \beta^t (1-\gamma)^{t-1} \pi^*(s^t)(1-\gamma)C_t(s^t)$$
$$\leq \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^{*,t}} \hat{\beta}^t (1-\gamma)^{t-1} \pi^*(s^t)(1-\gamma)\hat{C}_t(s^t) + B_0$$
$$- \sum_{t=1}^{\infty} \sum_{s^t \in \mathcal{S}^{*,t-1} \times (\mathcal{S} \backslash \mathcal{S}^*)} \hat{\beta}^t (1-\gamma)^{t-1} \pi^*(s^t)\gamma B_t(s^t) \tag{18}$$

Similarly, for $s_t \in \mathcal{S} \backslash \mathcal{S}^*$, because $V_0\left(K_t(s^{t-1}), B_t(s^t), s_t\right) \geq 0$,

$$\sum_{t=1}^{\infty} \sum_{s^t \in \mathcal{S}^{*,t}} \beta^t (1-\gamma)^{t-1} \pi^*(s^t)\gamma V_0\left(K_t(s^{t-1}), B_t(s^t), s_t\right).$$
$$\leq \sum_{t=1}^{\infty} \sum_{s^t \in \mathcal{S}^{*,t-1} \times (\mathcal{S} \backslash \mathcal{S}^*)} \hat{\beta}^t (1-\gamma)^{t-1} \pi^*(s^t)\gamma V_0\left(K_t(s^{t-1}), B_t(s^t), s_t\right)$$
$$= \sum_{t=1}^{\infty} \sum_{s^t \in \mathcal{S}^{*,t-1} \times (\mathcal{S} \backslash \mathcal{S}^*)} \hat{\beta}^t (1-\gamma)^{t-1} \pi^*(s^t)\gamma\left(V_0^{FB}\left(K_t(s^{t-1}), s_t\right) + B_t(s^t)\right) \tag{19}$$

Adding up side-by-side, (18) and (19), we obtain

$$u(\{C_t, K_{t+1}, B_{t+1}\}) \leq V^{FB}(K_0, B_0, s_0),$$

for all $\{C_t, K_{t+1}, B_{t+1}\}$ satisfying the constraints on the entrepreneurs. Therefore $V(K_0, B_0, s_0; \bar{b}) \leq V^{FB}(K_0, B_0, s_0)$. □

Proof of Lemma 2. The first property is immediate because G is linearly homogenous. Let $x = (s_0, K_0, B_0) \in X_{\bar{b}}$. By definition, $K_0 > 0$ and there exists a sequence

$$\left\{C_t(s^t), K_{t+1}(s^t), B_{t+1}(s^{t+1})\right\}_{t=0}^{\infty}$$

with (2), (3), and (4) satisfied for all t, s^t. It is also immediate that $(s', K', B'(s')) = (s_1, K_1(s_1), B_1(s_1)) \in X_{\bar{b}}$ and (6),(7), and (8) are satisfied. The reverse statement is also immediate.

It is also easy to see that if $(s, 1, b) \in X_{\bar{b}}$, then $(s, 1, b') \in X_{\bar{b}}$ for all $b' > b$. In addition, for $s \in \mathcal{S}^*$, $(s, 1, -A(s)) \in X_{\bar{b}}$, with $B_{t+1}(s^{t+1}) = 0$ for all $t \geq 0$, $K_{t+1}(s^t)$ is determined such that

$$G(K_{t+1}, K_t) = 0,$$

and $C_t(s^t) = 0$. For $s \in \mathcal{S} \backslash \mathcal{S}^*$: $(s, 1, b) \in X_{\bar{b}}$ if and only if $b \geq -\bar{b}(s)$.

Let $\underline{b}(s) = \inf\{b : (s,1,b) \in X_{\bar{b}}\}$. So $(s,1,b) \in X_{\bar{b}}$ for all $b > \underline{b}(s)$, and $\underline{b}(s) \leq \max\{-A(s), -\bar{b}(s)\}$. For $s \in \mathcal{S}\backslash\mathcal{S}^*$: $\underline{b}(s) = -\bar{b}(s)$. For $s \in \mathcal{S}^*$, $\bar{b}(s)$ is determined by

$$\underline{b}(s) = \inf_{\substack{k' > 0 \\ (s',1,b'(s')) \in X_{\bar{b}} \\ b'(s') \geq -\bar{b}(s')}} G(k',1) + k'\hat{\beta}\sum_{s'|s} \pi\left(s'|s\right) b'\left(s'\right) - A(s).$$

or equivalently for each $s \in \mathcal{S}^*$,

$$\underline{b}(s) = \inf_{k'>0} G(k',1) + k'\hat{\beta}\sum_{s'|s} \pi\left(s'|s\right) \max\left\{\underline{b}\left(s'\right), -\bar{b}(s')\right\} - A(s). \tag{20}$$

Lemma 10 below shows that $\{\underline{b}(s)\}_{s\in\mathcal{S}^*}$ is the smallest solution to the Bellman equation (20). In addition, for each s, there exists $\underline{k}(s) > 0$ such that (20) holds at $k' = \underline{k}(s)$, i.e. the inf can be achieved.

Given that inf can be achieved, $(s,1,\underline{b}(s)) \in X_{\bar{b}}$. Moreover,

$$V(1,\underline{b}(s),s) \geq \beta\gamma\underline{k}(s) \sum_{s'\in\mathcal{S}\backslash\mathcal{S}^*} \pi(s'|s)(A(s') - \bar{b}(s')) > 0. \tag{21}$$

□

Lemma 10. *$(\underline{b}(s'))_{s\in\mathcal{S}}$ ($\underline{b}(s) = -\bar{b}(s)$ for all $s \in \mathcal{S}\backslash\mathcal{S}^*$) is the smallest solution to the system of equation (20). Moreover the* inf *is achieved, i.e., it can be replaced by* min.

Proof. Let $\underline{b}^n(s)$ be determined recursively as $\underline{b}^0(s) = -\bar{b}(s')$ and

$$\underline{b}^{n+1}(s) = \inf_{k'>0} G(k',1) + k'\hat{\beta}\sum_{s'|s} \pi\left(s'|s\right) \max\left\{\underline{b}^n\left(s'\right), -\bar{b}(s')\right\} - A(s).$$

For any $\tilde{b}(s)$ satisfies (20), it is easy to show that $\underline{b}^n(s) < \tilde{b}(s)$ for all $s \in \mathcal{S}^*$. Therefore $\lim_{n\to\infty} \bar{b}^n = \underline{b}(s)$ is the smallest solution to (20). The inf can be achieved because G is convex and satisfies that Inada conditions. Let $\underline{k}(s)$ denote the minimizer from (20). □

Proof of Lemma 3. Proof of Part 1: Consider a sequence $\{(K_n, B_n)\} \to (K, B)$, and $(K_n, B_n), (K, B) \in X_{\bar{b}}$, we show that

$$\limsup_{n\to\infty} V^M(K_n, B_n, s) \leq V^M(K, B, s). \tag{22}$$

Indeed for each $\epsilon > 0$, there exists a sequence

$$\left\{C_t^n(s^t), K_{t+1}^n(s^t), B_{t+1}^n(s^{t+1})\right\}_{t=0}^{\infty}$$

that satisfies (2), (3), (4) and (11) and (12) such that

$$u\left(\left\{C_t^n(s^t), K_{t+1}^n(s^t), B_{t+1}^n(s^{t+1})\right\}_{t=0}^{\infty}\right) > V^M(K_n, B_n, s) - \epsilon.$$

Now, because of (11) and (12), for $t > 0$

$$\overline{M}^t K_n \geq K_t \geq K_n \underline{M}^t$$

and

$$-\bar{b}(s_t)K_t \leq B_t \leq MK_t.$$

Therefore, we can find a subsequence $\{n_k\}$ such that

$$\left\{C_t^{n_k}(s^t), K_{t+1}^{n_k}(s^t), B_{t+1}^{n_k}(s^{t+1})\right\}_{t=0}^{\infty} \to \left\{C_t(s^t), K_{t+1}(s^t), B_{t+1}(s^{t+1})\right\}_{t=0}^{\infty}, \tag{23}$$

for each t. First, this implies, $K_0 = K$ and $B_0 = B$. Second, $\{C_t(s^t), K_{t+1}(s^t), B_{t+1}(s^{t+1})\}_{t=0}^{\infty}$ satisfies (2), (3), (4) and (11) and (12). Therefore,

$$u\left(\left\{C_t(s^t), K_{t+1}(s^t), B_{t+1}(s^{t+1})\right\}_{t=0}^{\infty}\right) \leq V^M(K, B, s).$$

For each $T > 0$,

$$\begin{aligned}
V^{FB}(K_{n_k}) + B_{n_k} &\geq \sum_{t=0}^{\infty} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \hat{\beta}^t (1-\gamma)^{t-1} \pi^*\left(s^t\right) \begin{pmatrix} (1-\gamma)\mathbf{1}_{\{\iota(s_t)=1\}} C_t^{n_k}\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0\left(K_t^{n_k}(s^{t-1}), B_t^{n_k}\left(s^t\right), s_t\right) \end{pmatrix} \\
&\geq \sum_{t=0}^{\infty} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \hat{\beta}^t (1-\gamma)^{t-1} \pi^*\left(s^t\right) \left(\gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0\left(K_t^{n_k}(s^{t-1}), B_t^{n_k}\left(s^t\right), s_t\right) \right) \\
&\geq \sum_{t=0}^{\infty} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \hat{\beta}^t (1-\gamma)^{t-1} \pi^*\left(s^t\right) \left(\gamma \mathbf{1}_{\{\iota(s_t)=0\}} \left(A(s_t) - \bar{b}(s_t)\right) K_t^{n_k}(s^{t-1}) \right) \\
&\geq \sum_{s^T : s^{T-1} \in \mathcal{S}^{*,T-1}} \hat{\beta}^T (1-\gamma)^{T-1} \pi^*\left(s^T\right) \left(\gamma \mathbf{1}_{\{\iota(s_T)=0\}} \left(A(s_T) - \bar{b}(s_T)\right) K_T^{n_k}(s^{T-1}) \right).
\end{aligned}$$

Let

$$\alpha = \gamma \min_{s \in \mathcal{S}^*} \min_{s' \in \mathcal{S} \setminus \mathcal{S}^*} \pi(s'|s) \left(A(s) - \bar{b}(s)\right) > 0.$$

The last inequality implies

$$V^{FB}(K_{n_k}) + B_{n_k} \geq \alpha \sum_{s^T : s^{T-1} \in \mathcal{S}^{*,T-1}} \hat{\beta}^T (1-\gamma)^{T-1} \pi^*\left(s^{T-1}\right) \ K_T^{n_k}(s^{T-1}) \ .$$

We choose T sufficiently high such that

$$\left(\frac{\hat{\beta}}{\beta}\right)^T > \frac{\sup_n \left(V^{FB}(K_n) + B_n\right)}{\alpha} \left(\begin{array}{c} (1-\gamma) \max_{s \in \mathcal{S}^*} \left(v^{FB}(s) + M\right) + \\ \gamma \max_{s \in \mathcal{S} \setminus \mathcal{S}^*} \left(A(s) + M\right) \end{array} \right) \frac{1}{\epsilon}. \tag{24}$$

So

$$\begin{aligned}
&\sum_{t=0}^{\infty} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \beta^t (1-\gamma)^{t-1} \pi^* \left(s^t\right) \left(\begin{array}{c} (1-\gamma) \mathbf{1}_{\{\iota(s_t)=1\}} C_t^{n_k}\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0 \left(K_t^{n_k}(s^{t-1}), B_t^{n_k}\left(s^t\right), s_t\right) \end{array} \right) \\
&= \sum_{t=0}^{T-1} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \beta^t (1-\gamma)^{t-1} \pi^* \left(s^t\right) \left(\begin{array}{c} (1-\gamma) \mathbf{1}_{\{\iota(s_t)=1\}} C_t^{n_k}\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0 \left(K_t^{n_k}(s^{t-1}), B_t^{n_k}\left(s^t\right), s_t\right) \end{array} \right) \\
&+ \sum_{t=T}^{\infty} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \beta^t (1-\gamma)^{t-1} \pi^* \left(s^t\right) \left(\begin{array}{c} (1-\gamma) \mathbf{1}_{\{\iota(s_t)=1\}} C_t^{n_k}\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0 \left(K_t^{n_k}(s^{t-1}), B_t^{n_k}\left(s^t\right), s_t\right) \end{array} \right) \\
&\leq \sum_{t=0}^{T-1} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \beta^t (1-\gamma)^{t-1} \pi^* \left(s^t\right) \left(\begin{array}{c} (1-\gamma) \mathbf{1}_{\{\iota(s_t)=1\}} C_t^{n_k}\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0 \left(K_t^{n_k}(s^{t-1}), B_t^{n_k}\left(s^t\right), s_t\right) \end{array} \right) \\
&+ \sum_{s^T : s^{T-1} \in \mathcal{S}^{*,T-1}} \beta^T (1-\gamma)^{T-1} \pi^* \left(s^T\right) K_T^{n_k}(s^{T-1}) \left(\begin{array}{c} (1-\gamma) \mathbf{1}_{\{\iota(s_T)=1\}} \left(v^{FB}(s_T) + M\right) + \\ \gamma \mathbf{1}_{\{\iota(s_T)=0\}} \left(A(s_T) + M\right) \end{array} \right).
\end{aligned}$$

By the definition of T in equation (24),

$$\begin{aligned}
&\sum_{s^T : s^{T-1} \in \mathcal{S}^{*,T-1}} \beta^T (1-\gamma)^{T-1} \pi^* \left(s^T\right) K_T^{n_k}(s^{T-1}) \left(\begin{array}{c} (1-\gamma) \mathbf{1}_{\{\iota(s_T)=1\}} \left(v^{FB}(s_T) + M\right) + \\ \gamma \mathbf{1}_{\{\iota(s_T)=0\}} \left(A(s_T) + M\right) \end{array} \right) \\
&\leq \epsilon.
\end{aligned}$$

So the last inequality yields

$$\begin{aligned}
&\sum_{t=0}^{T-1} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \beta^t (1-\gamma)^{t-1} \pi^* \left(s^t\right) \left(\begin{array}{c} (1-\gamma) \mathbf{1}_{\{\iota(s_t)=1\}} C_t^{n_k}\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0 \left(K_t^{n_k}(s^{t-1}), B_t^{n_k}\left(s^t\right), s_t\right) \end{array} \right) \\
&\geq \sum_{t=0}^{\infty} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \beta^t (1-\gamma)^{t-1} \pi^* \left(s^t\right) \left(\begin{array}{c} (1-\gamma) \mathbf{1}_{\{\iota(s_t)=1\}} C_t^{n_k}\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0 \left(K_t^{n_k}(s^{t-1}), B_t^{n_k}\left(s^t\right), s_t\right) \end{array} \right) \\
&- \epsilon \\
&\geq V^M(K_n, B_n, s) - 2\epsilon.
\end{aligned}$$

Because of the subsequent convergence (23), there exists K such that for all $k \geq K$

$$\begin{aligned}
&\sum_{t=0}^{T-1} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \hat{\beta}^t (1-\gamma)^{t-1} \pi^* \left(s^t\right) \left(\begin{array}{c} (1-\gamma)\mathbf{1}_{\{\iota(s_t)=1\}} C_t^{n_k}\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0 \left(K_t^{n_k}(s^{t-1}), B_t^{n_k}\left(s^t\right), s_t\right) \end{array} \right) \\
&\leq \sum_{t=0}^{T-1} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \hat{\beta}^t (1-\gamma)^{t-1} \pi^* \left(s^t\right) \left(\begin{array}{c} (1-\gamma)\mathbf{1}_{\{\iota(s_t)=1\}} C_t\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0 \left(K_t(s^{t-1}), B_t\left(s^t\right), s_t\right) \end{array} \right) \\
&+ \epsilon.
\end{aligned}$$

Thus, for all $k \geq K$:

$$\begin{aligned}
V^M(K, B, s) \geq & u\left(\left\{C_t(s^t), K_{t+1}(s^t), B_{t+1}(s^{t+1})\right\}_{t=0}^{\infty}\right) \\
\geq & \sum_{t=0}^{T-1} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \hat{\beta}^t (1-\gamma)^{t-1} \pi^* \left(s^t\right) \left(\begin{array}{c} (1-\gamma)\mathbf{1}_{\{\iota(s_t)=1\}} C_t\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0 \left(K_t(s^{t-1}), B_t\left(s^t\right), s_t\right) \end{array} \right) \\
\geq & \sum_{t=0}^{T-1} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \hat{\beta}^t (1-\gamma)^{t-1} \pi^* \left(s^t\right) \left(\begin{array}{c} (1-\gamma)\mathbf{1}_{\{\iota(s_t)=1\}} C_t^{n_k}\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0 \left(K_t^{n_k}(s^{t-1}), B_t^{n_k}\left(s^t\right), s_t\right) \end{array} \right) \\
& - \epsilon \\
\geq & V^M(K_{n_k}, B_{n_k}, s) - 3\epsilon.
\end{aligned}$$

Sending ϵ to zero, we obtain the desired inequality (22).

Proof of Part 2: The proof is standard. The weak concavity of V^M comes from the the fact that $G(K', K)$ is convex. □

Proof of Lemma 4. The proof that v^M satisfies the Bellman equation (13) is standard. Now, we show that if $v : X_{\bar{b}}^0 \to \mathbb{R}^+$ satisfies the Bellman equation and is upper semi-continuous, then $Kv(\frac{B}{K}, s)$ is indeed the value function. First, we show that for any sequence

$$\left\{C_t^n(s^t), K_{t+1}^n(s^t), B_{t+1}^n(s^{t+1})\right\}_{t=0}^{\infty}$$

that satisfies (2), (3), (4) and (11) and (12), we have $u\left(\left\{C_t^n(s^t), K_{t+1}^n(s^t), B_{t+1}^n(s^{t+1})\right\}_{t=0}^{\infty}\right) \leq K_0 v(\frac{B_0}{K_0}, s_0)$. Indeed, from the Bellman equation,

$$\begin{aligned}
K_0 v\left(\frac{B_0}{K_0}, s_0\right) \geq & C_0 + \\
& \beta \left(\begin{array}{c} \gamma \sum_{s_1 \in \mathcal{S} \setminus S^*} \pi^*(s_1|s_0) \left(A(s_1) - \bar{b}(s_1)\right) K_1 \\ +(1-\gamma) \sum_{s_1 \in S^*} \pi^*(s_1|s_0) K_1(s_1) v\left(\frac{B_1(s_1, s_0)}{K_1(s_0)}, s_1\right) \end{array} \right). \qquad (25)
\end{aligned}$$

Iterating this inequality forward using the Bellman equation (25), and the fact that $v \geq 0$, we obtain the desired inequality. So

$$K_0 v\left(\frac{B_0}{K_0}, s_0\right) \geq V^M(K_0, B_0, s_0).$$

Because v is upper continuous and $X_{\bar{b}}^0$ is closed, together with the bounds in $k', b'(s')$, the supremum can be attained. Therefore, for each K_0, B_0, there exists $(s_1, K_1(s_0), B_1(s_1, s_0))$, such that (25) is satisfied with equality. By iterating this procedure forward, we obtain a sequence

$$\left\{C_t^n(s^t), K_{t+1}^n(s^t), B_{t+1}^n(s^{t+1})\right\}_{t=0}^{\infty},$$

that satisfies all the constraints, and conditional on s^{t-1}, the Bellman (25) is satisfied with equality. Because $v \geq 0$,

$$u\left(\left\{C_t^n(s^t), K_{t+1}^n(s^t), B_{t+1}^n(s^{t+1})\right\}_{t=0}^{\infty}\right) \leq K_0 v(\frac{B_0}{K_0}, s_0). \tag{26}$$

Also by construction,

$$\begin{aligned}
K_0 v(\frac{B_0}{K_0}, s_0) = &\sum_{t=0}^{T-1} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \beta^t (1-\gamma)^{t-1} \pi^*\left(s^t\right) \begin{pmatrix} (1-\gamma)\mathbf{1}_{\{\iota(s_t)=1\}} C_t\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} \left(A(s_{t-1}) - \bar{b}\left(s_{t-1}\right)\right) K_t(s^{t-1}) \end{pmatrix} \\
&+ \sum_{s^T : s^{T-1} \in \mathcal{S}^{*,T-1}} \beta^T (1-\gamma)^{T-1} \pi^*\left(s^T\right) K_T(s^{T-1}) \begin{pmatrix} (1-\gamma)\mathbf{1}_{\{\iota(s_T)=1\}} \left(v(s_T, \frac{B_T(s^T)}{K_T(s^{T-1})})\right) + \\ \gamma \mathbf{1}_{\{\iota(s_T)=0\}} \left(A(s_T) - \bar{b}(s_T)\right) \end{pmatrix} \\
\leq &u\left(\left\{C_t^n(s^t), K_{t+1}^n(s^t), B_{t+1}^n(s^{t+1})\right\}_{t=0}^{\infty}\right) \\
&+ \sum_{s^T : s^{T-1} \in \mathcal{S}^{*,T-1}} \beta^T (1-\gamma)^{T-1} \pi^*\left(s^T\right) K_T(s^{T-1}) \begin{pmatrix} (1-\gamma)\mathbf{1}_{\{\iota(s_T)=1\}} \left(v^{FB}(s_T) + M\right) + \\ \gamma \mathbf{1}_{\{\iota(s_T)=0\}} \left(A(s_T) - \bar{b}(s_T)\right) \end{pmatrix} \\
\leq &u\left(\left\{C_t^n(s^t), K_{t+1}^n(s^t), B_{t+1}^n(s^{t+1})\right\}_{t=0}^{\infty}\right) \\
&+ \alpha \sum_{s^{T-1} \in \mathcal{S}^{*,T-1}} \beta^{T-1} (1-\gamma)^{T-1} \pi^*\left(s^{T-1}\right) K_T(s^{T-1}),
\end{aligned} \tag{27}$$

where

$$\alpha = \beta \max_{s \in \mathcal{S}^*} \sum_{s' \in \mathcal{S}^*} \pi^*(s'|s) \left(1-\gamma\right) \left(v^{FB}(s') + M\right) + \sum_{s' \in \mathcal{S} \setminus \mathcal{S}^*} \pi^*(s'|s) \gamma \left(A(s') - \bar{b}(s')\right).$$

Now, from inequality (26),

$$K_0 v(\frac{B_0}{K_0}, s_0) \geq \sum_{t=0}^{\infty} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*,t-1}} \beta^t (1-\gamma)^{t-1} \pi^* \left(s^t\right) \gamma \left(A(s_{t-1}) - \bar{b}\left(s_{t-1}\right)\right) K_t(s^{t-1}).$$

Therefore,

$$\lim_{T\to\infty} \sum_{s^{T-1} \in \mathcal{S}^{*,T-1}} \beta^{T-1} (1-\gamma)^{T-1} \pi^* \left(s^{T-1}\right) K_T(s^{T-1}) = 0.$$

Combining this limit with (27) implies,

$$u\left(\left\{C_t^n(s^t), K_{t+1}^n(s^t), B_{t+1}^n(s^{t+1})\right\}_{t=0}^{\infty}\right) \geq K_0 v(\frac{B_0}{K_0}, s_0). \tag{28}$$

From (26) and (28),

$$u\left(\left\{C_t^n(s^t), K_{t+1}^n(s^t), B_{t+1}^n(s^{t+1})\right\}_{t=0}^{\infty}\right) \geq K_0 v(\frac{B_0}{K_0}, s_0).$$

Therefore

$$K_0 v\left(\frac{B_0}{K_0}, s_0\right) = V^M(K_0, B_0, s_0).$$

Moreover, the supremum can be attained by $\left\{C_t^n(s^t), K_{t+1}^n(s^t), B_{t+1}^n(s^{t+1})\right\}_{t=0}^{\infty}$. □

Proof of Lemma 5. Let $k^*(s)$ be defined by

$$G_1(k^*(s), 1) = \hat{\beta}\gamma \sum_{s' \in \mathcal{S} \backslash S^*} \pi^*(s'|s)\bar{b}(s') + \beta\gamma \sum_{s' \in \mathcal{S} \backslash S^*} \pi^*(s'|s)\left(A(s') - \bar{b}(s')\right).$$

First we show that if $k' < k^*(s)$, then $c = 0$. Assume by contradiction that $c > 0$. We increase k' by $\Delta k > 0$, this requires a decrease in c of

$$\Delta k \left(G_1(k', 1) + \hat{\beta}(1-\gamma) \sum_{s' \in S^*} \pi^*(s'|s) b'(s') - \hat{\beta}\gamma \sum_{s' \in \mathcal{S} \backslash S^*} \pi^*(s'|s)\bar{b}(s')\right)$$

So the change in the objective function is

$$\begin{aligned} &- \Delta k \left(G_1(k', 1) + \hat{\beta}(1-\gamma) \sum_{s' \in S^*} \pi^*(s'|s) b'(s') - \hat{\beta}\gamma \sum_{s' \in \mathcal{S} \backslash S^*} \pi^*(s'|s)\bar{b}(s')\right) \\ &+ \Delta k \beta \left(\gamma \sum_{s' \in \mathcal{S} \backslash S^*} \pi^*(s'|s)\left(A(s') - \bar{b}(s')\right) + (1-\gamma) \sum_{s' \in S^*} \pi^*(s'|s) v(b'\left(s'\right), s')\right) \end{aligned}$$

If $k' < k^*$, we show that

$$G_1(k',1) + \hat{\beta}(1-\gamma) \sum_{s' \in S^*} \pi^*(s'|s) b'(s') - \hat{\beta}\gamma \sum_{s' \in \mathcal{S} \setminus S^*} \pi^*(s'|s) \bar{b}(s')$$
$$< \beta \left(\gamma \sum_{s' \in \mathcal{S} \setminus S^*} \pi^*(s'|s) \left(A(s') - \bar{b}(s') \right) + (1-\gamma) \sum_{s' \in S^*} \pi^*(s'|s) v(b'(s'), s') \right) \tag{29}$$

Indeed, by Lemma 11, if $\hat{\beta} b'(s') \leq v(b'(s'), s')$. Therefore, if $k' < k^*(s)$, i.e., $G_1(k',1) < \hat{\beta}\gamma \sum_{s' \in \mathcal{S} \setminus S^*} \pi^*(s'|s) \bar{b}(s') + \beta\gamma \sum_{s' \in \mathcal{S} \setminus S^*} \pi^*(s'|s) \left(A(s') - \bar{b}(s') \right)$, (29) holds. Then we can find strict improvement by increasing k' and decreasing c, a contradiction.

Thus if $k' < k^*$, $c = 0$. Now if we choose $\underline{M} < k^*$ such that

$$\beta \underline{M} \left(\gamma \sum_{s' \in \mathcal{S} \setminus S^*} \pi^*(s'|s) \left(A(s') - \bar{b}(s') \right) + (1-\gamma) \sum_{s' \in S^*} \pi^*(s'|s) \left(v^{FB}(s') + M \right) \right)$$

strictly less than the minimum value $\underline{v}(s)$ defined in Lemma 2, then $k' > \underline{M}$.

Let $\lambda \geq 1$ denote the multiplier on the budget constraint. The F.O.C in k' implies that

$$\begin{aligned} G_1(k',1) =& -\hat{\beta}(1-\gamma) \sum_{s' \in S^*} \pi^*(s'|s) b'(s') + \hat{\beta}\gamma \sum_{s' \in \mathcal{S} \setminus S^*} \pi^*(s'|s) \bar{b}(s') \\ & + \frac{1}{\lambda} \beta \left(\gamma \sum_{s' \in \mathcal{S} \setminus S^*} \pi^*(s'|s) \left(A(s') - \bar{b}(s') \right) + (1-\gamma) \sum_{s' \in S^*} \pi^*(s'|s) v(b'(s'), s') \right). \\ \leq & \hat{\beta}(1-\gamma) \sum_{s' \in S^*} \pi^*(s'|s) \bar{b}(s') + \hat{\beta}\gamma \sum_{s' \in \mathcal{S} \setminus S^*} \pi^*(s'|s) \bar{b}(s') \\ & + \beta(1-\gamma) \sum_{s' \in S^*} \pi^*(s'|s) \left(v^{FB}(s') + M \right) + \beta\gamma \sum_{s' \in \mathcal{S} \setminus S^*} \pi^*(s'|s) \left(A(s') - \bar{b}(s') \right). \end{aligned}$$

Let $\overline{M}$ denote the level of k' such that $G_1(k',1)$ strictly greater than the last expression (which is a function of M). Then, $k' < \overline{M}$, strictly. □

Lemma 11. *Given* $\left(c, k', (b'(s'))_{s' \in \mathcal{S}} \right)$ *that solves of the optimization problem* (13), *then for any* $s' \in \mathcal{S}^*$

$$\hat{\beta} b'(s') \leq \beta v(b'(s'), s').$$

Proof. This is obvious if $b'(s') \leq 0$ since $v \geq 0$. Now if $b'(s') > 0$, we increase c by $\Delta c > 0$ and decrease $b'(s')$ by $\frac{\Delta c}{\hat{\beta}(1-\gamma)k'\pi^*(s'|s)}$, which leads to a change in the objective function

(13) by

$$+\Delta c - \beta(1-\gamma)\pi^*(s'|s)\left(v\left(b'\left(s'\right),s'\right) - v\left(b'\left(s'\right) - \frac{\Delta c}{\hat{\beta}(1-\gamma)k'\pi^*(s'|s)}, s'\right)\right),$$

which is a strict improvement if

$$v\left(b'\left(s'\right),s'\right) - v\left(b'\left(s'\right) - \Delta b, s'\right) < \frac{\hat{\beta}}{\beta}\Delta b, \tag{30}$$

where

$$\Delta b = \frac{\Delta c}{\hat{\beta}(1-\gamma)k'\pi^*(s'|s)}.$$

So, since $b'(s') > 0$ for $\Delta b > 0$ sufficiently small, (30) must not be satisfied, i.e.

$$v\left(b'\left(s'\right) - \Delta b, s'\right) - v\left(b'\left(s'\right),s'\right) \geq \frac{\hat{\beta}}{\beta}\Delta b. \tag{31}$$

By Lemma 12, and (31), we then have

$$\hat{\beta}b'(s') \leq \beta v(b'\left(s'\right), s').$$

□

Lemma 12. *Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$, strictly increasing and weakly concave, and, $x > 0$ such that there exists $d, \Delta > 0$ such that*

$$f(x+\Delta) - f(x) \geq d\Delta.$$

Then $f(x) \geq dx$.

Proof. Because f is weakly concave,

$$\frac{f(x) - f(0)}{x} \geq \frac{f(x) - f(x-\Delta)}{\Delta} \geq d.$$

Moreover, because f is positive,

$$\frac{f(x)}{x} \geq \frac{f(x) - f(0)}{x}.$$

Thus we obtain the desired inequality. □

Proof of Lemma 6. Let λ denote the multiplier on the budget constraint (16). From the

F.O.C. in c, $\lambda \geq 1$. From the F.O.C. in k' (because the constraints (14) on k' do not bind):

$$\lambda = \frac{\beta\left(\gamma\sum_{s'\in\mathcal{S}\setminus S^*}\pi^*(s'|s)\left(A(s')-\bar{b}(s')\right)+(1-\gamma)\sum_{s'\in S^*}\pi^*(s'|s)v(b'(s'),s')\right)}{G_1(k',1)-\hat{\beta}\gamma\sum_{s'\in\mathcal{S}\setminus S^*}\pi^*(s'|s)\bar{b}(s')+\hat{\beta}(1-\gamma)\sum_{s'\in S^*}\pi^*(s'|s)b'(s')}$$

Because $v(b'(s'),s') \leq v^{FB}(s') + b'(s')$, this implies

$$\lambda \leq \frac{\beta\left(\gamma\sum_{s'\in\mathcal{S}\setminus S^*}\pi^*(s'|s)\left(A(s')-\bar{b}(s')\right)+(1-\gamma)\sum_{s'\in S^*}\pi^*(s'|s)v^{FB}(s')+(1-\gamma)b(s)\right)}{G_1(k',1)-\hat{\beta}\gamma\sum_{s'\in\mathcal{S}\setminus S^*}\pi^*(s'|s)\bar{b}(s')+\hat{\beta}(1-\gamma)b(s)},$$

where $b(s) = \sum_{s'\in S^*}\pi^*(s'|s)b'(s')$. Because $G_1 \geq 0$ and $\lambda \geq 1$,

$$1 \leq \frac{\beta\left(\gamma\sum_{s'\in\mathcal{S}\setminus S^*}\pi^*(s'|s)\left(A(s')-\bar{b}(s')\right)+(1-\gamma)\sum_{s'\in S^*}\pi^*(s'|s)v^{FB}(s')+(1-\gamma)b(s)\right)}{-\hat{\beta}\gamma\sum_{s'\in\mathcal{S}\setminus S^*}\pi^*(s'|s)\bar{b}(s')+\hat{\beta}(1-\gamma)b(s)}.$$

Because $\beta < \hat{\beta}$, there exists b^* such that for all $b \geq b^*$

$$1 > \frac{\beta\left(\gamma\sum_{s'\in\mathcal{S}\setminus S^*}\pi^*(s'|s)\left(A(s')-\bar{b}(s')\right)+(1-\gamma)\sum_{s'\in S^*}\pi^*(s'|s)v^{FB}(s')+(1-\gamma)b(s)\right)}{-\hat{\beta}\gamma\sum_{s'\in\mathcal{S}\setminus S^*}\pi^*(s'|s)\bar{b}(s')+\hat{\beta}(1-\gamma)b(s)}.$$

Thus $b(s) < b^*$. So if $M \geq \frac{b^*+\max_{s\in\mathcal{S}^*}\sum_{s'\in\mathcal{S}^*}\pi(s'|s)\bar{b}(s')}{\min_{s,s'\in\mathcal{S}^*}\pi(s'|s)}$, then $b'(s') < M$ strictly. □

2 Extensions

In this section, we consider two extensions of the analysis in the previous section. In the first extension, we allow for unbounded first-best and in the second extension, we allow for decreasing return to scale in the production function of the entrepreneurs.

2.1 Unbounded First-Best

Numerically, we find that even when the first best yield infinite value, i.e. Assumption 3 is not satisfied, but financial friction is severe, i.e. θ is sufficiently small, there exists an equilibrium in which the value function of the entrepreneurs is well-defined and is finite.

The following theorem provides sufficient condition for a function satisfying the Bellman equation (5) to be the value function given the borrowing limits $\bar{b}(s)$.

Theorem 3. *Suppose $Kv\left(\frac{B}{K}, s\right) \geq 0$ is a solution of the Bellman equation (5), in which the* sup *is attained and can be replaced by* max.

In addition, v satisfies:

A1) There exists a long run ergodic intervals $\left[\underline{b}(s), \bar{b}(s)\right]$, i.e. $\underline{b}(s') \leq b'(b, s, s') \leq \bar{b}(s')$ for all $b \in [\underline{b}(s), \bar{b}(s)]$.

A2) There exist $\underline{v}, \overline{v} > 0$ such that $\underline{v} < v(b, s) < \overline{v}$ for all $b \in \left[\underline{b}(s), \bar{b}(s)\right]$

Then $Kv\left(\frac{B}{K}, s\right)$ is the value function $V(K, B, s; \bar{b})$.

Proof. We first need to show that, for all sequence $\left\{C_t\left(s^t\right), K_{t+1}\left(s^t\right), B_{t+1}\left(s^{t+1}\right)\right\}_{t,s^t}$ satisfies (2), (3), and (4), we have

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) C_t\left(s^t\right) \leq K_0 v\left(\frac{B_0}{K_0}, s_0\right). \tag{32}$$

Indeed, by the definition of Bellman equation (5), we have

$$\begin{aligned}
K_0 v\left(\frac{B_0}{K_0}, s_0\right) &\geq C_0\left(s_0\right) + \beta \sum_{s^1|s_0} \pi\left(s_1|s_0\right) K_1 v\left(\frac{B_1\left(s_1\right)}{K_1}, s_1\right) \\
&\geq C_0\left(s_0\right) + \beta \sum_{s^1|s_0} \pi\left(s_1|s_0\right) C_1\left(s_1\right) + \beta \sum_{s^2|s_0} \pi\left(s^2\right) K_2 v\left(\frac{B_2\left(s^2\right)}{K_2\left(s^1\right)}, s_1\right) \\
&\geq \ldots \\
&\geq \sum_{t=0}^{T} \sum_{s^t|s_0} \beta^t \pi\left(s^t\right) C_t\left(s^t\right) \\
&+ \sum_{s^{T+1}|s_0} \beta^{T+1} \pi\left(s^{T+1}\right) K_{T+1}\left(s^T\right) v\left(\frac{B_{T+1}\left(s^{T+1}\right)}{K_{T+1}\left(s^T\right)}, s_{T+1}\right)
\end{aligned}$$

As T goes to infinity and $v \geq 0$, we obtain (32).

Second, we need to show that v is actually attainable, i.e., there exists a sequence

$$\left\{C_t\left(s^t\right), K_{t+1}\left(s^t\right), B_{t+1}\left(s^{t+1}\right)\right\}_{t,s^t}$$

that satisfies (2), (3), and (4) such that

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) C_t\left(s^t\right) = K_0 v\left(\frac{B_0}{K_0}, s_0\right).$$

Indeed, by definition of v, we can find $\{c_t, k_t, b_{t+1}\}_{t=0}^{\infty}$ such that for each T:

$$v(b,s)=\mathbb{E}_0\left[\begin{array}{c}\sum_{s^T\in\mathcal{S}^{*,T}}\{\sum_{t=0}^{T}(\beta(1-\gamma))^t\left(\prod_{t'=0}^{t}k_{t'}\right)c_t\\ +(\beta(1-\gamma))^{T+1}\left(\prod_{t'=0}^{T+1}k_{t'}\right)v(b_{T+1},s_{T+1})\}\end{array}\right]$$
$$+\sum_{t=0}^{T}\mathbb{E}_0\left[\sum_{s^t\in\mathcal{S}^{*,t-1}\times(\mathcal{S}\backslash\mathcal{S}^*)}(\beta(1-\gamma))^t\gamma\left(\prod_{t'=0}^{t+1}k_{t'}\right)v(b_{t+1},s_{t+1})\right]$$

Because of condition A1, we can choose T^* such that $b_T\in\left[\underline{b},\overline{b}\right]$ for all $T\geq T^*$. And because of condition A2, for $T\geq T^*$, we have $v(b_{T+1},s_{T+1})\geq\underline{v}>0$ and $c_t\geq 0$, so

$$v(b,s)>\sum_{t=0}^{T}\mathbb{E}_0\left[\sum_{s^T\in\mathcal{S}^{*,T-1}\times(\mathcal{S}\backslash\mathcal{S}^*)}(\beta(1-\gamma))^t\gamma\left(\prod_{t'=0}^{t+1}k_{t'}\right)v(b_{t+1},s_{t+1})\right]$$
$$>\sum_{t=0}^{T}\mathbb{E}_0\left[\sum_{s^{T-1}\in\mathcal{S}^{*,T-1}}(\beta(1-\gamma))^{T-1}\gamma\left(\prod_{t'=0}^{T}k_{t'}\right)\underline{v}\right].$$

Therefore

$$\lim_{T\longrightarrow\infty}\mathbb{E}_0\left[\sum_{s^{T-1}\in\mathcal{S}^{*,T-1}}(\beta(1-\gamma))^{T-1}\left(\prod_{t'=0}^{T}k_{t'}\right)\right]=0.$$

Combining this limit with the bounds for $v(b_{T+1},s_{T+1})$ in condition A2, we obtain

$$\lim_{T\longrightarrow\infty}\mathbb{E}_0\left[\sum_{s^T\in\mathcal{S}^{*,T}}(\beta(1-\gamma))^{T+1}\left(\prod_{t'=0}^{T+1}k_{t'}\right)v(b_{T+1},s_{T+1})\right]=0.$$

Therefore

$$v(b,s)=\lim_{T\longrightarrow\infty}\mathbb{E}_0\left[\sum_{s^T\in\mathcal{S}^{*,T}}\sum_{t=0}^{T}(\beta(1-\gamma))^t\left(\prod_{t'=0}^{t}k_{t'}\right)c_t\right]$$
$$+\sum_{t=0}^{T}\mathbb{E}_0\left[\sum_{s^t\in\mathcal{S}^{*,t-1}\times(\mathcal{S}\backslash\mathcal{S}^*)}(\beta(1-\gamma))^t\gamma\left(\prod_{t'=0}^{t+1}k_{t'}\right)v(b_{t+1},s_{t+1})\right]$$
$$=\lim_{T\longrightarrow\infty}\mathbb{E}_0\left[\sum_{t=0}^{T}\beta^t\left(\prod_{t'=0}^{t}k_{t'}\right)c_t\right].$$

Thus $v(b,s)$ is attainable. $\left\{C_t\left(s^t\right),K_{t+1}\left(s^t\right),B_{t+1}\left(s^{t+1}\right)\right\}_{t,s^t}$ is obtained from $\{c_t,k_t,b_{t+1}\}_{t=0}^{\infty}$ using $K_t=K_0\prod_{t_1=1}^{t}k_{t_1}$ and $C_t=K_tc_t$, $B_t=K_tb_t$. $\square$

Therefore, if in addition, if v satisfies

$$\bar{b}(s) = \inf\left\{b : (1, b, s) \in X_{\bar{b}} \text{ and } v(b, s) \geq (1-\theta)\, v(0, s)\right\},$$

then v is an equilibrium value function. We can verify this property numerically.

2.2 Decreasing Return to Scale

If the production function exhibits decreasing return to scale: $F(K_t, s_t)$, then the value function is no longer homogeneous.[3] The first-best problem is characterized by the following Bellman equation. For $s \in \mathcal{S}^*$

$$\begin{aligned} v^{FB}(K, s) = \max_{K' > 0} F(K, s) - G(K', K) + \hat{\beta}(1-\gamma) \sum_{s' \in S^*} \pi^*(s'|s) v^{FB}(K', s') \\ + \hat{\beta}\gamma \sum_{s' \in \mathcal{S} \setminus \mathcal{S}^*} \pi^*(s'|s) F(K', s'). \end{aligned} \tag{33}$$

Since F has decreasing return to scale, we can choose $\bar{K}$ such that $F(K, s) - G(K, K) < 0$ for all $K \geq \bar{K}$ and for all $s \in \mathcal{S}$. In this case we can restrict the entrepreneurs' optimization problem to a bounded interval of capital $[0, \bar{K}]$.

For the entrepreneurs, the borrowing constraint now depends on existing capital, i.e. $\bar{B}(K, s)$.

$$V(K_0, B_0, s_0; \overline{B}) = \sup_{\left\{C_t(s^t), K_{t+1}(s^t), B_{t+1}\left(s^{t+1}\right)\right\}_{t, s^t}} u\left(\left\{C_t\left(s^t\right), K_{t+1}\left(s^t\right), B_{t+1}\left(s^{t+1}\right)\right\}_{t, s^t}\right) \tag{34}$$

$$u = \sum_{t=0}^{\infty} \sum_{s^t : s^{t-1} \in \mathcal{S}^{*, t-1}} \beta^t (1-\gamma)^{t-1} \pi^*\left(s^t\right) \left(\begin{array}{c} (1-\gamma)\mathbf{1}_{\{\iota(s_t)=1\}} C_t\left(s^t\right) + \\ \gamma \mathbf{1}_{\{\iota(s_t)=0\}} V_0\left(K_t(s^{t-1}), B_t\left(s^t\right), s_t\right) \end{array}\right)$$

$s.t.$

$$C_t\left(s^t\right) + G\left(K_{t+1}\left(s^t\right), K_t\left(s^{t-1}\right)\right) \leq F\left(K_t\left(s^{t-1}\right), s_t\right) + B_t\left(s^t\right) - \hat{\beta} \sum_{s^{t+1}|s^t} \pi\left(s^{t+1}|s^t\right) B_{t+1}\left(s^{t+1}\right) \tag{35}$$

$$C_t(s^t) \geq 0, \bar{K} > K_{t+1}\left(s^t\right) > 0, \tag{36}$$

$$B_{t+1}(s^{t+1}) \geq -\overline{B}\left(K_{t+1}\left(s^t\right), s_{t+1}\right), \tag{37}$$

[3] Rampini and Viswanathan (2013) is a special case without adjustment cost.

where for each $s \in \mathcal{S} \backslash \mathcal{S}^*$:

$$V_0(K, B, s) = F(K, s) + B.$$

Since the entrepreneurs' optimization problem is defined over a finite interval $[0, \bar{K}]$, the standard contraction mapping approach applies. The value function $V(K, B; \bar{B})$ of the entrepreneurs is the unique fixed point of a Bellman equation.

As in Subsection 1.2, we look for $\overline{B}$ such that

$$\overline{B}(K, s) = \inf \left\{ B : (K, B, s) \in X_{\bar{B}} \text{ and } V(K, B, s) \geq V((1-\theta) K, 0, s) \right\}.$$

A priori, this borrowing limit is a nonlinear function of K, unlike the result in Theorem 2.

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